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Homological symbols and the Quillen conjecture

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ABSTRACT

We formulate a "correct" version of the Quillen conjecture on linear group homology for certain arithmetic rings and provide evidence for the new conjecture. In this way we predict that the linear group homology has a direct summand looking like an unstable form of Milnor K-theory and we call this new theory "homological symbols algebra". As a byproduct, we prove the Quillen conjecture in homological degree two for the rank two and the prime 5.

1. Introduction

Let R be a subring with identity of the complex numbers $\mathbb{C}$ and resp. GL_n, SL_n the discrete group of $n \times n$ matrices over R with determinant resp. nonzero, 1. If $H(GL_n) := H^*(GL_n; \mathbb{F}_\ell)$ denotes the mod ℓ group cohomology of GL_n, then the canonical inclusion $R \subset \mathbb{C}$ induces a module structure of $H(GL_n)$ over the singular mod ℓ cohomology ring of Chern classes $P_n := H^*(BGL_n(\mathbb{C}); \mathbb{F}_\ell)$ where $BGL_n(\mathbb{C})$ denotes the classifying space of the Lie group $GL_n(\mathbb{C})$ of invertible $n \times n$ matrices over $\mathbb{C}$. In [16, p. 591] Quillen conjectured that for certain primes ℓ and rings R the module $H(GL_n)$ is *free* over P_n. We call this statement *the strong Quillen conjecture for the rank n and the prime* ℓ.

In particular, if we fix $R = \mathbb{Z}[\frac{1}{\ell}, \xi_\ell]$ where ℓ is a *regular* prime and $\xi_\ell \in \mathbb{C}$ is a primitive ℓ-th root of unity, then it has been shown in [12, p. 51] that the strong Quillen's conjecture implies that *the homomorphism*

$$\iota_{np} : H_p(GL_1^{\times n}; \mathbb{F}_\ell) \to H_p(GL_n; \mathbb{F}_\ell)$$

induced by the canonical inclusion $GL_1^{\times n} \subset GL_n$ *on mod* ℓ *homology is surjective for all* p. We call the statement that ι_{np} is surjective *the weak Quillen conjecture in homological degree p for the rank n and the prime* ℓ. This weak conjecture was disproved in [7] for $n \geq 32$, $\ell = 2$, and in [1] for $n \geq 27$, $\ell = 3$, in the sense that there is an unspecified p depending on n and ℓ for which the statement fails.

In this article we formulate yet another conjecture for ℓ *odd* and regular (Conjecture 5.1) which together with the weak Quillen conjecture for the rank n, the prime ℓ and *all* homological degrees p implies the strong Quillen conjecture for the same rank n and prime ℓ (see Section 5.1 for a full discussion). More specifically, by Proposition 5.3, this new conjecture states that a certain *finite* set of homological classes in $H_*(GL_1; \mathbb{F}_\ell)$ vanish in $H_*(SL_2; \mathbb{F}_\ell)$ under the map induced from embedding GL_1 in SL_2 via $u \mapsto \begin{pmatrix} u^{-1} & 0 \\ 0 & u \end{pmatrix}$. These classes are called *étale obstruction classes* since they originate from studying étale models [3] for the classifying spaces BGL_n. The bar complex cycles representing these classes are given explicitly in Definition 4.3.

As evidence for the Conjecture 5.1 we remark that this conjecture and the weak Quillen conjecture for $n = 2$ and all p are true for $\ell = 3$ by direct calculations [1] and thus, the strong Quillen conjecture holds in this case. Also the case $\ell = 2$ fits into the same pattern for the ranks $n = 2$ [15] and $n = 3$ [11]. In this article we prove a new result stating that

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Theorem 1.1. $H_2(SL_2(\mathbb{Z}[\frac{1}{5}, \xi_5]; \mathbb{F}_5)) = 0$.

As a corollary, our conjecture is true in homological degree two for $\ell = 5$ in the sense that the étale obstruction classes from $H_2(GL_1; \mathbb{F}_5)$ obviously vanish in $H_2(SL_2; \mathbb{F}_5)$. As a byproduct, we obtain that the weak Quillen conjecture in homological degree two for the rank two and the prime 5 is also true and

Theorem 1.2. $H_2(GL_2(\mathbb{Z}[\frac{1}{5}, \xi_5]; \mathbb{F}_5)) \approx \mathbb{F}_5 \oplus \mathbb{F}_5 \oplus \mathbb{F}_5 \oplus \mathbb{F}_5$.

The technique used in proving Theorem 1.1 is based on proving a key result in Theorem 6.2 regarding the structure of the group SL_2 as a finitely presented group and using GAP [9] in a clever way. The main difficulties reside in the complexity of the combinatorial group problems associated with Hopf's formula and its generalizations [17].

Another feature of this article is a characterization of a direct summand (as a vector space) of the bi-graded algebra

$$A := \bigoplus_{i,j=0}^{\infty} H_i(GL_j; \mathbb{F}_\ell) \tag{1.1}$$

where the algebra structure is induced from the matrix block multiplication. This summand is the bi-graded subalgebra $KA \subset A$ generated by the linear subspace $H_*(GL_1; \mathbb{F}_\ell) \subset A$ and its structure is predicted by the new conjecture in the sense that the relations in KA come from $H_*(GL_1^{\times 2}; \mathbb{F}_\ell)$ in a certain explicit way (see Remark 5.2).

We recall [14] that the (naive) Milnor K-theory of the ring R is the tensor algebra generated by the group of units GL_1 modulo the Steinberg relations $u \otimes (1-u) = 0$ coming from $GL_1^{\otimes 2}$ for $u, 1-u \in GL_1$. By replacing GL_1 with $H_*(GL_1; \mathbb{F}_\ell)$, $GL_1^{\otimes 2}$ with $H_*(GL_1^{\times 2}; \mathbb{F}_\ell)$ and the Steinberg relations with those relations predicted by our conjecture, we obtain the conjectural structure of KA. For this reason, we call KA the *algebra of homological symbols at* ℓ associated with the ring R.

The paper is organized as follows. After reviewing some basic group homology facts in Section 2 and introducing some algebra terminology in Section 3, we describe the direct summand of the algebra (1.1) and estimate from "above" the relations of this summand in Theorem 4.6. The conjecture on the exact relations is formulated in Section 5. In Section 6 we estimate the relations in SL_2 from "below" for *any* regular odd prime and use them in Section 7 to prove Theorem 1.1 (see Corollary 7.5). Theorem 1.2 follows now from Theorems 1.1 and 4.6 by a spectral sequence argument.

