Discrete Applied Mathematics 3 (1981) 175–186
North-Holland Publishing Company

SUBSTITUTES AND COMPLEMENTS IN NETWORK FLOW PROBLEMS*

David GALE
Department of Industrial Engineering and Operations Research University of California, Berkeley, CA 94720, USA

Themistocles POLITOF
Operations Research Center, University of California, Berkeley, CA 94720, USA

Received 31 October 1980

If f is a function of several variables, one calls a pair of variables *substitutes* (*complements*) if the change of the value of the function when both variables are increased is at most (at least) equal to the sum of the changes when each is increased separately. We here consider the case where f is the value of a maximum weight circulation on a network and the variables are the upper and lower bounds and the weights of a pair of arcs. We introduce a simple combinatorial criterion for two arcs to be in "series" or "parallel" and show that these two cases correspond to the variables being complements or substitutes respectively. This generalizes results of Shapley for the special case of the maximum flow and optimal assignment problems. We also show that our result is best possible in that if two arcs are neither in series nor parallel, then the corresponding variables can be either substitutes or complements or both.

1. Introduction

The notion of substitutes and complements is a familiar one in economics. Butter and margarine are substitutes since there is something to be gained by having one or the other but not much additional gain from having both. A lock and key, on the other hand, are complements since neither is much use without the other. These ideas are easily made quantitative. Let $f(x_1, y_1)$ measure the welfare from having x_1 units of one good and y_1 of another, and suppose[1] $x_2 > x_1$ and $y_2 > y_1$. The goods are substitutes (complements) if

$$f(x_2,y_2)-f(x_1,y_1) \leq (\geq) \, (f(x_2,y_1)-f(x_1,y_1))+(f(x_1,y_2)-f(x_1,y_1)) \quad (1.1)$$

where the left side above measures the increase in welfare from increasing both x and y while the right side gives the sum of the increases if the quantities are increased one

* This research has been partially supported by the National Science Foundation under Grant SES-7805196 and the Office of Naval Research under Contract N00014-76-C-0134 with the University of California.

[1] If the x_i are vectors, then ">" denotes the usual partial order.

at a time. Note that (1.1) simplifies to

$$f(x_2,y_2)+f(x_1,y_1)\le(\ge)f(x_1,y_2)+f(x_2,y_1). \tag{1.2}$$

Functions satisfying (1.2) are called ***sub* (*super*) *modular***.

In 1961, Shapley [1][2] and [2] proved a number of interesting results concerning substitutes and complements in network and assignment problems. The first paper studies the value of the maximum flow in a network as a function of arc capacities. The results are

(A) Given any pair of arcs α,β the maximum flow μ is either a submodular or supermodular function of the arc capacities c_α, c_β over the entire range of capacity values.

(B) If arcs α and β are "in parallel" meaning that either the heads or tails of the arcs lie on a common node, then they are substitutes. If they are "in series" meaning the head of α is the tail of β, then they are complements.

The second paper treats the classical optimal assignment problem. With usual interpretation of jobs and applicants it is shown that any two jobs or any two applicants are substitutes. More precisely the increase in the value of the optimal assignment when two new jobs (or applicants) are brought in, does not exceed the sum of the increases due to each job separately. On the other hand, a pair consisting of a job and an applicant act as complements.

In the present paper we consider a general network model which covers both the cases of [1] and [2]. We are concerned with a directed graph $\mathcal{N}$ in which there are upper and lower bounds and a weight assigned to each arc. Thus, given an arc α we have a vector $q_\alpha=(\underline{c}_\alpha,\bar{c}_\alpha,w_\alpha)$ where $\underline{c}_\alpha\le\bar{c}_\alpha$. The function to be studied is the value of the maximum weight circulation on $\mathcal{N}$. It is easy to see how the two cases considered by Shapley can be formulated in this way. The maximum flow problem is changed to an equivalent circulation problem by identifying source and sink, and assigning a weight of one to all arcs out of the "source-sink" and a weight of zero to all other arcs. The assignment problem is, of course, a special case of the transportation problem which is converted to a circulation problem in a standard way by introducing an additional node z. Each sink s_j' is connected to z by an arc with lower bound equal to the demand at s_j', and z is connected to each source s_i by an arc with upper bound equal to the supply at s_i.

