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Quasi-Isometrically Embedded Subgroups of Thompson's Group F

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The goal of this paper is to study subgroups of Thompson's group F which are isomorphic to $F \times \mathbb{Z}^n$ and $F \times F$. A result estimating the norm of an element of Thompson's group is found, and this estimate is used to prove that these particular subgroups are quasi-isometrically embedded. © 1999 Academic Press

The interesting properties of Thompson's group F have made it a favorite object of study among group theorists and topologists. It was first used by McKenzie and Thompson to construct finitely presented groups with unsolvable word problems [7]. It is of interest also in homotopy theory in work related to homotopy idempotents, due to its universal conjugacy idempotent map ϕ, also used to see that F is an infinitely iterated HNN extension. In [2] Brown and Geoghegan found F to be the first torsion-free infinite-dimensional FP_∞ group. Also, F contains an abelian free group of infinite rank, but it does not admit a free nonabelian subgroup [1]. These properties suggest that, even though F is finitely presented, it has certain features which are usually found in infinitely generated groups. The fact that it admits quasi-isometrically embedded subgroups isomorphic to $F \times \mathbb{Z}^n$ and $F \times F$ is another example of these tendencies.

Many questions about F are still open, in particular it is not known whether F is automatic, or what is its Dehn function—although some estimates have been found by Guba, who proves it is polynomial in [5]. The amenability of F is also unknown, a fact that is of considerable interest

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since F is expected to be a counterexample to the von Neumann conjecture (a group is amenable if and only if it does not contain a free nonabelian subgroup) for finitely presented groups.

Questions about the geometric properties of F have also been proposed. Bridson raised the question of whether F and $F \times \mathbb{Z}$ could be quasi-isometric. In this paper we provide a partial answer to this question, proving that F admits a quasi-isometrically embedded subgroup isomorphic to $F \times \mathbb{Z}$. As a corollary, F is the first example of a finitely presented group whose asymptotic cone is infinite-dimensional (see [4]).

There are in the literature several interpretations of F that are useful to study it. Cannon *et al.* provide two of these interpretations in [3], one as a group of homeomorphisms of the interval $[0, 1]$, and another in terms of tree diagrams. In [2] Brown and Geoghegan prove that F is isomorphic to a certain group of piecewise linear homeomorphisms of $\mathbb{R}$, a fact that will be used extensively in this paper. This construction allows us to translate group-theoretical questions to the setting of these homeomorphisms of $\mathbb{R}$. Another interesting geometric interpretation for F has been described recently by Guba and Sapir in [6]. Some of the results below—mainly Proposition 4—admit alternate geometric proofs using the diagrams constructed in [6].

The organization of this paper is as follows: after a summary of results about F in Section 1, including a description of the geometric realizations for F, an estimate of the norm of an element of F is found in Section 2. The last sections are dedicated to stating and proving the results about the different subgroups of F.

1. GENERALITIES ON THOMPSON'S GROUP F

Thompson's group F is the group defined by the infinite presentation

$$\mathscr{P} = \langle x_k, k \geq 0 \mid x_i^{-1} x_j x_i = x_{j+1}, \text{ if } i < j \rangle.$$

Even though this is an infinite presentation, it follows from the relations that the generators x_k, for $k > 1$, are a consequence of x_0 and x_1. In fact, F admits a finite presentation given by

$$\mathscr{F} = \left\langle x_0, x_1 \mid \left[x_0 x_1^{-1}, x_0^{-1} x_1 x_0 \right], \left[x_0 x_1^{-1}, x_0^{-2} x_1 x_0^2 \right] \right\rangle$$

(see [2]). Throughout this paper, every time we refer to the word metric, or the norm, or the distance in F, it will make reference to the finite presentation $\mathscr{F}$ of F.

It is a consequence of the presentation $\mathscr{P}$ that the map $\phi\colon F \to F$ defined by $\phi(x_i) = x_{i+1}$ is a conjugacy idempotent, i.e., satisfies $\phi^2(x) = x_0^{-1} \phi(x) x_0$, for all $x \in F$. Also, ϕ is injective, mapping F to the copy of itself generated by x_k, for $k \geq 1$. Moreover, this map is shown in [2] to be universal among conjugacy idempotents.