2. Group homology preliminaries

We recall some standard facts about group homology as in [4]. Let G be a multiplicative group with neutral element $1 \in G$ and k a commutative ring with identity.

2.1. The shuffle product

Let $\mathcal{B}_*(G; k)$ be the normalized bar complex:

$$\mathcal{B}_0(G; k) \overset{\partial}{\leftarrow} \mathcal{B}_1(G; k) \ldots \overset{\partial}{\leftarrow} \mathcal{B}_{s-1}(G; k) \overset{\partial}{\leftarrow} \mathcal{B}_s(G; k) \overset{\partial}{\leftarrow} \ldots \tag{2.1}$$

where $\mathcal{B}_s(G; k)$ is the free k-module generated by the set of symbols $[x_1|\ldots|x_s]$ with $x_1, \ldots, x_s \in G \setminus \{1\}$ and ∂ is the k-homomorphism given by the formula:

$$\partial[x_1|\ldots|x_s] = [x_2|\ldots|x_s] + \sum_{j=1}^{s-1} (-1)^j [x_1|\ldots|x_j x_{j+1}|\ldots|x_s] + (-1)^s [x_1|\ldots|x_{s-1}]$$

with $[x_1|\ldots|x_j x_{j+1}|\ldots|x_s] = 0$ by convention if $x_j x_{j+1} = 1$. By definition, the group homology $H_*(G; k)$ with k-coefficients is the homology of the chain complex (2.1).

On the other hand, the chain complex (2.1) can be regarded as a graded algebra $\mathcal{B}(G; k)$ over k which is anti-commutative, associative, and unital with respect to the shuffle product

$$[x_1|\ldots|x_i] \wedge [x_{i+1}|\ldots|x_{i+s}] = \sum (-1)^\sigma [x_{\sigma(1)}|\ldots|x_{\sigma(i+s)}] \tag{2.2}$$

where the sum is over all the permutations σ of $i+s$ letters that shuffle $\{1, \ldots, i\}$ with $\{i+1, \ldots, i+s\}$ i.e. $\sigma^{-1}(1) < \cdots < \sigma^{-1}(i)$ and $\sigma^{-1}(i+1) < \cdots < \sigma^{-1}(i+s)$ and $(-1)^\sigma$ is the signature of σ.

Nevertheless, $\mathcal{B}(G; k)$ is not necessarily a differential graded algebra since the Leibniz formula

$$\partial([x_1|\ldots|x_i] \wedge [x_{i+1}|\ldots|x_{i+s}]) = (\partial[x_1|\ldots|x_i]) \wedge [x_{i+1}|\ldots|x_{i+s}] + (-1)^i [x_1|\ldots|x_i] \wedge (\partial[x_{i+1}|\ldots|x_{i+s}]) \tag{2.3}$$

holds if and only if $x_j x_k = x_k x_j$ for all $j \le i < k$. As an immediate consequence of (2.3) we have the following

Lemma 2.1. *If $x_1, \ldots, x_i$ are elements of G commuting with one another, then the element of $\mathcal{B}_i(G; k)$ given by formula*

$$\langle x_1, x_2, \ldots, x_i \rangle = [x_1] \wedge [x_2] \wedge \cdots \wedge [x_i]$$

is a cycle representing a homological class in $H_i(G; k)$ which is i-linear and skew-symmetric in $x_1, \ldots, x_i$.

2.2. The Bockstein homomorphism

If ℓ is a prime number and $\zeta \in G$ such that $\zeta^\ell = 1$, then for each nonnegative integer s we define an element of $B_{2s}(G; k)$ given by the formula

$$[\zeta]^{(s)} = \sum_{i_1,\dots,i_s=1}^{\ell-1} [\zeta^{i_1}|\zeta|\zeta^{i_2}|\zeta|\dots|\zeta^{i_s}|\zeta]$$

where $[\zeta]^{(0)} = [\,]$ is the generator of $\mathcal{B}_0(G; k)$. By an inductive argument we can verify that

$$[\zeta]^{(s)} \wedge [\zeta]^{(i)} = \binom{s+i}{i} [\zeta]^{(s+i)} \tag{2.4}$$

for all nonnegative integers s, i. Again by an inductive argument using (2.3) and (2.4) we can verify the formula

$$\partial([\zeta]^{(s)}) = \ell[\zeta]^{(s-1)} \wedge [\zeta] \tag{2.5}$$

for all positive integers s. In this context, recall [10, p. 303] that the short exact sequence of chain complexes

$$0 \to \mathcal{B}_*(G; \mathbb{Z}/\ell) \xrightarrow{\times\ell} \mathcal{B}_*(G; \mathbb{Z}/\ell^2) \to \mathcal{B}_*(G; \mathbb{Z}/\ell) \to 0 \tag{2.6}$$

associated with the multiplication by ℓ map induces a homology long exact sequence

$$\cdots \to H_i(G; \mathbb{Z}/\ell) \to H_i(G; \mathbb{Z}/\ell^2) \to H_i(G; \mathbb{Z}/\ell) \xrightarrow{\beta} H_{i-1}(G; \mathbb{Z}/\ell) \to \cdots$$

where β is the Bockstein homomorphism. In particular, if $\mathbb{F}_\ell$ denotes the field of order ℓ then by a diagram chasing using (2.6) and (2.5) we obtain the following

Lemma 2.2. *If $\zeta \in G$ such that $\zeta^\ell = 1$ and s is a positive integer, then $[\zeta]^{(s)}$ is a cycle representing a homology class $\omega \in H_{2s}(G; \mathbb{F}_\ell)$ such that $[\zeta]^{(s-1)} \wedge [\zeta]$ is a cycle representing the class $\beta(\omega) \in H_{2s-1}(G; \mathbb{F}_\ell)$.*

2.3. The Pontryagin ring

If G is an abelian group then, according to (2.3), $\mathcal{B}(G; k)$ is a differential graded algebra with respect to the shuffle product (2.2) inducing a graded algebra structure on homology $H_*(G; k)$. If ${}_\ell G$ denotes the ℓ-torsion subgroup of G and $\Gamma({}_\ell G)$ the algebra of divided powers [4, p. 119] over $\mathbb{F}_\ell$ generated in degree two by ${}_\ell G$, then the homomorphism of graded algebras

$$\Gamma({}_\ell G) \to H_*(G; \mathbb{F}_\ell) \tag{2.7}$$

sending each element ζ of ${}_\ell G$ to the class of $[\zeta]^{(1)}$ in $H_2(G; \mathbb{F}_\ell)$ is well defined according to (2.4). Similarly, if $\Lambda(G \otimes \mathbb{Z}/\ell)$ denotes the exterior algebra over $\mathbb{F}_\ell$ generated in degree one by $G \otimes \mathbb{Z}/\ell$ then the homomorphism of graded algebras

$$\Lambda(G \otimes \mathbb{Z}/\ell) \to H_*(G; \mathbb{F}_\ell) \tag{2.8}$$

sending each element $g \otimes 1$ of $G \otimes \mathbb{Z}/\ell$ to the class of $[g]$ in $H_1(G; \mathbb{F}_\ell)$ is also well defined according to Lemma 2.1.

Proposition 2.3 (*[4, p. 126]*). *If ℓ is a prime number and G is an abelian group, then the maps (2.7) and (2.8) induce an isomorphism of graded algebras*

$$\Gamma({}_\ell G) \otimes \Lambda(G \otimes \mathbb{Z}/\ell) \approx H_*(G; \mathbb{F}_\ell).$$

If G_1, G_2 are two groups then the Künneth isomorphism [5, p. 218]

$$\kappa : H_*(G_1; \mathbb{F}_\ell) \otimes H_*(G_2; \mathbb{F}_\ell) \xrightarrow{\approx} H_*(G_1 \times G_2; \mathbb{F}_\ell) \tag{2.9}$$

is induced by the map sending

$$[x_1|\dots|x_i] \otimes [x_{i+1}|\dots|x_{i+s}] \mapsto [x_1 \times 1|\dots|x_i \times 1] \wedge [1 \times x_{i+1}|\dots|1 \times x_{i+s}]$$

where x_j is an element of G_1 for $j \le i$ and an element of G_2 for $j > i$. In particular, if both G_1 and G_2 are abelian, then κ is a graded algebra isomorphism with respect to the product

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{|b_1||a_2|}(a_1 \wedge a_2) \otimes (b_1 \wedge b_2)$$

defined for homogeneous elements $a_i, b_i \in H_*(G_i; \mathbb{F}_\ell)$ of degrees $|a_i|$ and $|b_i|$ for $i = 1, 2$.

Remark 2.4. If G is an abelian group and $\mu : G \times G \to G$ is its group law homomorphism, then the composition between the induced homomorphism

$$\mu_* : H_*(G \times G; \mathbb{F}_\ell) \to H_*(G; \mathbb{F}_\ell)$$

and the Künneth isomorphism (2.9) for $G_1 = G_2 = G$ defines a product on $H_*(G; \mathbb{F}_\ell)$ that can be easily checked to be induced by the shuffle product. In this case, $H_*(G; \mathbb{F}_\ell)$ is called the *Pontryagin ring* and its structure is given by Proposition 2.3.

3. Algebras of homological symbols

Let k be a fixed commutative ring with identity.

3.1. Algebras of symbols

If $A = \bigoplus_{i,n=0}^{\infty} A_{in}$ is an associative bi-graded k-algebra, denote by

$$A_{*n} := \bigoplus_{i=0}^{\infty} A_{in} \subset A \tag{3.1}$$

the k-submodule of all elements with the second degree n. Also, let

$$q : T(A_{*1}) = \bigoplus_{n=0}^{\infty} A_{*1}^{\otimes n} \to A \tag{3.2}$$

be the canonical bi-graded algebra homomorphism where $T(A_{*1})$ is the bi-graded tensor k-algebra generated by the k-submodule $A_{*1} \subset A$. Here $\otimes n$ denotes the n-fold graded tensor product over k.

Definition 3.1. The *algebra of symbols* associated with an associative bi-graded k-algebra $A = \bigoplus_{i,n=0}^{\infty} A_{in}$ is the quotient bi-graded algebra

$$KA := T(A_{*1})/\ker q$$

with respect to the kernel of the canonical homomorphism (3.2).

Definition 3.2. An associative bi-graded k-algebra $A = \bigoplus_{i,n=0}^{\infty} A_{in}$ is *quadratic* with respect to the second degree if the canonical homomorphism (3.2) is surjective and its kernel can be generated as a two-sided ideal by a subset of $A_{*1}^{\otimes 2}$.

According with the above definitions the algebra of symbols KA associated with an associative bi-graded k-algebra A comes with a *natural* bi-graded algebra monomorphism

$$q' : KA \hookrightarrow A. \tag{3.3}$$

Some questions of interest will be to study when q' is an isomorphism and when KA is a quadratic algebra with respect to the second degree.

3.2. Graded H-spaces

We say that a topological space $X = \bigsqcup_{n=0}^{\infty} X_n$ decomposed into a disjoint union of non-empty open subspaces $X_n \subset X$ is a *graded H-space* if there is a continuous map $h : X \times X \to X$ with

$$h(X_n \times X_m) \subset X_{n+m} \quad \text{for all } n, m \geq 0$$

such that X is an associative H-space relative h in the sense of [10, p. 281] with the homotopy unit in X_0. A continuous map between graded H-spaces

$$f : X = \bigsqcup_{n=0}^{\infty} X_n \to Y = \bigsqcup_{n=0}^{\infty} Y_n$$

is a *graded H-map* if $f(X_n) \subset Y_n$ for all $n \geq 0$ and f is an H-map.

Definition 3.3. The *k-algebra of homological symbols* associated with a graded H-space $X = \bigsqcup_{n=0}^{\infty} X_n$ is the algebra of symbols $KH_*(X; k)$ associated in the sense of the Definition 3.1 with the bi-graded k-algebra

$$H_*(X; k) \approx \bigoplus_{i,n=0}^{\infty} H_i(X_n; k)$$

where $H_*(\ ; k)$ is the singular homology functor with k-coefficients.

In the above definition, the bi-graded algebra structure on $H_*(X; k)$ is induced from the graded H-structure on X via the Künneth homomorphisms

$$H_*(X_n; k) \otimes H_*(X_m; k) \to H_*(X_n \times X_m; k)$$

and the assignment $X \mapsto KH_*(X;k)$ is obviously natural with respect to graded H-maps. Also we have a natural monomorphism

$$q' : KH_*(X;k) \hookrightarrow H_*(X;k). \tag{3.4}$$

given by (3.3) applied to $A = H_*(X;k)$.

Notation 3.4. For the rest of this article, if not otherwise stated, we fix $\ell := 2r+1$ a *regular* odd prime number, ξ is a primitive ℓ-root of unity, and $R := \mathbb{Z}[\frac{1}{\ell}, \xi]$ the ring of cyclotomic ℓ-integers. Also GL_n, SL_n will denote the groups of matrices over R as defined in the Introduction.

4. The main examples

In this article we are concerned with examples of algebras of homological symbols arising from linear groups.

4.1. Approximations to BGL_n

The mod ℓ homology of the group GL_n is naturally isomorphic to the singular mod ℓ homology of its classifying space BGL_n. The classifying space BGL_n can be approximated by the classifying space $BGL_1^{\times n}$ of the n-fold direct product $GL_1^{\times n}$ and by a topological space BGL_n^{et} called the *étale model at* ℓ, defined in [8, p. 3]. These spaces are connected by natural continuous maps

$$BGL_1^{\times n} \xrightarrow{\iota_n} BGL_n \xrightarrow{f_n} BGL_n^{\acute{e}t} \tag{4.1}$$

where ι_n is the classifying space map induced by the canonical inclusion $GL_1^{\times n} \subset GL_n$ and f_n is a map defined in [8, p. 3]. By taking the disjoint union of the diagrams (4.1) we obtain a diagram of topological spaces and continuous maps

$$X := \coprod_{n=0}^{\infty} BGL_1^{\times n} \xrightarrow{\iota} Y := \coprod_{n=0}^{\infty} BGL_n \xrightarrow{f} Z := \coprod_{n=0}^{\infty} BGL_n^{\acute{e}t} \tag{4.2}$$

such that each disjoint union has a graded H-space structure induced by the matrix block-multiplication and the maps $\iota = \sqcup \iota_n$ and $f = \sqcup f_n$ are graded H-maps. On mod ℓ homology, the diagram (4.2) induces a commutative diagram of bi-graded algebras and homomorphisms

$$\begin{array}{ccccc} KH_*(X;\mathbb{F}_\ell) & \xrightarrow{K\iota_*} & KH_*(Y;\mathbb{F}_\ell) & \xrightarrow{Kf_*} & KH_*(Z;\mathbb{F}_\ell) \\ \downarrow{\scriptstyle q_1} & & \downarrow{\scriptstyle q_2} & & \downarrow{\scriptstyle q_3} \\ H_*(X;\mathbb{F}_\ell) & \xrightarrow{\iota_*} & H_*(Y;\mathbb{F}_\ell) & \xrightarrow{f_*} & H_*(Z;\mathbb{F}_\ell) \end{array} \tag{4.3}$$

where the algebras of homological symbols in the first row are given by the Definition 3.3 and the monomorphisms q_i are given by (3.4) for $i = 1, 2, 3$. The second row of the diagram (4.3) can written as a diagram of bi-graded algebras

$$T := \bigoplus_{i,n=0}^{\infty} T_{i,n} \xrightarrow{\iota_*} A := \bigoplus_{i,n=0}^{\infty} A_{i,n} \xrightarrow{f_*} A^{\acute{e}t} := \bigoplus_{i,n=0}^{\infty} A_{i,n}^{\acute{e}t} \tag{4.4}$$

where for each bi-degree (i,n), we define

$$T_{i,n} := H_i(GL_1^{\times n};\mathbb{F}_\ell), \qquad A_{i,n} := H_i(GL_n;\mathbb{F}_\ell), \qquad A_{i,n}^{\acute{e}t} := H_i(BGL_n^{\acute{e}t};\mathbb{F}_\ell).$$

The first degree i is called the *homological degree* and the second degree n is called the *rank*.

Theorem 4.1 ([8, Lemma 6.2]). *The composed homomorphism $f_* \circ \iota_*$ in the diagram (4.4) is surjective.*

The rank n elements of T form the linear subspace $T_{*n} \subset T$ (see (3.1)) such that:

$$T_{*n} = H_*(GL_1^{\times n};\mathbb{F}_\ell) \approx H_*(GL_1;\mathbb{F}_\ell)^{\otimes n}$$

by the Künneth isomorphism. In particular,

$$T = H_*(X;\mathbb{F}_\ell) \approx T(H_*(GL_1;\mathbb{F}_\ell))$$

is the tensor algebra generated by $T_{*1} = H_*(GL_1;\mathbb{F}_\ell)$ and thus, q_1 in (4.3) is an isomorphism. From Theorem 4.1 we deduce the following

Corollary 4.2. *The monomorphism q_3 in the diagram (4.3) is an isomorphism.*

4.2. Étale obstruction classes

To describe the kernel of $f_* \circ \iota_*$ we observe that according to [18] the group of units GL_1 of the ring R is the abelian group generated by the set of cyclotomic units

$$\{-\xi, 1-\xi, 1-\xi^2, \dots, 1-\xi^r\} \tag{4.5}$$

subject to the relation $(-\xi)^{2\ell} = 1$. By applying Proposition 2.3 to GL_1, we deduce that T_{*1} is a vector space over $\mathbb{F}_\ell$ with basis the set of homology classes represented by cycles of the form

$$[\xi]^{(s)} \wedge \langle v_1, \dots, v_i \rangle \tag{4.6}$$

where s runs over all nonnegative integers and $\{v_1, \dots, v_i\}$ over all subsets of the set (4.5). In this context, the following definition is a slight modification of [3, p. 2336]:

Definition 4.3. A class $\epsilon \in T_{*1}$ represented by a cycle of the form (4.6) is called an *étale obstruction class* if s is a nonnegative integer and $\{v_1, \dots, v_i\}$ is a subset of the set (4.5) of cardinality i such that $i = s + 2j$ for some integer $j > 0$.

Definition 4.4. A class $\omega \in T_{1*}$ represented by a cycle of the form (4.6) is called a *homogeneous class of weight* $\|\omega\| := s + i$.

Remark 4.5. For each integer $i \geq 2$ let $e(i)$ denote the cardinality of the set of all integers $s \equiv i \mod 2$ such that $0 \leq s \leq i-2$. Then the number e of étale obstruction classes is finite and given by the formula

$$e = \sum_{i=2}^{r+1} e(i) \binom{r+1}{i}.$$

The following group homomorphisms:

$$GL_1 \xrightarrow{t} GL_1^{\times 2} \xleftarrow{\rho} GL_1^{\times 3}$$

given by the formulas

$$t(u) = u^{-1} \times u, \qquad \rho(u \times v \times w) = uw \times vw, \tag{4.7}$$

for $u, v, w \in GL_1$ induce homomorphisms on mod ℓ homology:

$$t_* : T_{*1} \to T_{*2}, \qquad \rho_* : T_{*2} \otimes T_{*1} \approx T_{*3} \to T_{*2},$$

where the source T_{*3} of ρ_* has been identified with $T_{*2} \otimes T_{*1}$ via the Künneth isomorphism. With these preparations, we have the following important result:

Theorem 4.6 ([3]). *The kernel of the bi-graded algebra homomorphism:*

$$f_* \circ \iota_* : T \to A^{ét}$$

is the two-sided ideal of T generated by the set of elements of the form:

$$\rho_*(t_*(\eta) \otimes z), \tag{4.8}$$

where $\eta, z \in T_{*1}$ such that η runs over all the étale obstruction classes and the homogeneous classes of odd weight $\|\eta\|$, and z runs over a vector space basis for T_{*1}.

The proof of this theorem is a direct translation using Lemmas 2.1 and 2.2 of the calculations made in [3, p. 2338]. Also, via the Künneth isomorphisms, T can be regarded as the tensor algebra on T_{*1} and T_{*1} can be identified via $f_* \circ \iota_*$ with $A^{ét}_{*1}$ (see [8, Proposition 5.2]). Thus, combining the Theorems 4.1 and 4.6 we obtain the structure of the bi-graded algebra $A^{ét}$ as a quadratic algebra with respect to the rank:

Corollary 4.7. *The bi-graded algebra $A^{ét}$ in (4.4) is a quadratic algebra with respect to the rank in the sense of the Definition 3.2.*

Remark 4.8. The homomorphism ρ_* defines a graded module structure on T_{*2} over the Pontryagin ring T_{*1} (see Remark 2.4). The Theorem 4.6 says that the kernel of $f_* \circ \iota_*$ is generated as a two-sided ideal by a submodule of T_{*2} of finite rank e over T_{*1} modulo the classes (4.8) with $\|\eta\|$ odd, where e is given by Remark 4.5.

5. The main conjecture

5.1. The statement

The maps in the diagram (4.3) have the following known properties:

(1) $K\iota_*$ and Kf_* are surjective. This is immediate from the fact that $K\iota_*$ and Kf_* are bijective in rank 1 and their targets are generated as algebras by rank 1 elements.

(2) f_* is surjective but not an isomorphism. The first part follows from the Theorem 4.1 while the last part was proven in [2].

(3) ι_* is surjective *if* the Quillen conjecture [16, p. 591] holds true for the ring R and all the ranks n. This fact was proven in [12, p. 51].

(4) q_1 and q_3 are isomorphisms. These facts follow from the Corollary 4.2 and its preceding proof.

(5) q_2 is an isomorphism *if* ι_* is surjective. This follows from (4) by chasing the diagram (4.3).

(6) f_* is an isomorphism *if* Kf_* is bijective and ι_* is surjective. This follows from (4) and (5) by chasing the diagram (4.3).

In this article we conjecture that:

Conjecture 5.1. *The map $Kf_* : KA \to KA^{ét} \approx A^{ét}$ in the diagram (4.3) is an isomorphism.*

By (2), (3), (6), our Conjecture 5.1 implies that the Quillen conjecture [16, p. 591] for the ring R defined in Notation 3.4 cannot be true in all the ranks n. In this sense, our conjecture can be regarded as a "correction" of the Quillen conjecture. Also our conjecture implies that ι_* is not surjective and q_2 is not an isomorphism.

Remark 5.2. The Conjecture 5.1 and the Theorems 4.1 and 4.6 (see also Remark 4.8) compute the direct summand KA of the mysterious algebra (1.1). This summand is an algebra of homological symbols which is quadratic with respect to the rank by the Corollary 4.7.

5.2. A useful reduction

Recalling t, ρ defined in (4.7), we have a commutative diagram

$$\begin{array}{ccc} GL_1 \times GL_1 & \xrightarrow{t\times Id} & GL_1^{\times 2} \times GL_1 \\ {\scriptstyle \tau\times Id}\downarrow & & \downarrow{\scriptstyle \widetilde{\rho}} \\ SL_2 \times GL_1 & \xrightarrow{\mu} & GL_2 \end{array}$$

where Id is the identity map, $\widetilde{\rho}$ is ρ composed with the canonical inclusion $GL_1^{\times 2} \subset GL_2$,

$$\tau(u) = \begin{pmatrix} u^{-1} & 0 \\ 0 & u \end{pmatrix} \quad \text{and} \quad \mu(A \times u) = A \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} \text{ (matrix product)} \tag{5.1}$$

for all $u \in GL_1$ and $A \in SL_2$. By passing to mod ℓ homology we have the following

Proposition 5.3. *The Conjecture 5.1 is true if and only if $\tau_*(\epsilon) = 0$ in $H_*(SL_2; \mathbb{F}_\ell)$ for all étale obstruction classes $\epsilon \in H_*(GL_1; \mathbb{F}_\ell)$.*

Proof. The cycle $[\xi^{-1}]^{(1)}$ is homologous to $-[\xi]^{(1)}$ as we deduce from

$$\partial[\xi^i|\xi^{-1}|\xi] = [\xi^{-1}|\xi] - [\xi^{i-1}|\xi] - [\xi^i|\xi^{-1}]$$

by taking the sum over $i = 1, \ldots, \ell$. If $\sigma_* : T_{*1} \to T_{*1}$ is the homomorphism induced by $\sigma : GL_1 \to GL_1, u \mapsto u^{-1}$, and η is represented by (4.6) then we can prove inductively that $\sigma_*(\eta) = (-1)^{\|\eta\|}\eta$ where $\|\eta\|$ is given by Definition 4.4. Because σ extends to an inner automorphism of SL_2 via τ, we conclude that $\tau_* \circ \sigma_*$ is the identity map on $H_*(SL_2; \mathbb{F}_\ell)$. Hence, the classes $\tau_*(\eta)$ with $\eta \in T_{*1}$ and $\|\eta\|$ odd vanish in $H_*(SL_2; \mathbb{F}_\ell)$. The necessity follows now from the equation

$$\widetilde{\rho}_*(t_*(\eta) \otimes z) = \mu_*(\tau_*(\eta) \otimes z)$$

by chasing the diagram (4.3) and using the Theorem 4.6. The sufficiency follows by a spectral sequence argument as in [3, Lemma 4.8]. □

6. A group theoretical approach

The aim of this section is to provide a group theoretical method producing evidence for the Conjecture 5.1. This method is based on a finitely presented group defined next.

6.1. A finitely presented group

Let SE_2 be the group generated by the symbols $D(u)$ and $E(x)$ subject to the following relations [6]:

$$\begin{aligned}
&\text{Type I.} \quad E(x)E(0)E(y) = D(-1)E(x+y)\\
&\text{Type II.} \quad E(x) = D(u)E(xu^2)D(u)\\
&\text{Type III.} \quad E(u^{-1})E(u)E(u^{-1}) = D(-u)\\
&\text{Type IV.} \quad D(u)D(v) = D(uv)
\end{aligned} \tag{6.1}$$

where $u, v \in GL_1$ and $x, y \in R$ run over all elements. We introduce the following labels:

$$z := D(\xi), \qquad u_i := D(\epsilon_i), \qquad a := E(0), \qquad b := E(1) \tag{6.2}$$

where $\epsilon_i := 1 - \xi^i$ for $i = 1, 2, \ldots, r$ are given by (4.5), and we define:

$$b_t := z^{rt} b z^{rt} a, \qquad w := z^c u_1 u_2 \ldots u_r \tag{6.3}$$

where $t = 0, 1, 2, \ldots, 2r$ and $c \geq 0$ is the smallest integer such that

$$2c \equiv r^2 + \frac{r(r+1)}{2} \bmod \ell.$$

We will occasionally use b_t with t an *arbitrary* integer where $b_t = b_s$ if $t \equiv s \bmod \ell$ and the following notation $[x, y] = xyx^{-1}y^{-1}$.

Definition 6.1. For each non-empty subset $I \subset \{1, 2, \ldots, r\}$ define

$$c(I) := \left(\prod_{t=0}^{2r} b_t^{c_t(I)}\right) a^{-1} \prod_{i \in I} u_i$$

where $c_t(I) \in \mathbb{Z}$ such that in R we have the following identity:

$$\epsilon_I := \prod_{i \in I} \epsilon_i = \prod_{i \in I} (1 - \xi^i) = \sum_{t=0}^{2r} c_t(I) \xi^t.$$

For instance, if $I = \{i\}$ is a singleton, then $c(I) = b_0 b_i^{-1} a^{-1} u_i$ and if $I = \{i, j\}$ has two elements then $c(I) = b_0 b_i^{-1} b_j^{-1} b_{i+j} a^{-1} u_i u_j$.

Theorem 6.2. *The group SE_2 defined above is generated by*

$$z, u_1, u_2, \ldots, u_r, a, b$$

subject to the following relations:

$$z^\ell = [z, u_i] = [u_i, u_j] = 1 \tag{6.4}$$

$$a^4 = [a^2, z] = [a^2, u_i] = 1 \tag{6.5}$$

$$a = zaz = u_i a u_i \tag{6.6}$$

$$[b_s, b_t] = 1 \tag{6.7}$$

$$b^3 = a^2 = b_0 b_1 \ldots b_{2r} \tag{6.8}$$

$$b_t^\ell = w^{-1} b_t^{(-1)^r} w \tag{6.9}$$

$$c(I)^3 = 1 \tag{6.10}$$

$$ba^2 = u_i b z^{-ri} b^{-1} b_0^{-1} z^{ri} b z^{-i} u_i \tag{6.11}$$

where $i, j \in \{1, 2, \ldots, r\}$, $s, t \in \{0, 1, 2, \ldots, 2r\}$, and $I \subset \{1, 2, \ldots, r\}$ runs over all nonempty subsets.

The theorem implies that SE_2 has a finite presentation with $r + 3$ generators and $6 + 6.5r + 2.5r^2 + 2^r$ relators. Its proof will be given as a sequence of lemmas. For convenience, we will refer to the relations (6.1) only by type. Also we will tacitly use (6.2) and (6.3), the relations in GL_1 given at the beginning of Section 4.2, and when appropriately, Type IV.

Lemma 6.3. *$z, u_1, u_2, \ldots, u_r, a, b$ generate SE_2.*

Proof. By Type II with $u = -1$, it follows that $D(-1)$ is central and by Type I with $x = y = 0$, we have

$$a^2 = D(-1). \tag{6.12}$$

Since each $v \in GL_1$ can be written as $v = (-\xi)^j \epsilon_1^{a_1} \ldots \epsilon_r^{a_r}$ for some integers $j, a_1, \ldots, a_r$, we have

$$D(v) = a^{2j} z^j u_1^{a_1} \ldots u_r^{a_r}. \tag{6.13}$$

By Type II with $u = \xi^{rt}$ and $x = \xi^{-2rt} = \xi^t$,

$$b_t = E(\xi^t)E(0). \tag{6.14}$$

By Type I with $y = -x$ and (6.12),

$$E(x)E(0)E(-x)E(0) = 1$$

and hence,

$$b_t^{-1} = E(-\xi^t)E(0). \tag{6.15}$$

If $x' = \sum_{t=0}^{2r} m_t \xi^t$ in R with m_t integers, then, by Type I,

$$E(x') = \left[\prod_{t=0}^{2r} (E(\xi^t)E(0))^{m_t^+} (E(-\xi^t)E(0))^{m_t^-}\right] E(0)^{-1} D(-1)^{m-1}$$

where $m = \sum_{t=0}^{2r} m_t$, and $m_t = m_t^+ - m_t^-$ with m_t^+, m_t^- nonnegative integers. Combining (6.12), (6.14) and (6.15) with the equation above, we deduce that

$$E(x') = \left(\prod_{t=0}^{2r} b_t^{m_t}\right) a^{2m-3}. \tag{6.16}$$

We remark that a permutation of the ξ^t-terms in x' corresponds to a permutation of the b_t-factors in $E(x')$. Any ring element $x \in R$ can be written in the form $x = x'v^{-2}$ for some $x' \in \mathbb{Z}[\xi]$ and $v \in GL_1$. By Type II, we have

$$E(x) = D(v)E(x')D(v) \tag{6.17}$$

with $D(v)$ given by (6.13) and $E(x')$ by (6.16), concluding the proof. $\square$

Lemma 6.4. *The relations (6.4)–(6.8) are necessary.*

Proof. We have the following list of short arguments:
(6.4) follows from Type IV.
(6.5) follows from (6.12).
(6.6) follows from Type II with $x = 0$.
(6.7) follows from (6.16) with $x' = \xi^t + \xi^s = \xi^s + \xi^t$ in R.
(6.8) *the first part* follows from Type III with $u = 1$ and (6.12).
(6.8) *the second part* follows by (6.16) with $x' = \sum_{j=0}^{2r} \xi^j = 0$ in R.

Lemma 6.5 ([18]). *In R we have $\ell = (-1)^r \lambda^2$ where $\lambda := \xi^c \epsilon_1 \epsilon_2 \dots \epsilon_r$.*

Lemma 6.6. *(6.9) is necessary.*

Proof. By Lemma 6.5 we can apply (6.17) to

$$x = \ell\xi^t, \qquad x' = (-1)^r \xi^t, \qquad v = \lambda^{-1}$$

and get

$$E(\ell\xi^t) = D(\lambda)^{-1} E((-1)^r \xi^t) D(\lambda)^{-1}.$$

By (6.13) and (6.16), the equation above can be rewritten as

$$b_t^{\ell} a^{4r-1} = w^{-1} b_t^{(-1)^r} a^{-1} w^{-1}.$$

Now we can use (6.5) and (6.6) proven in Lemma 6.4. $\square$

Lemma 6.7. *(6.10) is necessary.*

Proof. For $I \subset \{1, 2, \dots, r\}$ recall that $\epsilon_I := \prod_{i \in I} \epsilon_i$. Then, the Definition 6.1 gives by (6.16) with $x' = \epsilon_I$ the following formula

$$c(I) = D(-1)E(\epsilon_I)D(\epsilon_I).$$

By (6.17) with $x = \epsilon_I^{-1}$ and $x' = v = \epsilon_I$ we have

$$E(\epsilon_I^{-1}) = D(\epsilon_I)E(\epsilon_I)D(\epsilon_I).$$

By Type III with $u = \epsilon_I$, we have

$$D(-1)E(\epsilon_I)E(\epsilon_I^{-1})E(\epsilon_I)D(\epsilon_I) = 1.$$

The conclusion follows by combining the three equations above. □

Lemma 6.8. (6.11) *is necessary.*

Proof. We start with $\epsilon_i^2 = \xi^i(\xi^{-i} - 2 + \xi^i)$ in R and by Type II with $u = \epsilon_i^{-1}$, $x = \epsilon_i^2$ and (6.17) with

$$x' = \xi^{-i} - 2 + \xi^i, \qquad v = \xi^{ri}, \qquad x = \xi^i x'$$

we get

$$u_i^{-1} b u_i^{-1} = z^{ri} b_{-i} b_0^{-1} b_0^{-1} b_i a^{-3} z^{ri}.$$

The desired relation now follows by (6.3) and (6.5). □

Lemma 6.9. *The relations* (6.4)–(6.11) *are sufficient to verify that* (1) *the relation* (6.13) *is well defined for* $v \in GL_1$, (2) *Type* IV *holds true, and* (3) $a^2 = D(-1)$ *is central.*

The proof is immediate by (6.4), (6.5) and (6.8) *the first part*. In what follows we will use this lemma tacitly.

Lemma 6.10. *The relations* (6.4)–(6.11) *are sufficient to verify that* (6.16) *is well defined for* $x' \in \mathbb{Z}[\xi]$.

Proof. Let $x' = \sum_{t=0}^{2r} m_t \xi^t = \sum_{t=0}^{2r} n_t \xi^t$ in R with $m_t,\ n_t \in \mathbb{Z}$. Then $m_t - n_t = j$ is independent of t. From (6.7) and (6.8) *the second part* we deduce that the right hand side of (6.16) remains unchanged under the transformation $m_t = n_t + j$ or a permutation of the b_t-factors. □

Lemma 6.11. *The relations* (6.4)–(6.11) *are sufficient for Type* I *with* $x, y \in \mathbb{Z}[\xi]$.

Proof. Let $x = \sum_{t=0}^{2r} m_t \xi^t$ and $y = \sum_{t=0}^{2r} n_t \xi^t$ with m_t, n_t integers. By Lemma 6.10 we can choose $x + y = \sum_{t=0}^{2r}(m_t + n_t)\xi^t$ and Type I follows from (6.16) and (6.7).

Lemma 6.12. *The relations* (6.4)–(6.11) *are sufficient to verify that* (6.17) *is well defined for* $x = x'v^{-2}$ *with* $x' \in \mathbb{Z}[\xi]$ *and* $v \in GL_1$.

Proof. It suffices to prove that the following statement

$P(x', v)$: *If* $y' := x'v^{-2} \in \mathbb{Z}[\xi]$ *then* $D(v)E(x')D(v) = E(y')$ *is a consequence of the relations* (6.4)–(6.11).

is true for all $x' \in \mathbb{Z}[\xi]$ and $v \in GL_1$ where $E(x')$, $E(y')$, and $D(v)$ are given by (6.16) and (6.13). By Lemmas 6.10 and 6.9, these formulas are independent of the way x', y', and v are presented. Also, we recall that $b_t = b_s$ if $t \equiv s \bmod \ell$.

$P(\pm\xi^t, -\xi)$ *is true*. If $x' = \xi^t$ and $y' = \xi^{t-2}$, we check that

$$z b_t a^{-1} z = b_{t-2} a^{-1}$$

holds true by definitions. The case $x' = -\xi^t$ is similar.

$P(\pm\xi^t, \epsilon_i^{-1})$ *is true*. If $x' = \xi^t$ and $y' = \xi^t - 2\xi^{t+i} + \xi^{t+2i}$, we use (6.6) to reduce the equation

$$u_i^{-1} b_t a^{-1} u_i^{-1} = b_s b_{t+i}^{-2} b_{t+2i} a^{-3}$$

to (6.11) as in the proof of Lemma 6.8. The case $x' = -\xi^t$ is similar.

$P(\pm\ell\xi^t, \lambda)$ *is true*. Here λ is defined in Lemma 6.5 such that

$$y' = x'\ell\lambda^{-2} = \pm(-1)^r \xi^t.$$

The statement now follows from (6.6) and (6.9).

By (6.6) and Lemma 6.11, $P(x_1', v)$ and $P(x_2', v)$ imply $P(x_1' + x_2', v)$. So, $P(x', -\xi)$, $P(x', \epsilon_i^{-1})$, and $P(\ell x', \lambda)$ are true for all $x' \in \mathbb{Z}[\xi]$ and $i = 1, 2, \ldots, r$. If $v_1^{-1}, v_2^{-1} \in \mathbb{Z}[\xi]$ such that $P(x', v_1)$ and $P(x', v_2)$ are true for all $x' \in \mathbb{Z}[\xi]$, then $P(x', v_1 v_2)$ is also true for all $x' \in \mathbb{Z}[\xi]$. Since (4.5) is a generating set for GL_1 it follows that $P(x', v)$ is true for all $x' \in \mathbb{Z}[\xi]$ and all $v \in GL_1$ such that $v^{-1} \in \mathbb{Z}[\xi]$. The proof can now be concluded by the observation that every element of GL_1 is of the form $v\lambda^s$ with $v^{-1} \in \mathbb{Z}[\xi]$ and s a nonnegative integer. □

Lemma 6.13. *The relations* (6.4)–(6.11) *are sufficient for Type* I, *Type* II, *and Type* III.

Proof. Type I: Given two ring elements $x, y \in R$ there exists $v \in GL_1$ such that $x = x'v^{-2}$ and $y = y'v^{-2}$ with $x', y' \in \mathbb{Z}[\xi]$. By (6.6) and Lemma 6.12 we get

$$E(x)E(0)E(y) = D(v)E(x')E(0)E(y')D(v).$$

So Type I is reduced to Lemma 6.11.

Type II follows from Lemma 6.12.

Type III: By Lemmas 6.11 and 6.12 we can reverse the proof of Lemma 6.7 to conclude that Type III with $u = \epsilon_I = \prod_{i\in I} \epsilon_i$ follows from (6.10) for $I \subset \{1, 2, \ldots, r\}$ non-empty and from (6.8) *the first part* if I is empty i.e. $u = 1$. Combining this with Type II, we deduce that Type III holds with $u = \epsilon_I v^2$ for any $v \in GL_1$ and any subset I. Moreover, the Type I implies

$$E(-u) = E(0)^{-1}E(u)^{-1}E(0)^{-1}$$

and hence, if Type III holds for $u \in GL_1$ then it holds for $-u$ as well. Since $\pm\epsilon_I$'s form a set of coset representatives for GL_1 modulo the squares, Type III holds in general. □

7. Hopf's formula calculations

There is a group homomorphism $\pi : SE_2 \to SL_2$ given by

$$D(u) \mapsto \begin{pmatrix} u^{-1} & 0 \\ 0 & u \end{pmatrix}, \qquad E(x) \mapsto \begin{pmatrix} x & 1 \\ -1 & 0 \end{pmatrix}$$

for all $u \in GL_1$ and $x \in R$. Regarding $D : GL_1 \to SE_2$ as a group homomorphism, we have the following commutative diagram

$$\begin{tikzcd} H_p(GL_1; \mathbb{F}_\ell) \arrow[r, "D_*"] \arrow[d, "\tau_*"] & H_p(SE_2; \mathbb{F}_\ell) \arrow[d, equal] \\ H_p(SL_2; \mathbb{F}_\ell) & H_p(SE_2; \mathbb{F}_\ell) \arrow[l, "\pi_*"'] \end{tikzcd} \tag{7.1}$$

where p is a positive integer and τ_* is induced by (5.1). Chasing this diagram, by Proposition 5.3 and Definition 4.3 we deduce that

Proposition 7.1. *The Conjecture 5.1 is true if for each subset $\{e_1, \ldots, e_i\}$ of $\{z, u_1, \ldots, u_r\}$ with $2 \le i \le r + 1$ elements and for each pair (s, j) of nonnegative integers with $i = s + 2j$ and $j > 0$, the standard cycle*

$$[z]^{(s)} \wedge \langle e_1, \ldots, e_i\rangle \tag{7.2}$$

represents the zero class in $H_p(SE_2; \mathbb{F}_\ell)$ where $z, u_1, \ldots, u_r$ are elements of SE_2 defined by (6.2) and $p = 3s + 2j$.

According to this proposition, for each prime $\ell = 2r+1$, Conjecture 5.1 follows from a verification that a certain *finite* set of explicitly given cycles (7.2) represent the zero class in $H_*(SE_2; \mathbb{F}_\ell)$. In particular, this set of cycles in $H_2(SE_2; \mathbb{F}_\ell)$ is given by $\langle e_1, e_2\rangle$ for e_1, e_2 in $\{z, u_1, \ldots, u_r\}$. Theorem 6.2 gives a short exact sequence

$$1 \to K \to F \to SE_2 \to 1$$

where F is the free group generated by z, u_i, a, b, b_t, and w for $1 \le i \le r$ and $0 \le t \le 2r$, and $K \subset F$ is the normal subgroup generated by the relators associated with the relations (6.3) and (6.4)–(6.11). Associated with this free presentation, Hopf's formula [4, p. 42] identifies

$$H_2(SE_2; \mathbb{Z}) \approx \frac{K \cap [F, F]}{[F, K]} \tag{7.3}$$

such that the standard cycle $\langle e_1, e_2\rangle$ with integer coefficients corresponds to the commutator $[e_1, e_2] \bmod [F, K]$. Here $[X, Y]$ denotes the group generated by the commutators $[x, y]$ with $x \in X$ and $y \in Y$.

Lemma 7.2. *SE_2 is a perfect group.*

Proof. For $x, y \in SE_2$, let $x \equiv y$ mean that xy^{-1} is a product of commutators in SE_2. By (6.4) and (6.6) we deduce that $z^\ell \equiv z^2 \equiv 1$ and hence, $z \equiv 1$ since ℓ is odd. Now (6.3) implies $b_t \equiv ba$ for all t. Combining this with (6.6) and (6.11), we get $ba^3 \equiv u_i^2 \equiv 1$. Since $a^4 \equiv 1$ by (6.5), we conclude that $b \equiv a$, and since $b^3 \equiv a^2$ by (6.8) we conclude that $b \equiv a \equiv 1$. Finally, from (6.10) with $I = \{i\}$ *singleton* (see Definition 6.1) we get $u_i^3 \equiv 1$ and since $u_i^2 \equiv 1$ we deduce that $u_i \equiv 1$ for all $i = 1, 2, \ldots, r$. Thus, all generators of SE_2 are $\equiv 1$. □

By Lemma 7.2 and the universal coefficients, from (7.3) we have

$$H_2(SE_2; \mathbb{F}_\ell) \approx \frac{(K \cap [F, F])K^\ell}{[F, K]K^\ell} \tag{7.4}$$

where K^ℓ is the normal subgroup generated by the relators that are the ