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A note on bounds for the derivatives of continued fractions

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Abstract

Although it is difficult to differentiate analytic functions defined by continued fractions, it is relatively easy in some cases to determine uniform bounds on such derivatives by perceiving the continued fraction as an infinite composition of linear fractional transformations and applying an infinite chain rule for differentiation.

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1. Introduction

The principal goal of this short paper is to develop an elementary procedure that will give both pointwise and uniform bounds on $|\mathrm{d}F(z)/\mathrm{d}z|$ for certain continued fractions of the form

$$F(z) = \frac{A_1(z)}{B_1(z)} \underset{+}{} \frac{A_2(z)}{B_2(z)} \underset{+}{} \cdots \tag{1.1}$$

in terms of the derivatives of $\{A_n(z)\}$ and $\{B_n(z)\}$. Examples show that in several applications the derived bounds are sharp. The approach used here is related to, although different from, that employed by Waadeland in several recent papers describing a kind of "Taylor's theorem" for certain continued fractions that are close to being limit periodic (see, e.g., [4]). Waadeland starts with the interpretation of a special case of the continued fraction (1.1) as a function of an infinite number of variables, and obtains a derivative formula in terms of these variables. Here we view the continued fraction essentially as a function of a single variable and expand the derivative $F'(z)$ using the chain rule from multivariable complex calculus.

Two other fairly straightforward procedures for finding these kinds of bounds involve Cauchy's inequality and an extended version of Schwarz's lemma. The former inequality, when routinely

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applied, gives rather poor results. The latter approach is illustrated by the following inequality for a function bounded by $M > 0$ on $|z| < 1$ [3]:

$$|F'(z)| \leqslant \frac{M^2 - |F(z)|^2}{M(1 - |z|^2)}.$$

Detailed knowledge of the values of $F(z)$ are required in order for this method to be productive. The procedure and resulting formulas derived in this paper require no such knowledge. It is not necessary to evaluate the continued fraction at any point of its domain — only a general bound such as the M mentioned above is needed.

2. The basic theorem

Theorem 2.1. *Suppose $w(z)$ is analytic on a domain $D \subseteq C$ with $w(D) \subseteq D$. Given a sequence of functions $\{f_n(z,w(z))\}$ analytic on D with $f_n(D,D) \subseteq D$, set $w := w(z)$ and form the sequence of analytic functions $\{F_n\}$:*

$$F_1(z,w) := f_1(z,w), \quad F_n(z,w) := F_{n-1}(z,f_n(z,w)) \qquad \text{for } n > 1.$$

Set $F_n^j := F_n^j(z,w) := F_{n-1}^j(z,f_n(z,w))$, with $F_j^j := f_j(z,w)$ and $n \geqslant j$. In addition, set $F^j := \mathrm{Lim}_{n\to\infty} F_n^j(z,w)$. Assume $F(z,w) = \lim_{n\to\infty} F_n(z,w)$ exists, with convergence being uniform on compact subsets of D.

Then

$$\frac{\mathrm{d}F_n(z,w(z))}{\mathrm{d}z} = \sum_{k=0}^{n-2} \left\{ \left(\prod_{j=1}^{k} \frac{\partial f_j(z,F_n^{j+1})}{\partial F_n^{j+1}} \right) \frac{\partial f_{k+1}(z,F_n^{k+2})}{\partial z} \right\}$$
$$+ \prod_{j=1}^{n-1} \frac{\partial f_j(z,F_n^{j+1})}{\partial F_n^{j+1}} \left[\frac{\partial f_n(z,w)}{\partial z} + \frac{\partial f_n(z,w)}{\partial w} \frac{\mathrm{d}w}{\mathrm{d}z} \right]. \qquad (2.1)$$