The purpose of this paper is to give a complete characterization of when a pair of arcs are substitutes or complements in the context of our general model. We say a pair of arcs is *in parallel* if there is no simple undirected cycle in the graph in which both arcs have the same direction, i.e., if one traverses any cycle in the graph containing both of the arcs, one must traverse one arc in the forward, the other in the backward direction. Clearly a pair of arcs having heads or tails on a common node will have this property. A pair of arcs is *in series* if there is no simple undirected cycle in which the two arcs have opposite directions. This is clearly a generalization

[2] See also [3].

of the case when the head of one arc is the tail of the other. Our main result states that arcs in parallel are always substitutes while those in series are always complements. The characterization is "complete" in that if two arcs are neither in series nor in parallel, then they can be either substitutes or complements depending on the parameters of the rest of the graph. In fact we will show that for this case it can happen that the two arcs are complements over part of their domain and substitutes over another. Thus, property (A) of [1] does not generalize to arbitrary weighted networks.

The final section treats the special case of the Optimal Assignment Problem when we consider the optimal value as a function of some pair of entries in the assignment matrix. For this case it is shown that property (A) does hold. The property, however, does not generalize even to the simple transportation problem. Counter example is given in an appendix.

2. Networks and Circulations

A *network* $\mathscr{N}$ consists of a set N of *nodes*, a set $\mathscr{A}$ of arcs and a pair of mappings h and t from $\mathscr{A}$ to N. For α in $\mathscr{A}$ we write $t(\alpha)=u$ and $h(\alpha)=v$ (u is the "tail", v the "head" of α).

A *circulation* x is a real valued function on $\mathscr{A}$ such that

$$\sum_{t(\alpha)=u} x(\alpha)=\sum_{h(\alpha)=u} x(\alpha) \quad \text{for all } u \text{ in } N. \tag{2.1}$$

A simple (undirected) path P in $\mathscr{N}$ is a sequence of nodes and arcs $(u_0, \alpha_0, u_1, \alpha_1, \ldots, u_n, \alpha_n, u_{n+1})$ where the u_i are distinct and either

(i) $t(\alpha_k)=u_k, h(\alpha_k)=u_{k+1}$ in which case α_k is called a *forward arc* of P, or

(ii) $h(\alpha_k)=u_k, t(\alpha_k)=u_{k+1}$ and α_k is called a *backward arc* of P.

A cycle is a path in which $u_{n+1}=u_0$.

A pair of arcs is said to be *in parallel* if there is no simple cycle containing both as forward arcs. They are *in series* if no simple cycle contains one as forward, the other as a backward arc.

With any cycle Γ we associate a circulation x_Γ by the rule $x_\Gamma(\alpha)=1$ (-1) if α is a forward (backward) arc of Γ, and $x_\Gamma(\alpha)=0$ otherwise. We call x_Γ a *cycle circulation*.

The following lemma seems to be well known.

Lemma (Cylic Decomposition Lemma). *Any circulation x is a positive linear combination of cycle circulations, x_{Γ_i}, $i=1,\ldots,n$, that is*

$$x=\sum_{i=1}^{n} k_i x_{\Gamma_i}, \quad k_i \ge 0,$$

where

$$x_{\Gamma_i}(\alpha)x(\alpha)\ge 0 \quad \textit{for all } \alpha \textit{ in } \mathscr{A}. \tag{2.2}$$

An easy proof is by induction on the number of arcs of $\mathscr{A}$. Without loss of generality we may suppose $x(\alpha)\neq 0$ for all α in $\mathscr{A}$ (i.e., we may ignore all arcs α with $x(\alpha)=0$). Starting from some α_0 with say $x(\alpha_0)>0$, let $u_0=t(\alpha_0)$ and $u_1=h(\alpha_0)$. Then from (2.1) there exists α_1 such that either $t(\alpha_1)=u_1$ and $x(\alpha_1)>0$ or $h(\alpha_1)=u_1$ and $x(\alpha_1)<0$. Continuing in this way there is a path $(u_0,\alpha_0,\ldots,u_i,\alpha_i,\ldots)$ where $x(\alpha_i)$ is positive or negative according as α_i is a forward or backward arc. Eventually one gets a repetition of some u_i, say, $u_m=u_n, m<n$ so $\Gamma=(u_m,\alpha_m,\ldots,\alpha_{n-1},u_n)$ is a cycle. Let $k=\min_{m\leq i\leq n}|x(\alpha_i)|$. Then $x'=x-kx_\Gamma$ is a circulation whose value is zero on at least one arc of Γ. By induction, x' is a linear combination of cycle circulations, $\Gamma_1,\ldots,\Gamma_{n-1}$ and $x=x'+kx_\Gamma$ satisfies (2.2) by construction.

Our key result is the following

Corollary (Parallel Decomposition Lemma). *If α and β are in parallel and x is a circulation on $\mathscr{N}$ and $x(\alpha),x(\beta)>0$, then there exist circulations x^α and x^β such that*

$$x=x^\alpha+x^\beta, \tag{2.3}$$

$$x^\alpha(\gamma)x^\beta(\gamma)\geq 0 \quad \text{for all } \gamma \text{ in } \mathscr{A}, \tag{2.4}$$

$$x^\alpha(\beta)=x^\beta(\alpha)=0. \tag{2.5}$$

Proof. From the lemma

$$x=\sum_{i=1}^{n} k_i x_{\Gamma_i} \quad \text{for some cycles } \Gamma_i$$

and say $x_{\Gamma_i}(\alpha)>0$ for $i\leq r$, $x_{\Gamma_i}(\alpha)=0$ for $i>r$. Define $x^\alpha=\sum_{i=1}^{r} k_i x_{\Gamma_i}$ and note that $x_{\Gamma_i}(\beta)=0$ for $i\leq r$, for from the lemma $x_{\Gamma_i}(\beta)\geq 0$ but if β is in Γ it must be a backward arc (definition of parallel) so that $x_{\Gamma_i}(\beta)\leq 0$. Thus, $x^\alpha(\beta)=0$. Letting $x^\beta=x-x^\alpha$ we see that $x^\beta(\alpha)=0$ by the construction of x^α. Finally (2.4) follows from the (2.2) since each $x_{\Gamma_i}(\gamma)$ has the same sign as $x(\gamma)$. □

3. Capacitated, weighted networks

A network $\mathscr{N}$ is capacitated if there are real valued functions $\underline{c}$ and $\bar{c}$ on $\mathscr{A}$ such that $\underline{c}\leq\bar{c}$.

A circulation x on $\mathscr{N}$ is *feasible* if $\underline{c}\leq x\leq\bar{c}$.

A network is *weighted* if there is a real valued function w on $\mathscr{A}$.

The *value* $w(x)$ of a circulation x is defined by

$$w(x)=\sum_{\alpha\in\mathscr{A}} w(\alpha)x(\alpha).$$

The feasible circulation x is called *optimal* if $w(x)\geq w(x')$ for every feasible circulation x' on $\mathscr{N}$.

From now on we will assume all networks are capacitated and weighted. We denote by q the function from $\mathscr{A}$ to R^3 defined by $q(\gamma)=(\underline{c}(\gamma), \bar{c}(\gamma), w(\gamma))$. We will be concerned with the case where q is constant on all arcs except α and β. Let $\mu(q(\alpha), q(\beta))$ be the value of an optimal circulation for the given values $q(\alpha), q(\beta)$. We now state

Main Theorem. (A) *If α and β are in parallel, then $\mu(q(\alpha), q(\beta))$ is submodular,*

(B) *If α and β are in series, then $\mu(q(\alpha), q(\beta))$ is supermodular,*

(C) *In all other cases one can choose values $q(\gamma)$ for $\gamma \neq \alpha, \beta$, so that α and β are complements over one part of their domain and substitutes over another.*

In this section we will prove the equivalence of (A) and (B) and will establish (C). The proof of (A) will be given in the next section.

The proof that (A) and (B) are equivalent depends on the following observation: if in any network $\mathscr{N}$ we replace the arc γ by its *reverse* γ' meaning $h(\gamma')=t(\gamma)$, $t(\gamma')=h(\gamma)$ and define $q(\gamma')=(-\bar{c}(\gamma), -\underline{c}(\gamma), -w(\gamma))$, then the set of values $w(x)$ of this new network $\mathscr{N}_{\gamma'}$ is the same as that of $\mathscr{N}$ as x ranges over all feasible circulations. To see this note that if x is a feasible circulation on $\mathscr{N}$, then x' is feasible on $\mathscr{N}_{\gamma'}$ where $x'(\gamma')=-x(\gamma)$ and $x=x'$ otherwise. Further, $w(x)=w(x')$ since $w'(\gamma')x'(\gamma')=(-w(\gamma))(-x(\gamma))$. It follows that $\mu(\mathscr{N})=\mu(\mathscr{N}_{\gamma'})$ where these numbers denote the value of an optimal flow on $\mathscr{N}$ and $\mathscr{N}_{\gamma'}$.

Now from the definitions one sees at once that α and β are in parallel if and only if α and β' are in series. Also $q_1(\beta)\leq q_2(\beta)$ if and only if $q_2(\beta')\leq q_1(\beta')$. Suppose then $q_1(\alpha)\leq q_2(\alpha)$ and $q_1(\beta)\leq q_2(\beta)$. We then have

$$(\mu(q_1(\alpha), q_1(\beta))+\mu(q_2(\alpha), q_2(\beta)))-(\mu(q_1(\alpha), q_2(\beta))+\mu(q_2(\alpha), q_1(\beta))) \tag{3.2}$$

$$=(\mu(q_1(\alpha), q_1(\beta'))+\mu(q_2(\alpha), q_2(\beta')))-(\mu(q_1(\alpha), q_2(\beta'))+\mu(q_2(\alpha), q_1(\beta'))). \tag{3.3}$$

If this expression is non-positive, then α and β are substitutes from (3.2) while α and β' are complements from (3.3). Thus, if we establish that arcs in parallel are substitutes it will follow that arcs in series are complements.

To prove (C) note that if α and β are neither in series nor in parallel, then there is a cycle in which α and β are forward arcs and another in which α is forward and β is backward.. We claim that $\mathscr{N}$ must contain a subnetwork "equivalent" to the one shown in Fig. 1.

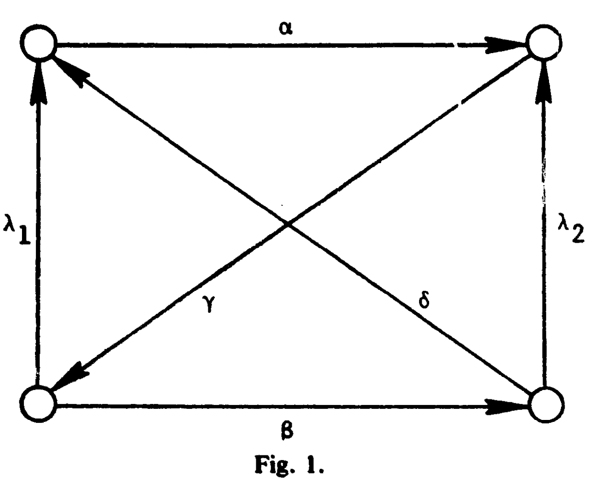

Fig. 1.

To see this, observe that since α and β are neither in series nor in parallel there must be four paths connecting the heads, the tails and heads to tails of α and β (e.g., since α and β are not in parallel there is a cycle having α and β as forward arcs, hence a path from $h(\alpha)$ to $t(\beta)$. This path must contain at least one arc since otherwise α and β would be in series). In Fig. 1 these paths have been replaced by single arcs, but this is equivalent to selecting one arc out of each path and making all other arcs "dummies", that is giving them zero weight and infinite upper and lower capacities. We may suppose the arcs of the figure have the direction shown by the arrows for if not they can be replaced by their reverses as described in the preceeding paragraph without affecting the sub or super modularity of $\mu(q(\alpha), q(\beta))$. We set the lower capacities of all arcs of Figure 1 equal to zero. The λ_i arcs have infinite upper capacity and zero weight. Arcs α and β have zero weight while γ has capacity 2 weight 2 and δ has capacity 1 and weight 3, as illustrated in Fig. 2.

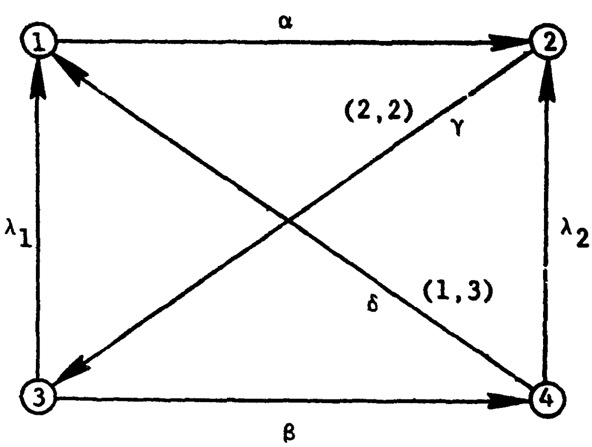

Fig. 2.

Now

(a) if $\bar{c}_\alpha = \bar{c}_\beta = 0$, then $\mu = 0$,

(b) if $\bar{c}_\alpha = 1$, $\bar{c}_\beta = 0$ let $x_\alpha = x_\gamma = x_{\lambda_1} = 1$ giving $\mu = 2$,

(c) if $\bar{c}_\alpha = 0$, $\bar{c}_\beta = 1$ let $x_\beta = x_\gamma = x_{\lambda_2} = 1$ giving $\mu = 2$,

(d) if $\bar{c}_\alpha = \bar{c}_\beta = 1$, let $x_\alpha = x_\beta = x_\gamma = x_\delta = 1$ giving $\mu = 5$,

so that μ is supermodular for these values.

Next

(e) if $\bar{c}_\alpha = 2$, $\bar{c}_\beta = 0$ let $x_\alpha = x_\gamma = x_{\lambda_1} = 2$ giving $\mu = 4$,

(f) if $\bar{c}_\alpha = 0$, $\bar{c}_\beta = 2$ let $x_\beta = x_\gamma = x_{\lambda_2} = 2$ giving $\mu = 4$,

(g) if $\bar{c}_\alpha = \bar{c}_\beta = 2$, let $x_\alpha = x_\gamma = 2$, $x_{\lambda_1} = x_\beta = x_\delta = 1$ giving $\mu = 7$,

so μ is submodular for these values.

Of course one needs to show in each case that the given circulation is optimal. For this purpose we use the standard.

Optimality Theorem. *A feasible circulation x is optimal if and only if there is a function P on N such that*

$$x(\alpha)=\bar{c}(\alpha) \quad \text{if } P(h(\alpha))-P(t(\alpha))<w(\alpha),$$

$$x(\alpha)=\underline{c}(\alpha) \quad \text{if } P(h(\alpha))-P(t(\alpha))>w(\alpha).$$

The appropriate functions for the six cases above are

$$\text{(a), (c), (d), (f)} \quad P_1=3,\ P_2=0,\ P_3=2,\ P_4=0,$$

$$\text{(b), (e)} \quad P_1=3,\ P_2=1,\ P_2=3,\ P_4=0,$$

$$\text{(g)} \quad P_1=P_2=P_3=P_4=0.$$

4. Proof of the Main Theorem

It remains to prove part (A) of the Main Theorem. We are given α and β in parallel. Let q be defined on $\mathcal{N}$ and suppose $q'(\alpha)\geq q(\alpha)$ and $q'(\beta)\geq q(\beta)$. We denote by $\mathcal{N}^{\alpha}$, $\mathcal{N}^{\beta}$, $\mathcal{N}^{\alpha\beta}$ respectively the network with $q(\alpha)$ replaced by $q'(\alpha)$, with $q(\beta)$ replaced by $q'(\beta)$ and with both replacements. The quantities c^{α}, c^{β}, $c^{\alpha\beta}$, w^{α}, w^{β}, $w^{\alpha\beta}$ are defined correspondingly. Finally we abbreviate, writing μ for $\mu(q(\alpha),q(\beta))$, μ^{α} for $\mu(q'(\alpha),q(\beta))$, μ^{β} for $\mu(q(\alpha),q'(\beta))$ and $\mu^{\alpha\beta}$ for $\mu(q'(\alpha),q'(\beta))$. In this notation, we must show for any $\tilde{x}$ which is feasible on $\mathcal{N}^{\alpha\beta}$

$$w^{\alpha\beta}(\tilde{x})+\mu\leq\mu^{\alpha}+\mu^{\beta}. \tag{4.1}$$

Let x^* be optimal on $\mathcal{N}$.

Case 1. $\tilde{x}(\alpha)\leq x^*(\alpha)$. Then

$$\underline{c}(\alpha)\leq\underline{c}^{\alpha}(\alpha)\leq\tilde{x}(\alpha)\leq x^*(\alpha)\leq\bar{c}(\alpha)\leq\bar{c}^{\alpha}(\alpha) \tag{4.2}$$

so that $\tilde{x}$ is feasible on $\mathcal{N}^{\beta}$ so

$$w^{\beta}(\tilde{x})\leq\mu^{\beta}. \tag{4.3}$$

Also from (4.2) x^* is feasible on $\mathcal{N}^{\alpha}$ so

$$w^{\alpha}(x^*)\leq\mu^{\alpha}, \tag{4.4}$$

giving

$$w^{\beta}(\tilde{x})+w^{\alpha}(x^*)\leq\mu^{\alpha}+\mu^{\beta}. \tag{4.5}$$

Now from the definition we have

$$w^{\beta}(\tilde{x})=w^{\alpha\beta}(\tilde{x})+(w(\alpha)-w'(\alpha))\tilde{x}(\alpha),$$

$$w^{\alpha}(x^*)=w(x^*)+(w'(\alpha)-w(\alpha))x^*(\alpha).$$

Adding gives

$$w^\alpha(x^*)+w^\beta(\tilde{x})=w^{\alpha\beta}(\tilde{x})+w(x^*)+(w'(\alpha)-w(\alpha))(x^*(\alpha)-\tilde{x}(\alpha))$$
$$\geq w^{\alpha\beta}(\tilde{x})+w(x^*) \tag{4.6}$$

by Case 1 assumption. Combining (4.6) with (4.5) and noting that $w(x^*)=\mu$, we get (4.1).

The case $\tilde{x}(\beta)\leq x^*(\beta)$ is proved symmetrically. This leaves

Case 2. $\tilde{x}(\alpha)>x^*(\alpha)$, $\tilde{x}(\beta)>x^*(\beta)$. Letting $y=\tilde{x}-x^*$, we have from the Parallel Decomposition Lemma

$$y=y^\alpha+y^\beta \tag{4.7}$$

where

$$y^\alpha y^\beta\geq 0, \tag{4.8}$$

$$y^\alpha(\alpha)=y(\alpha), \qquad y^\beta(\beta)=y(\beta), \tag{4.9}$$

since $y^\alpha(\beta)=y^\beta(\alpha)=0$. Now let $x^\alpha=x^*+y^\alpha$. Then from (4.9)

$$x^\alpha(\alpha)=\tilde{x}(\alpha) \tag{4.10}$$

and

$$x^\alpha(\beta)=x^*(\beta). \tag{4.11}$$

Further, from (4.7) and (4.8) we see that $y^\alpha(\gamma)$ has the same sign and smaller absolute value that $y(\gamma)$ from which we get either

$$x^*(\gamma)\leq x^\alpha(\gamma)\leq\tilde{x}(\gamma) \quad \text{or} \quad \tilde{x}(\gamma)\leq x^\alpha(\gamma)\leq x^*(\gamma)$$

and in either case

$$\underline{c}(\gamma)\leq x^\alpha(\gamma)\leq\bar{c}(\gamma) \tag{4.12}$$

so from (4.10), (4.11) and (4.12) x^α is feasible on $\mathcal{N}^\alpha$, hence

$$w^\alpha(x^\alpha)\leq\mu^\alpha. \tag{4.13}$$

Symmetrically

$$w^\beta(x^\beta)\leq\mu^\beta. \tag{4.14}$$

Now

$$x^\alpha+x^\beta=x^*+y^\alpha+x^*+y^\beta=2x^*+\tilde{x}-x^*=x^*+\tilde{x}$$

so

$$w^{\alpha\beta}(x^*)+w^{\alpha\beta}(\tilde{x})=w^{\alpha\beta}(x^\alpha)+w^{\alpha\beta}(x^\beta) \tag{4.15}$$

but

$$w^{\alpha\beta}(x^*)=w(x^*)+(w'(\alpha)-w(\alpha))x^*(\alpha)+(w'(\beta)-w(\beta))x^*(\beta)$$
$$w^{\alpha\beta}(x^\alpha)=w^\alpha(x^\alpha)+(w'(\beta)-w(\beta))x^*(\beta) \quad \text{from (4.11)}, \tag{4.16}$$
$$w^{\alpha\beta}(x^\beta)=w^\beta(x^\beta)+(w'(\alpha)-w(\alpha))x^*(\alpha) \quad \text{symmetrically.}$$

Substituting in (4.15) gives

$$w(x^*)+w^{\alpha\beta}(\bar{x})=w^{\alpha}(x^{\alpha})+w^{\beta}(x^{\beta})\leq\mu^{\alpha}+\mu^{\beta}$$

from (4.13) and (4.14) which gives (4.1).

5. The optimal assignment problem

Although this problem is a special case of our general network problem, it is simpler, because of its special structure, to consider it separately.

Problem. Given an $n\times n$ matrix $A=(a_{ij})$, choose n entries one in each row and column so that the sum of the entries is a maximum.

If all entries except a_{ij} and a_{kl} are held fixed, this maximum value will be denoted by $\mu(a_{ij},a_{kl})$. We wish to determine whether μ is sub or super modular.

Now if a_{ij} and a_{kl} are in the same row or column, they must clearly be substitutes, for since both entries cannot be in any optimal solution it follows that the value when both take higher values is the same as when only one of them does. If the entries are not in the same row or column, then they may be either substitutes or complements. A typical example is shown in Fig. 3.

x	0	0
0	y	0
0	0	a

Fig. 3.

The claim is that $\mu(x,y)$ is sub or super modular according as a is negative or positive. We now show that the general case can be reduced to this example. Suppose the variable entries are a_{11} and a_{22}. Let $\mu_0=\mu(-\infty,-\infty)$ and let $\mu_1(\mu_2)$ be the value of the assignment problem with row and column 1 (2) deleted and $a_{22}(a_{11})=-\infty$ and let μ_{12} be the value of the problem with rows and column 1 and 2 deleted. Now for any values of a_{11} and a_{22}, it follows that

$$\mu(a_{11},a_{22})=\max(\mu_0,\mu_1+a_{11},\mu_2+a_{22},\mu_{12}+a_{11}+a_{22}).$$

This corresponds to the obvious fact that any assignment must contain one, both or neither of the entries a_{11} and a_{22}.

Now the modularity of μ is not changed by subtracting the constant μ_0. If we then define $x=\mu_1-\mu_0+a_{11}$, $y=\mu_2-\mu_0+a_{22}$, (5.1) becomes

$$\mathcal{N}(x,y)=\max(0,x,y,\mu_{12}+\mu_0-\mu_1-\mu_2+x+y) \qquad (5.2)$$

any letting $a=\mu_{12}+\mu_0-\mu_1-\mu_2$ we have the same situation as that of the example.

Theorem. *The entries a_{11} and a_{22} are substitutes (complements) if and only if*

$$(\mu_{12}+\mu_0)-(\mu_1+\mu_2)\leq (\geq)\ 0.$$

Proof. We are considering the function

$$\mu(x,y)=\max(0,x,y,a+x+y)$$

and claim it is sub or super modular as a is negative or positive. The proof consists of an analysis of various case and is precisely the same as that given in [1] for maximum flows. The reader is referred to that paper. □

References

[1] L.S. Shapley, On network flow functions, Naval Res. Logist. Quart. 8 (1961) 154–158.

[2] L.S. Shapley, Complements and substitutes in the optimal assignment problem, Naval Res. Logist. Quart. 9 (1962) 45–48.

[3] L.R. Ford and D.R. Fulkerson, Flows in Networks (Princeton University Press, Princeton, NJ, 1962) 30–35.

Appendix

We give here an example of a simple transportation problem, in which two arcs, α and β, are substitutes over part of their domain and complements over another. This shows that property (A) (see page 176) which is true for the assignment problem does not extend to the transportation problem.

Remark. The following example is a minimum-cost transportation problem. Costs can be considered as negative profits, the latter corresponding to the weights in our previous treatment of the model. Increasing the weights is equivalent to decreasing the costs, and since our objective function, being the sum of costs, is the negative of the sum of profits, the inequalities (1.1) and (1.2) for substitutes and complements have to be reversed.

In the following tableaus the supplies (s_i) appear at the first column and the demands (d_j) at the top row. The left hand entry in each cell is the cost a_{ij}, the right is the number of units x_{ij} shipped from i to j. If the right entry does not appear, it is assumed to be zero. All blank cells are assumed to have infinite (very big) costs and zero flow. The last column and the bottom row show correspondingly the dual variables associated with the supplies (p_i) and the demands (q_j). The optimality of a given transportation schedule is verified with the use of the well known optimality criterion:

$$q_j-p_i\leq a_{ij}\quad \text{all } i,j$$

$$q_j-p_i=a_{ij}\quad \text{if } x_{ij}>0.$$

s \ d	5	5	2	2	2	2	3	3	
5	α 6		5,2				4,3		0
5		β 6		5,2				4,3	0
4	5,2	5,2			5				0
2					5,2	5			0
2			5	5		5,2			0
3	4,3							4	1
3		4,3					4		1
	5	5	5	5	5	5	4	4	q \ p

(a)

s \ d	5	5	2	2	2	2	3	3	
5	α 4,2		5				4,3		0
5		β 6		5,2				4,3	0
4	5	5,2			5,2				0
2					5	5,2			0
2			5,2	5		5			0
3	4,3							4	0
3		4,3					4		1
	4	5	5	5	5	5	4	4	q \ p

(b)

s \ d	5	5	2	2	2	2	3	3	
5	α 6		5,2				4,3		0
5		β 4,2		5				4,3	0
4	5,2	5			5,2				0
2					5	5,2			0
2			5	5,2		5			0
3	4,3							4	1
3		4,3					4		0
	5	4	5	5	5	5	4	4	q \ p

(c)

s \ d	5	5	2	2	2	2	3	3	
5	α 4,2		5				4,3		1
5		β 4		5,2				4,3	1
4	5	5,2			5,2				0
2					5	5,2			0
2			5,2	5		5			1
3	4,3							4	1
3		4,3					4		1
	5	5	6	6	5	5	5	5	q \ p

(d)

s \ d	5	5	2	2	2	2	3	3	
5	α 6		5,2				4,3		0
5		β 6		5,2				4,3	0
4	5,2	5,2			5				0
2					5,2	5			0
2			5	5		5,2			0
3	4,3							4	1
3		4,3					4		1
	5	5	5	5	5	5	4	4	q \ p

(e)

s \ d	5	5	2	2	2	2	3	3	
5	α 3,2		5				4,3		1
5		β 6		5,2				4,3	0
4	5	5,2			5,2				0
2					5	5,2			0
2			5,2	5		5			0
3	4,3							4	0
3		4,3					4		1
	4	5	5	5	5	5	5	4	q \ p

(f)

s \ d	5	5	2	2	2	2	3	3	
5	α 6		5,2				4,3		0
5		β 3,2		5				4,3	1
4	5,2	5			5,2				0
2					5	5,2			0
2			5	5,2		5			0
3	4,3							4	1
3		4,3					4		0
	5	4	5	5	5	5	4	5	q \ p

(g)

s \ d	5	5	2	2	2	2	3	3	
5	α 3,5		5				4		2
5		β 3,3		5,2				4	2
4	5	5,2			5,2				0
2					5	5,2			0
2			5,2	5		5			2
3	4							4,3	1
3		4					4,3		1
	5	5	7	7	5	5	5	5	q \ p

(h)

The entries to be varied appear at the upper left hand corner and are denoted by α and β. The total cost of the transportation schedule for the different cases will be denoted by μ, μ^α, μ^β, $\mu^{\alpha\beta}$ in conformity with the notation in the rest of the paper.

We have for tableaus (a)–(d) the values

(a) $a_\alpha = a_\beta = 6$, then $\mu = 108$,
(b) $a_\alpha = 4$, $a_\beta = 6$, then $\mu^\alpha = 106$,
(c) $a_\alpha = 6$, $a_\beta = 4$, then $\mu^\beta = 106$,
(d) $a_\alpha = a_\beta = 4$, then $\mu^{\alpha\beta} = 106$,

so that α and β are substitutes for these values (see previous remark).

For (e)–(h)

(e) $a_\alpha = a_\beta = 6$, then $\mu = 108$ (same as (a)),
(f) $a_\alpha = 3$, $a_\beta = 6$, then $\mu^\alpha = 104$,
(g) $a_\alpha = 6$, $a_\beta = 3$, then $\mu^\beta = 104$,
(h) $a_\alpha = a_\beta = 3$, then $\mu^{\alpha\beta} = 98$,

so that α and β are complements for these values.