It is also seen in [2, 3] that the elements of F admit a unique normal form in the generators of $\mathscr{P}$. From the relations in $\mathscr{P}$ it is easy to see that any element of F admits an expression of the form

$$x_{i_1}^{r_1} x_{i_2}^{r_2} \cdots x_{i_n}^{r_n} x_{j_m}^{-s_m} \cdots x_{j_2}^{-s_2} x_{j_1}^{-s_1}$$

such that

(1) $r_1, \ldots, r_n, s_1, \ldots, s_m > 0$,
(2) $i_1 < i_2 < \cdots < i_n$,
(3) $j_1 < j_2 < \cdots < j_m$, and
(4) $i_n \neq j_m$.

This expression is not unique if we do not require an extra condition: if for some i both x_i and x_i^{-1} appear, then either x_{i+1} or x_{i+1}^{-1} must appear as well. For otherwise there would be a subproduct of the form $x_i \phi^2(x) x_i^{-1}$ that could be replaced by $\phi(x)$. From this construction it is clear that, given a word in the generators x_k of $\mathscr{P}$, using the relations and the extra condition, we can obtain the unique normal form, and the length of the word does not increase in this process. In other words, the unique normal form is the shortest of all the words that represent a given element in the generators of $\mathscr{P}$. This fact will be used later in this paper.

Geometric realizations of F as a group of homeomorphisms of the real line, or of a closed interval, can be used to deduce algebraic properties of F. Two realizations will be used in this paper to understand the different subgroups of F.

Brown and Geoghegan show in [2] that G admits an isomorphism with the group of certain piecewise linear homeomorphisms of $\mathbb{R}$. Let

$$f_k : \mathbb{R} \to \mathbb{R}$$

be the map defined by

$$f_k(t) = \begin{cases} t & \text{if } t \leq k, \\ 2t - k & \text{if } k \leq t \leq k+1, \\ t+1 & \text{if } t \geq k+1. \end{cases}$$

The group G generated by the maps f_k, for all integers $k \geq 0$, and with right action, is isomorphic to Thompson's group, with x_k identified with f_k. The right action means that the composition of two maps is written the opposite way: the element $x_i x_j$ of F is associated with the element $f_i f_j$ of G, which represents the map $f_j \circ f_i$ on $\mathbb{R}$. From now on we will identify the groups F and G. This construction will be extremely useful in Section 2.

Using this construction it is easy to see more properties of F: the subgroup of F generated by the elements $x_{2k} x_{2k+1}^{-1}$, for $k \geq 0$, is a free abelian subgroup of infinite rank. To see that two of these elements commute, observe that the map $f_{2k}f_{2k+1}^{-1}$ is the identity except in the interval $[2k, 2k+2]$. Also, due to this fact, it is clear that $x_0x_1^{-1}$ commutes with x_k for $k \geq 2$, so the subgroup generated by $x_0x_1^{-1}$, x_2, and x_3 is isomorphic to $F \times \mathbb{Z}$. In Section 3 it will be proved that this subgroup is nondistorted in F.

Another interesting subgroup of G can be constructed by considering those maps $f \in G$ which satisfy $f(1) = 1$. This subgroup is isomorphic to $G \times G$. The second component is the subgroup $\phi(G)$ of elements of G which are the identity in the interval $[0, 1]$, which is clearly isomorphic to G, generated by f_1 and f_2. To see that the first component, which is the subgroup of maps $f \in G$ which are the identity everywhere except in the interval $[0, 1]$, we will use another geometric realization of F as a group of homeomorphisms of $[0, 1]$. A complete description of this realization can be found in [3].

Let H be the group of orientation-preserving homeomorphisms g of $[0, 1]$ which satisfy the following properties:

(1) g is piecewise linear,

(2) the derivative of g, at any point where it is defined, is a power of 2, and

(3) the points where g is not differentiable are dyadic integers.

It is proved in [3] that this group is isomorphic to F. The authors of [3] write the compositions acting on the left and use A and B for the generators of F, so since we take compositions on the right, we will take as generators of H the elements

$$g_0(t) = \begin{cases} 2t & \text{if } t \in \left[0, \frac{1}{4}\right] \\ t + \frac{1}{4} & \text{if } t \in \left[\frac{1}{4}, \frac{1}{2}\right] \\ \frac{t}{2} + \frac{1}{2} & \text{if } t \in \left[\frac{1}{2}, 1\right] \end{cases} \qquad g_1(t) = \begin{cases} t & \text{if } t \in \left[0, \frac{1}{2}\right] \\ 2t - \frac{1}{2} & \text{if } t \in \left[\frac{1}{2}, \frac{5}{8}\right] \\ t + \frac{1}{8} & \text{if } t \in \left[\frac{5}{8}, \frac{3}{4}\right] \\ \frac{t}{2} + \frac{1}{2} & \text{if } t \in \left[\frac{3}{4}, 1\right], \end{cases}$$

which correspond to the elements A^{-1} and B^{-1} in [3].

If we consider the subgroup of H of those maps g such that $g(\frac{1}{2}) = \frac{1}{2}$, it is clear that this subgroup is isomorphic to $H \times H$, where the two components of $H \times H$ are the maps which are the identity in the intervals

$[\frac{1}{2}, 1]$ and $[0, \frac{1}{2}]$, respectively. Generators for the second component are g_1 and $g_2 = g_0^{-1} g_1 g_0$, and for the first component the generators are $g_0^2 g_1^{-1} g_0^{-1}$ and $g_0 g_1^2 g_2^{-1} g_1^{-1} g_0^{-1}$. So from this realization of F it is clear that F has a subgroup isomorphic to $F \times F$.

We can also identify H with the subgroup of G (see the first realization) of elements which are the identity everywhere except in $[0, 1]$: an element of H can be thought as the restriction to $[0, 1]$ of one of this maps in G. This way we see that this subgroup is also isomorphic to F. The elements $f_0^2 f_1^{-1} f_0^{-1}$ and $f_0 f_1^2 f_2^{-1} f_1^{-1} f_0^{-1}$ correspond to g_0 and g_1 under this identification, hence they generate this subgroup of G.

2. THE ESTIMATE OF THE NORM

The geometric interpretation given by the maps f_k provides a method to compute the word metric of F (with respect to the finite presentation $\mathscr{F}$). For instance, to compute the norm of an element x of F, we can study the corresponding map f in G: we know that f can be obtained as a composition of the maps f_0 and f_1 and their inverses, and we only need to estimate how many occurrences of f_0 and f_1 and their inverses we need to obtain f. This can be studied from properties of the graph of f.

Given a point (a, b) of the graph of f, with $b = f(a)$, since f is piecewise linear, we denote by $f'_+(a)$ and $f'_-(a)$ the right and left derivatives of f at a. If $f'_+(a) \neq f'_-(a)$, we say that the point (a, b) is a breaking point of the graph of f.

To completely understand the maps in G we need to study how multiplying by a generator affects a map. Let $f \in G$ be one of these piecewise linear homeomorphisms. Then, the map $ff_i = f_i \circ f$ has a graph that can be easily related to the graph of f. Since the map f_i has slope 2 only on those points with y-coordinate in the interval $[i, i+2]$, the graph of ff_i is obtained by stretching the portion of the graph that has y-coordinate in $[i, i+1]$ to the interval $[i, i+2]$, and all the graph is moved one unit up in all the points with $y \geq i+1$. A point (a, b) on the graph of f with $b \in (i, i+1)$ appears now as the point $(a, i + 2(b-i))$, and the derivatives satisfy

$$(ff_i)'_+(a) = 2f'_+(a) \qquad \text{and} \qquad (ff_i)'_-(a) = 2f'_-(a).$$

Similarly, the map ff_i^{-1} shrinks the interval $[i, i+2]$ down to $[i, i+1]$, and the derivatives get divided by 2. We will use these facts in the proof of the norm estimate below.

The following lemma is an example of how these maps can be used to obtain group-theoretical properties:

LEMMA 1. *Let $f \in G$, and let (a, b) be a point of the graph of f. Assume that one of the two derivatives $f'_+(a)$ and $f'_-(a)$ is different from* 1. *Then the norm of f can be bounded by*

$$|f|_G \geq \max\{1, a-2, b-2\}.$$

In particular this applies to any breaking point of the graph of f.

Proof. Since $|f|_G = |f^{-1}|_G$, we only need to prove that $|f|_G \geq b-2$. Clearly we can assume $b \geq 3$.

Observe that in f_0 and f_1 the highest point with a derivative different from 1 is the point $(2,3)$ in f_1, and further compositions with either f_0 or f_1 can only increase the y-coordinate by 1, and double the slope only at a point with y-coordinate in $[0,3]$. So to achieve a derivative different from 1 in (a,b) one can start with f_1 and compose it with $b-3$ generators more, at least. So one needs to compose at least $b-2$ generators to obtain a graph that has a derivative different from 1 in (a,b). ∎

The next result is the estimate of the norm of an element of G in terms of the unique normal form.

PROPOSITION 2. *Let $f \in G$ be an element with normal form*

$$f = f_{i_1}^{r_1} \cdots f_{i_n}^{r_n} f_{j_m}^{-s_m} \cdots f_{j_1}^{-s_1}.$$

Let $D(f) = r_1 + \cdots + r_n + s_1 + \cdots + s_m + i_n + j_m$. Then

$$\frac{D(f)}{6} - 2 \leq |f|_G \leq 3D(f).$$

Proof. Since $|f|_G = |f^{-1}|_G$, we can assume that $i_n > j_m$.

One of the inequalities is easy: rewrite the normal form in terms of f_0 and f_1 using $f_i = f_0^{-i+1} f_1 f_0^{i-1}$ to obtain a word representing f with only f_0, f_1 and their inverses. It is easy to see that the length of this word is less than $3D(f)$.

It is also not difficult to see that $|f|_G \geq r_1 + \cdots + r_n + s_1 + \cdots + s_m$: if $|f|_G < r_1 + \cdots + r_n + s_1 + \cdots + s_m$, there exists a word on f_0 and f_1 (and their inverses) that has length less than the unique normal form, contradicting the fact that the normal form is the shortest word for f. One could take this word and construct from it a normal form that would be shorter than the unique one.

The last step in the proof is to prove that

$$|f|_G \geq \frac{i_n}{2} - 2.$$

Assume that $r_1 + \cdots + r_n + s_1 + \cdots + s_m < i_n/2$. If not, the inequality follows from the previous paragraph. Consider the graph of the element

$$g = f_{i_1}^{r_1} \cdots f_{i_n}^{r_n},$$

the positive part of the normal form. Each one of these generators performs a stretching of the graph (see above), the last one being a stretching of the interval $[i_n, i_n + 1]$ into $[i_n, i_n + 2]$. So in the graph of g there is a point P with coordinates $(x, i_n + 1)$ such that the two derivatives at this point are equal to 2^N where $N \geq r_n$. We want to follow the movement down of P after composing with all the inverses that appear in the normal form. The desired conclusion is that at the end, in the graph of f, the point that corresponds to P is a breaking point, or else the function f still has derivatives 2^N at this point.

Observe the effect that composing with $f_{j_m}^{-1}$ has on P. If $j_m = i_n - 1$, in $gf_{j_m}^{-1}$ the point corresponding to P is now a breaking point: its derivatives are 2^N and 2^{N-1}. Further compositions with any f_i^{-1} will keep this point a breaking point, since $i_n > j_m > \cdots > j_1$. If $j_m < i_n - 1$, then in $gf_{j_m}^{-1}$ the point P has just seen its y-coordinate decreased by one and the derivatives are still both 2^N.

In the graph of $gf_{j_m}^{-s_m}$, the situation of the point P depends on the value of s_m:

(1) If $s_m < i_n - j_m - 1$, P is not a breaking point and the derivatives are both 2^N.

(2) If $s_m = i_n - j_m - 1$, the last composition by $f_{j_m}^{-1}$ has made P a breaking point.

(3) If $s_m > i_n - j_m - 1$, P is now a breaking point whose y-coordinate is not an integer, and in any case it will remain a breaking point throughout, so it will be a breaking point of f.

The key to this argument is to observe that since $i_n > j_m > \cdots > j_1$, a composition by an f_i^{-1} cannot decrease the right derivative at P without decreasing the left derivative. One of these compositions either divides the left derivative by 2 without touching the right derivative, or it divides both derivatives by 2. It divides both derivatives by 2 only after P has been already made a breaking point. So the conclusion is that in f, either the derivatives at this point are still 2^N (if the $s_1, \ldots, s_m$ are small enough) or else it is a breaking point. In any case one of the two derivatives at this point is not 1. We need now to compute the y-coordinate of this point to apply Lemma 1.

In g the y-coordinate of P was $i_n + 1$. Every application of an f_i^{-1} may decrease the y-coordinate at most by 1. So the y-coordinate is at least

$$i_n + 1 - s_1 - \cdots - s_m,$$

but from our assumption, $r_1 + \cdots + r_n + s_1 + \cdots + s_m \leq i_n/2$, we conclude that the y-coordinate of this point is at least $i_n/2$. From Lemma 1 it follows now that $|f|_G \geq i_n/2 - 2$.

Combining all inequalities (including $i_n > j_m$) we have

$$\begin{aligned} |f|_G &\geq \max\left\{r_1 + \cdots + r_n + s_1 + \cdots + s_m, \frac{i_n}{2} - 2, \frac{j_m}{2} - 2\right\} \\ &\geq \frac{r_1 + \cdots + r_n + s_1 + \cdots + s_m + i_n/2 + j_m/2 - 4}{3} \\ &\geq \frac{D(f)}{6} - 2, \end{aligned}$$

and the proof is complete. ∎

This estimate of the norm can be used, for instance, to recover the known fact that states that the growth of F is exponential (see [3]). One only needs to count how many normal forms have $D \leq n/3$ to make sure that the corresponding elements have norm less than n, and it is easy to find that this number of normal forms is exponential in n.

3. QUASI-ISOMETRICALLY EMBEDDED SUBGROUPS

Recall that a map

$$F\colon X \to Y$$

between metric spaces is called a quasi-isometric embedding if there exist constants $K, C > 0$ such that

$$\frac{d(x,x')}{K} - C \leq d(F(x), F(x')) \leq Kd(x,x') + C,$$

for all $x, x' \in X$. If G is a finitely generated group, and H is a finitely generated subgroup, then the fact that the inclusion is a quasi-isometric embedding is equivalent to saying that the distortion function

$$h(r) = \frac{1}{r}\max\{|x|_H | x \in H, |x|_G \leq r\}$$

is bounded. If a subgroup is quasi-isometrically embedded, then its own word metric is equivalent to the metric induced by the word metric of the ambient group, and the distortion is bounded.

Our goal is to prove that several subgroups of F are quasi-isometrically embedded, namely, those subgroups constructed in Section 1 which are isomorphic to $F \times \mathbb{Z}^n$ and $F \times F$. The generators are:

(1) $x_0 x_1^{-1}, \ldots, x_{2n-2} x_{2n-1}^{-1}, x_{2n}, x_{2n+1}$ for $F \times \mathbb{Z}^n$,

(2) $t_0 = x_0^2 x_1^{-1} x_0^{-1}$, $t_1 = x_0 x_1^2 x_2^{-1} x_1^{-1} x_0^{-1}$, x_1, x_2 for $F \times F$.

PROPOSITION 3. *The subgroup $\phi^n(F)$ of F is quasi-isometrically embedded.*

Proof. We have that $\phi^n(F)$ is isomorphic to F, and the normal form of an element in $\phi^n(F)$ is exactly the same when the element is considered in F, so we can use the norm estimate to establish the inequalities. ∎

Observe that for a particular $\phi^n(F)$ the constants for the quasi-isometric embedding will depend on n.

PROPOSITION 4. *Every cyclic subgroup of F is quasi-isometrically embedded.*

Proof. Let x be the generator of the cyclic subgroup, and let

$$x_{i_1}^{r_1} x_{i_2}^{r_2} \cdots x_{i_n}^{r_n} x_{j_m}^{-s_m} \cdots x_{j_2}^{-s_2} x_{j_1}^{-s_1},$$

be its normal form. Conjugating the generator does not change the property of being quasi-isometrically embedded, so we can assume that $i_1 \neq j_1$, and taking the inverse if necessary we can further assume that $i_1 < j_1$. It is easy to see now that the normal form for x^k starts with a term $x_{i_1}^{kr_1}$, and then $D(x^k)$ is greater than kr_1. ∎

An alternate proof of this result can be formulated as a corollary to Lemma 15.29 of [6].

PROPOSITION 5. *The subgroup $\mathbb{Z} \times F$ generated by*

$$x_0 x_1^{-1}, \qquad x_2, \qquad x_3$$

is quasi-isometrically embedded.

Proof. Let

$$x = x_{i_1}^{r_1} x_{i_2}^{r_2} \cdots x_{i_n}^{r_n} x_{j_m}^{-s_m} \cdots x_{j_2}^{-s_2} x_{j_1}^{-s_1}$$

be an element of $\phi^2(F)$, i.e., with all indices being at least 2. Let $\bar{x} = (x_0 x_1^{-1})^k x$ be an element of $\mathbb{Z} \times F$. Its distance in $\mathbb{Z} \times F$ can be estimated as $k + D(x)$. We will compute the estimate $D(\bar{x})$ of its distance in F and see that it does not differ much from $k + D(x)$.

To do that, we need to compute the normal form in F of $\bar{x}$. But

$$\begin{aligned}\bar{x} &= \left(x_0^k x_k^{-1} \cdots x_1^{-1} x_0^{-1}\right)\left(x_{i_1}^{r_1} x_{i_2}^{r_2} \cdots x_{i_n}^{r_n} x_{j_m}^{-s_m} \cdots x_{j_2}^{-s_2} x_{j_1}^{-s_1}\right) \\ &= x_0^k x_{i_1+k}^{r_1} \cdots x_{i_n+k}^{r_n} x_{j_m+k}^{-s_m} \cdots x_{j_1+k}^{-s_1} x_k^{-1} \cdots x_1^{-1} x_0^{-1}\end{aligned}$$

which is the normal form for $\bar{x}$. Then,

$$D(\bar{x}) = r_1 + \cdots + r_n + s_1 + \cdots + s_m + 2k + i_n + k + j_m + k$$

and clearly,

$$k + D(x) \le D(\bar{x}) \le 4(k + D(x))$$

which finishes the proof. ∎

COROLLARY 6. *The subgroup $F \times \mathbb{Z}^n$ generated by*

$$x_{2n}, x_{2n+1}, x_0 x_1^{-1}, x_2 x_3^{-1} \ldots, x_{2n-2} x_{2n-1}^{-1}$$

is quasi-isometrically embedded.

COROLLARY 7. (1) *The free abelian group generated by*

$$x_0 x_1^{-1}, x_2 x_3^{-1} \ldots, x_{2n-2} x_{2n-1}^{-1}$$

is quasi-isometrically embedded.

(2) *The asymptotic cone of F is infinite-dimensional.*

4. THE SUBGROUP $F \times F$

The first component F of this subgroup is generated by the elements

$$t_0 = x_0^2 x_1^{-1} x_0^{-1}, \qquad t_1 = x_0 x_1^2 x_2^{-1} x_1^{-1} x_0^{-1},$$

while the second component is just $\phi(F)$. We need to compute the normal form in F of a general element of $F \times F$. The following lemma computes this normal form for a particular type of elements of $\langle t_0, t_1 \rangle$. Let

$$t_k = t_0^{-(k-1)} t_1 t_0^{k-1} = x_0 x_1 \cdots x_{k-1} x_k^2 x_{k+1}^{-1} x_k^{-1} \cdots x_0^{-1}$$

be the generators for the infinite presentation for $F = \langle t_0, t_1 \rangle$.

LEMMA 8. *The element*

$$t_{i_1}^{r_1} \cdots t_{i_n}^{r_n}$$

has a normal form in F of the type

$$x_0 x_1 \cdots x_{i_1-1} x_{i_1}^{r_1+1} w_1 x_{i_2}^{r_2} w_2 x_{i_3}^{r_3} \cdots w_{n-1} x_{i_n}^{r_n} x_a^{-1} x_{a-1}^{-1} \cdots x_0^{-1},$$

where

(1) *for* $k = 1, \dots, n-1$, *the word* w_k *is either* 1 *or* $x_{l_k} x_{l_k+1} \cdots x_{i_{k+1}}$, *with* $l_k \geq i_k + 1$, *and*

(2) $i_n + r_n \leq a \leq i_n + r_1 + r_2 + \cdots + r_n$.

Proof of Lemma 8. The proof is by induction on n. If $n = 1$ the element $t_{i_1}^{r_1}$ has normal form

$$x_0 \cdots x_{i_1-1} x_{i_1}^{r_1+1} x_{i_1+r_1}^{-1} \cdots x_0^{-1}.$$

In this case there are no words w_k and $a = i_1 + r_1$.

Suppose now that the statement is true for n and consider an element of the form

$$t_{i_1}^{r_1} \cdots t_{i_n}^{r_n} t_i^r,$$

with $i > i_n$ and $r \geq 1$. To construct its normal form we multiply

$$\left(x_0 x_1 \cdots x_{i_1-1} x_{i_1}^{r_1+1} w_1 x_{i_2}^{r_2} \cdots w_{n-1} x_{i_n}^{r_n} x_a^{-1} x_{a-1}^{-1} \cdots x_0^{-1}\right)$$
$$\times \left(x_0 x_1 \cdots x_i x_i^r x_{i+r}^{-1} \cdots x_0^{-1}\right),$$

and the cancellations determine the new normal form:

(1) If $a < i$, then the new normal form is

$$x_0 x_1 \cdots x_{i_1-1} x_{i_1}^{r_1+1} w_1 x_{i_2}^{r_2} \cdots w_{n-1} x_{i_n}^{r_n} x_{a+1} \cdots x_i x_i^r x_{i+r}^{-1} \cdots x_0^{-1}.$$

We have $w_n = x_{a+1} \cdots x_i$ with $a \geq i_n + r_n \geq i_n$, and the new a is $i + r$.

(2) If $a = i$, then the new normal form is

$$x_0 x_1 \cdots x_{i_1-1} x_{i_1}^{r_1+1} w_1 x_{i_2}^{r_2} \cdots w_{n-1} x_{i_n}^{r_n} x_i^r x_{i+r}^{-1} \cdots x_0^{-1}$$

so $w_n = 1$ and the new a is $i + r$.

(3) If $a > i$, then we construct the new normal form by

$$x_0 x_1 \cdots x_{i_1-1} x_{i_1}^{r_1+1} w_1 x_{i_2}^{r_2} \cdots w_{n-1} x_{i_n}^{r_n} x_a^{-1} \cdots x_{i+1}^{-1} x_i^r x_{i+r}^{-1} \cdots x_0^{-1}$$
$$= x_0 x_1 \cdots x_{i_1-1} x_{i_1}^{r_1+1} w_1 x_{i_2}^{r_2} \cdots w_{n-1} x_{i_n}^{r_n} x_i^r x_{a+r}^{-1} \cdots x_{i+r+1}^{-1} x_{i+r}^{-1} \cdots x_0^{-1}$$

so $w_n = 1$ and we have $i + r \leq a + r \leq i_n + r_1 + \cdots + r_n + r \leq i + r_1 + \cdots + r_n + r$. The induction is complete. ∎

PROPOSITION 9. *The subgroup isomorphic to $F \times F$ generated by*

$$t_0 = x_0^2 x_1^{-1} x_0^{-1}, \qquad t_1 = x_0 x_1^2 x_2^{-1} x_1^{-1} x_0^{-1}, \qquad x_1, \qquad x_2$$

is quasi-isometrically embedded.

Proof. Let

$$t = t_{i_1}^{r_1} \cdots t_{i_n}^{r_n} t_{j_m}^{-s_m} \cdots t_{j_1}^{-s_1},$$
$$x = x_{\alpha_1}^{\delta_1} \cdots x_{\alpha_p}^{\delta_p} x_{\beta_q}^{-\varepsilon_q} \cdots x_{\beta_1}^{-\varepsilon_1}$$

be arbitrary elements of each component of $F \times F$. The element $(t, x) \in F \times F$ has norm (in $F \times F$) approximately equal to $D(t) + D(x)$, where

$$D(t) = r_1 + \cdots + r_n + s_1 + \cdots + s_m + i_n + j_m,$$
$$D(x) = \delta_1 + \cdots + \delta_p + \varepsilon_1 + \cdots + \varepsilon_q + \alpha_p + \beta_q.$$

We need to compute the normal form of the element tx in F. Since the elements t_i and x_j commute, we will compute the normal form of

$$tx = \left(t_{i_1}^{r_1} \cdots t_{i_n}^{r_n}\right)\left(x_{\alpha_1}^{\delta_1} \cdots x_{\alpha_p}^{\delta_p}\right)\left(x_{\beta_q}^{-\varepsilon_q} \cdots x_{\beta_1}^{-\varepsilon_1}\right)\left(t_{j_m}^{-s_m} \cdots t_{j_1}^{-s_1}\right).$$

By Lemma 8, we have

$$\begin{aligned}
&\left(t_{i_1}^{r_1} \cdots t_{i_n}^{r_n}\right)\left(x_{\alpha_1}^{\delta_1} \cdots x_{\alpha_p}^{\delta_p}\right) \\
&\qquad = \left(x_0 x_1 \cdots x_{i_1-1} x_{i_1}^{r_1+1} w_1 x_{i_2}^{r_2} \cdots w_{n-1} x_{i_n}^{r_n} x_a^{-1} x_{a-1}^{-1} \cdots x_0^{-1}\right)\left(x_{\alpha_1}^{\delta_1} \cdots x_{\alpha_p}^{\delta_p}\right) \\
&\qquad = \left(x_0 x_1 \cdots x_{i_1-1} x_{i_1}^{r_1+1} w_1 x_{i_2}^{r_2} \cdots w_{n-1} x_{i_n}^{r_n}\right)\left(x_{\alpha_1+a+1}^{\delta_1} \cdots x_{\alpha_p+a+1}^{\delta_p}\right) \\
&\qquad\qquad \times\left(x_a^{-1} \cdots x_0^{-1}\right),
\end{aligned}$$

and similarly,

$$\begin{aligned}
&\left(t_{j_1}^{s_1} \cdots t_{j_m}^{s_m}\right)\left(x_{\beta_1}^{\varepsilon_1} \cdots x_{\beta_q}^{\varepsilon_q}\right) \\
&\qquad = \left(x_0 x_1 \cdots x_{j_1-1} x_{j_1}^{s_1+1} v_1 x_{j_2}^{s_2} \cdots v_{m-1} x_{j_m}^{s_m} x_b^{-1} x_{b-1}^{-1} \cdots x_0^{-1}\right)\left(x_{\beta_1}^{\varepsilon_1} \cdots x_{\beta_q}^{\varepsilon_q}\right) \\
&\qquad = \left(x_0 x_1 \cdots x_{j_1-1} x_{j_1}^{s_1+1} v_1 x_{j_2}^{s_2} \cdots v_{m-1} x_{j_m}^{s_m}\right)\left(x_{\beta_1+b+1}^{\varepsilon_1} \cdots x_{\beta_q+b+1}^{\varepsilon_q}\right) \\
&\qquad\qquad \times\left(x_b^{-1} \cdots x_0^{-1}\right).
\end{aligned}$$

Taking the inverse if necessary, we can assume that $a \leq b$, and then, combining the two normal forms, we have

$$\begin{aligned} tx = {} & x_0 x_1 \cdots x_{i_1 - 1} x_{i_1}^{r_1 + 1} w_1 x_{i_2}^{r_2} \cdots w_{n-1} x_{i_n}^{r_n} \\ & x_{\alpha_1 + a + 1}^{\delta_1} \cdots x_{\alpha_p + a + 1}^{\delta_p} \\ & x_{a+1} \cdots x_b \\ & x_{\beta_q + b + 1}^{-\varepsilon_q} \cdots x_{\beta_1 + b + 1}^{-\varepsilon_1} \\ & x_{j_m}^{-s_m} v_{m-1}^{-1} \cdots x_{j_2}^{-s_2} v_1^{-1} x_{j_1}^{-s_1 + 1} x_{j_1 - 1}^{-1} \cdots x_1^{-1} x_0^{-1} \end{aligned}$$

which, after reordering, yields the normal form for tx:

$$\begin{aligned} tx = {} & x_0 x_1 \cdots x_{i_1 - 1} x_{i_1}^{r_1 + 1} w_1 x_{i_2}^{r_2} \cdots w_{n-1} x_{i_n}^{r_n} \\ & x_{a+1} \cdots x_b \\ & x_{\alpha_1 + b + 1}^{\delta_1} \cdots x_{\alpha_p + b + 1}^{\delta_p} \\ & x_{\beta_q + b + 1}^{-\varepsilon_q} \cdots x_{\beta_1 + b + 1}^{-\varepsilon_1} \\ & x_{j_m}^{-s_m} v_{m-1}^{-1} \cdots x_{j_2}^{-s_2} v_1^{-1} x_{j_1}^{-s_1 + 1} x_{j_1 - 1}^{-1} \cdots x_1^{-1} x_0^{-1}. \end{aligned}$$

We can now compute the norm in F of tx:

$$\begin{aligned} D(tx) = {} & i_1 + r_1 + 1 + l(w_1) + r_2 + \cdots + l(w_{n-1}) \\ & + r_n + (b - a) + \delta_1 + \cdots + \delta_p \\ & + j_1 + s_1 + 1 + l(v_1) + s_2 + \cdots + l(v_{m-1}) + s_m + \varepsilon_1 + \cdots + \varepsilon_q \\ & + \alpha_p + b + 1 + \beta_q + b + 1. \end{aligned}$$

Using the inequalities $0 \leq l(w_k) \leq i_{k+1} - i_k$ and $j_m \leq b \leq D(t)$, provided by Lemma 8, we can deduce that

$$D(t) + D(x) \leq D(tx) \leq 4D(t) + D(x) + 4$$

from which the desired result follows. ∎

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