As $n \to \infty$, $\mathrm{d}F_n(z,w(z))/\mathrm{d}z \to \mathrm{d}F(z,w(z))/\mathrm{d}z$ uniformly on compact subsets of D. If w is a constant, we have

$$\frac{\partial F(z,w)}{\partial z} = \frac{\mathrm{d}F(z)}{\mathrm{d}z} = \sum_{k=0}^{\infty} \left\{ \left(\prod_{j=1}^{k} \frac{\partial f_j(z,F^{j+1})}{\partial F^{j+1}} \right) \frac{\partial f_{k+1}(z,F_n^{k+1})}{\partial z} \right\}. \qquad (2.2)$$

Proof. First, write (2.1) as

$$\frac{\mathrm{d}F_n(z,w(z))}{\mathrm{d}z} = \frac{\partial f_1(z,F_n^2)}{\partial z} + \sum_{k=1}^{n-2} \left\{ \left(\prod_{j=1}^{k} \frac{\partial f_j(z,F_n^{j+1})}{\partial F_n^{j+1}} \right) \frac{\partial f_{k+1}(z,F_n^{k+2})}{\partial z} \right\}$$
$$+ \prod_{j=1}^{n-1} \frac{\partial f_j(z,F_n^{j+1})}{\partial F_n^{j+1}} \left[\frac{\partial f_n(z,w)}{\partial z} + \frac{\partial f_n(z,w)}{\partial w} \frac{\mathrm{d}w}{\mathrm{d}z} \right]. \qquad (2.1)^*$$

The pattern first becomes discernable for $n = 3$. From the chain rule one gets

$$\begin{aligned}\frac{\mathrm{d}F_3(z,w)}{\mathrm{d}z} &= \frac{\mathrm{d}f_1(z,F_3^2(z,w))}{\mathrm{d}z} \\ &= \frac{\partial f_1(z,F_3^2)}{\partial z} + \frac{\partial f_1(z,F_3^2)}{\partial F_3^2}\frac{\mathrm{d}F_3^2(z,w(z))}{\mathrm{d}z} \\ &= \frac{\partial f_1(z,F_3^2)}{\partial z} + \frac{\partial f_1(z,F_3^2)}{\partial F_3^2}\left[\frac{\mathrm{d}f_2(z,F_3^3(z,w(z)))}{\mathrm{d}z}\right] \\ &= \frac{\partial f_1(z,F_3^2)}{\partial z} + \frac{\partial f_1(z,F_3^2)}{\partial F_3^2}\left[\frac{\partial f_2(z,F_3^3)}{\partial z} + \frac{\partial f_2(z,F_3^3)}{\partial F_3^3}\frac{\mathrm{d}f_3(z,w(z))}{\mathrm{d}z}\right] \\ &= \frac{\partial f_1(z,F_3^2)}{\partial z} + \sum_{k=1}^{1}\left\{\left(\prod_{j=1}^{k}\frac{\partial f_j(z,F_3^{j+1})}{\partial F_3^{j+1}}\right)\frac{\partial f_{k+1}(z,F_n^{k+2})}{\partial z}\right\} \\ &\quad + \prod_{j=1}^{2}\frac{\partial f_j(z,F_3^{j+1})}{\partial F_3^{j+1}}\left[\frac{\partial f_3(z,w)}{\partial z} + \frac{\partial f_3(z,w)}{\partial w}\frac{\mathrm{d}w}{\mathrm{d}z}\right].\end{aligned}$$

Next, assume that (2.1)* is valid for some n and *any* suitable family $\{f_1, f_2, \ldots, f_n\}$. We show that (2.1)* holds for $n + 1$:

The pattern of (2.1)* is valid if we employ $\{f_2, f_3, \ldots, f_{n+1}\}$ rather than $\{f_1, f_2, \ldots, f_n\}$, i.e., consider $\{f_2 \circ f_3 \circ \cdots \circ f_{n+1}\}$ instead of $f_1 \circ f_2 \circ \cdots \circ f_n$. Eq. (2.1)* for n functions then becomes

$$\begin{aligned}\frac{\mathrm{d}F_{n+1}^2(z,w)}{\mathrm{d}z} &= \frac{\partial f_2(z,F_{n+1}^3)}{\partial z} + \sum_{k=2}^{n-1}\left\{\left(\prod_{j=2}^{k}\frac{\partial f_j(z,F_{n+1}^{j+1})}{\partial F_{n+1}^{j+1}}\right)\frac{\partial f_{k+1}(z,F_{n+1}^{k+2})}{\partial z}\right\} \\ &\quad + \prod_{j=2}^{n}\frac{\partial f_j(z,F_{n+1}^{j+1})}{\partial F_{n+1}^{j+1}}\left[\frac{\partial f_{n+1}(z,w)}{\partial z} + \frac{\partial f_{n+1}(z,w)}{\partial w}\frac{\mathrm{d}w}{\mathrm{d}z}\right].\end{aligned}$$

Thus,

$$\begin{aligned}&\frac{\mathrm{d}F_{n+1}(z,w(z))}{\mathrm{d}z} \\ &= \frac{\mathrm{d}f_1(z,F_{n+1}^2)(z,w(z)))}{\mathrm{d}z} \\ &= \frac{\partial f_1(z,F_{n+1}^2)}{\partial z} + \frac{\partial f_1(z,F_{n+1}^2)}{\partial F_{n+1}^2}\cdot\frac{\mathrm{d}F_{n+1}^2(z,w(z))}{\mathrm{d}z} \\ &\;\vdots \\ &= \frac{\partial f_1(z,F_{n+1}^2)}{\partial z} + \sum_{k=1}^{n-1}\left\{\left(\prod_{j=1}^{k}\frac{\partial f_j(z,F_{n+1}^{j+1})}{\partial F_{n+1}^{j+1}}\right)\frac{\partial f_{k+1}(z,F_{n+1}^{k+2})}{\partial z}\right\} \\ &\quad + \prod_{j=1}^{n}\frac{\partial f_j(z,F_{n+1}^{j+1})}{\partial F_{n+1}^{j+1}}\left[\frac{\partial f_{n+1}(z,w)}{\partial z} + \frac{\partial f_{n+1}(z,w)}{\partial w}\frac{\mathrm{d}w}{\mathrm{d}z}\right],\end{aligned}$$

and the expansions (2.1)* (and (2.1)) are established.

Since convergence of $\{F_n(z, w(z))\}$ is uniform on compact subsets of D, Weierstrass' theorem implies the analyticity of $F(z, w(z))$ and furthermore that $\mathrm{d}F_n(z, w(z))/\mathrm{d}z \to \mathrm{d}F(z, w(z))/\mathrm{d}z$ uniformly on compact subsets of D.

3. Applications to continued fractions

Critical to the use of (2.2) in Theorem 1.1 in this regard is knowledge of the approximant locations of each of the continued fraction tails F^j. Several classical continued fractions where this information is readily available will be studied, although these by no means exhaust the possibilities for applying Theorem 1.1.

In all the results that follow uniform convergence to $F(z)$ on compact subsets of D is guaranteed by the Stieltjes–Vitali theorem. Estimates of derivative bounds are valid, of course, only when the absolute series that are derived converge. Information on the various kinds of continued fractions described can be found in either [1] or [2].

Corollary 3.1 (A special Pringsheim case). *The continued fraction*

$$\frac{a_1(z)}{b_1} + \frac{a_2(z)}{b_2} + \cdots \tag{3.1}$$

where each $a_j(z)$ is analytic in D and $|a_j(z)| \leqslant |b_j| - 1$ for all j converges to $F(z)$ where $|F(z)| \leqslant 1$. The following estimates hold:

$$\begin{aligned}\left|\frac{\mathrm{d}F(z)}{\mathrm{d}z}\right| &\leqslant \sum_{k=0}^{\infty}\left(\prod_{j=1}^{k}\frac{|a_j(z)|}{(|b_j|-1)^2}\right)\frac{|\mathrm{d}a_{k+1}(z)/\mathrm{d}z|}{(|b_{k+1}|-1)}\\ &\leqslant \sum_{k=0}^{\infty}\left(\prod_{j=1}^{k+1}\frac{1}{|b_j|-1}\right)|\mathrm{d}a_{k+1}(z)/\mathrm{d}z|\\ &\leqslant \sum_{k=0}^{\infty}\left(\prod_{j=1}^{k+1}\frac{1}{|a_j(z)|}\right)|\mathrm{d}a_{k+1}(z)/\mathrm{d}z|.\end{aligned}$$

(i) *If $|b_j| \geqslant B+1$, $B>1$, then $|\mathrm{d}F(z)/\mathrm{d}z| \leqslant \sum_{k=0}^{\infty}(1/B)^{k+1}|\mathrm{d}a_{k+1}(z)/\mathrm{d}z|$*

(ii) *If $|b_j| \geqslant B+1 > 1$, and $|a_j'(z)| < A$ for all z in D and all j, then*

$$\left|\frac{\mathrm{d}F(z)}{\mathrm{d}z}\right| \leqslant \frac{A}{B-1}.$$

(iii) *If $a_j(z) = a_j z^p$, where $|b_j| \geqslant B+1$, $B>1$, $|a_j| \leqslant A$, $p>1$, and $|z| \leqslant R$ with $R < \inf \sqrt[p]{(|b_j|-1)/|a_j|}$. Then $|\mathrm{d}F(z)/\mathrm{d}z| \leqslant ApR^{p-1}/(B-1)$.*

Proof. Since $|F^j| \leqslant 1$ (see [1], e.g.), for all z in D and all j, we have

$$\left|\frac{\mathrm{d}F(z)}{\mathrm{d}z}\right| \leqslant \sum_{k=0}^{\infty}\left(\prod_{j=1}^{k}\frac{|a_j(z)|}{|b_j+F^{j+1}|^2}\right)\frac{|a_{k+1}(z)|}{|b_{k+1}+F^{k+1}|} \leqslant \sum_{k=0}^{\infty}\left(\prod_{j=1}^{k}\frac{|a_j(z)|}{(|b_j|-1)^2}\right)\frac{|a_{k+1}'(z)|}{|b_{k+1}|-1}$$

from which the results easily follow (the second inequality is used for (i), (ii), and (iii)). Assume $w=0$. □

Example 3.2.

$$F(z) = \frac{z}{2} + \frac{-1}{2} + \frac{-1}{2} + \cdots \quad \text{where } |z| \leqslant 2 - 1 = 1.$$

$F(z)$ reduces to the identity function. Hence $|F'(z)| \equiv 1$, and Corollary 3.1 (i) provides a sharp bound, giving precisely the same value. Cauchy's inequality $|F'(z)| \leqslant 1/(1 - |z|)$ is poor except at the origin.

Example 3.3.

$$\Phi_s(z) = \frac{z}{s+1} \frac{{}_0F_1(s+2;z)}{{}_0F_1(s+1;z)} = \frac{z}{s+1} + \frac{z}{s+2} + \frac{z}{s+3} + \cdots$$

for $s > 1$ and $|z| \leqslant s$. $\Phi_s(z)$ is a ratio of hypergeometric series. It easily follows from (i) (with $B = s$) that $|d\Phi_s(z)/dz| \leqslant 1/(s-1)$ for $|z| \leqslant s$.

Examples 3.4. From a standard continued fraction expansion of $\tan z$,

$$F(z) = 1 - \frac{z}{\tan z} = \frac{z^2}{3} - \frac{z^2}{5} - \frac{z^2}{7} - \cdots.$$

Assume $|z| \leqslant \sqrt{2}$. Then $1 + |a_j(z)| = 1 + |z|^2 \leqslant 3 \leqslant |b_j|$ for all j. Using the first inequality in Corollary 3.1 gives

$$\left|\frac{dF(z)}{dz}\right| \leqslant |z| \sum_{k=0}^{\infty} \frac{|z|^{2k}}{(k+1)4^k(k!)^2}.$$

Actual values of the derivative and predicted bounds are:

	True value	Bound
$\lvert F'(1.414)\rvert$	1.291	1.798
$\lvert F'(1.0)\rvert$	0.770	1.136
$\lvert F'(0.5)\rvert$	0.345	0.517
$\lvert F'(0.05)\rvert$	0.033	0.050
$\lvert F'(0.005)\rvert$	0.003	0.005

Examples 3.5.

$$F(z) = \frac{(5+\frac{1}{1})z^3}{6} + \frac{(5+\frac{1}{2})z^3}{7} + \frac{(5+\frac{1}{3})z^3}{8} + \cdots.$$

We have (from (iii))

$|dF(z)/dz| \leqslant 4.5$ for $|z| \leqslant 1$. Computations show $|F'(1)| = 2.41$.

Corollary 3.6 (The Worpitzky case). The continued fraction

$$\frac{a_1(z)}{1} + \frac{a_2(z)}{1} + \cdots, \tag{3.2}$$

where each $a_j(z)$ is analytical in a domain D and $|a_j(z)| \leqslant \frac{1}{4}$ for all j and all z in D converges to $F(z)$ where $|F(z)| \leqslant \frac{1}{2}$. In addition

$$\left|\frac{\mathrm{d}F(z)}{\mathrm{d}z}\right| \leqslant 2\sum_{k=0}^{\infty} 4^k\left(\prod_{j=1}^{k}|a_j(z)|\right)\left|\frac{\mathrm{d}a_{k+1}(z)}{\mathrm{d}z}\right|.$$

(i) *If $|a_j(z)| \leqslant r_j < \frac{1}{4}$ for all z in D and all j,*

$$\left|\frac{\mathrm{d}F(z)}{\mathrm{d}z}\right| \leqslant 2\sum_{k=0}^{\infty} 4^k\left(\prod_{j=1}^{k}|r_j|\right)\left|\frac{\mathrm{d}a_{k+1}(z)}{\mathrm{d}z}\right|,$$

(ii) *if $|a_j(z)| \leqslant r < \frac{1}{4}$ for all z in D and all j,*

$$\left|\frac{\mathrm{d}F(z)}{\mathrm{d}z}\right| \leqslant 2\sum_{k=0}^{\infty} (4r)^k\left|\frac{\mathrm{d}a_{k+1}(z)}{\mathrm{d}z}\right|.$$

Proof. Eq. (2.2) *can be written*

$$\left|\frac{\mathrm{d}F(z)}{\mathrm{d}z}\right| \leqslant \sum_{k=0}^{\infty}\left(\prod_{j=1}^{k}\frac{|a_j(z)|}{|1+F^{j+1}|^2}\right)\frac{|a'_{k+1}(z)|}{|1+F^{k+1}|} \leqslant \sum_{k=0}^{\infty}\left(\prod_{j=1}^{k}\frac{|a_j(z)|}{(1-\frac{1}{2})^2}\right)\frac{|a'_{k+1}(z)|}{(1-\frac{1}{2})},$$

since $|F^j| < \frac{1}{2}$ for each j. Here it is assumed that $|w| < \frac{1}{2}$. The remaining two inequalities are easily obtained.

Examples 3.7.

$$F(z) = \frac{\frac{1}{4}z}{1} \, \frac{\frac{-1}{4}}{+\;1} \, \frac{\frac{-1}{4}}{+\;1} + \cdots,$$

where $|z| \leqslant 1$. $F(z)$ reduces to $z/2$. Thus, $|F'(z)| \equiv \frac{1}{2}$, and the first formula in Corollary 3.6 is sharp, giving a uniform bound of $\frac{1}{2}$. Without special knowledge of the value of this continued fraction, a routine application of Cauchy's inequality is $|F'(z)| \leqslant (\frac{1}{2})/(1-|z|)$, which is accurate only when $z = 0$, and quite inaccurate for larger values of $|z|$.

Example 3.8.

$$F(z) = \frac{z\tanh z}{3} = \frac{z^2/3}{1} \, \frac{z^2/1.3}{+\;1} \, \frac{z^2/3.5}{+\;1} + \cdots$$

where $|z| \leqslant \sqrt{\rho} < \sqrt{3}/2$. Then, from (ii),

$$\left|\frac{\mathrm{d}F(z)}{\mathrm{d}z}\right| \leqslant 4\sqrt{\rho}\left[\frac{1}{3} + \sum_{k=1}^{\infty}\frac{(\frac{4}{3}\rho)^k}{4k^2-1}\right].$$

The uniform derivative bound on the set $|z| \leqslant 0.2$ from (ii) is approximately 0.281. Computation shows that $|F'(0.2)| = 0.130$. The uniform derivative bound on the set $|z| \leqslant 0.1$ from (ii) is 0.135, whereas $|F'(0.1)| = 0.066$. The rather severe restriction on $|z|$ stated above shows a limitation of the procedure, for if $|z|$ is larger than described the continued fraction fails to satisfy the Worpitzky criteria and the tails of the fraction may not meet the requirement $|F^j| < \frac{1}{2}$.

Corollary 3.9 (A special Van Vleck case). *The continued fraction*

$$\frac{1}{b_1(z)} + \frac{1}{b_2(z)} + \cdots, \tag{3.3}$$

where the $\{b_n(z)\}$ *are analytic in* D, $-\frac{1}{2}\pi + \varepsilon < \arg b_n(z) < \frac{1}{2}\pi - \varepsilon$ *and* $|b_n(z)| \geqslant R \geqslant 2$ *for all* z *in* D *and all* n, *converges to* $F(z)$, *where* $|F(z)| \leqslant r$ *with* $r = \frac{1}{2}(R - \sqrt{R^2 - 4}) \leqslant 1$. *In addition*,

$$\left|\frac{\mathrm{d}F(z)}{\mathrm{d}z}\right| \leqslant \sum_{k=0}^{\infty} r^{2(k+1)} \left|\frac{\mathrm{d}b_{k+1}(z)}{\mathrm{d}z}\right|.$$

Proof. Here $R = r + 1/r$, and Corollary 4.15 and Theorem 4.29 [1] insure both convergence of the continued fraction and the condition $|F^j| \leqslant r$. From (2.2),

$$\left|\frac{\mathrm{d}F(z)}{\mathrm{d}z}\right| \leqslant \sum_{k=0}^{\infty} \left(\prod_{j=1}^{k} \frac{1}{(|b_j(z)| - |F^j|)^2}\right) \frac{|\mathrm{d}b_{k+1}(z)/\mathrm{d}z|}{(|b_{k+1}(z)| - |F^{k+1}|)^2}$$

$$\leqslant \sum_{k=0}^{\infty} \left(\prod_{j=1}^{k} \frac{1}{(r + (1/r) - r)^2}\right) \frac{|\mathrm{d}b_{k+1}(z)/\mathrm{d}z|}{(r + (1/r) - r)^2} = \sum_{k=0}^{\infty} r^{2(k+1)} |\mathrm{d}b_{k+1}(z)/\mathrm{d}z|.$$

Example 3.10.

$$F(z) = \frac{-1}{z + \frac{1}{z}} + \frac{-1}{z + \frac{1}{z}} + \frac{-1}{z + \frac{1}{z}} + \cdots$$

$$= \frac{1}{-\left(z + \frac{1}{z}\right)} + \frac{1}{z + \frac{1}{z}} + \frac{1}{-\left(z + \frac{1}{z}\right)} + \frac{1}{z + \frac{1}{z}} + \cdots = -z$$

if $0 < |z| < 2^{1/2} - 1$.

Hence, $|F'(z)| \equiv 1$. Here $|b_j(z)| = |z + 1/z| \geqslant 1/|z| - |z| > 2$, and we assume the condition on the $\arg b_j(z)$ are met. Suppose, now, that $0 < z < 2^{1/2} - 1$, *giving* $|b_j(z)| = z + 1/z$. The series estimate in Corollary 3.9 is then sharp:

$$\sum_{k=0}^{\infty} r^{2(k+1)} |b'_{k+1}(z)| = \sum_{k=0}^{\infty} z^{2(k+1)} \left(\frac{1}{z^2} - 1\right) \equiv 1.$$

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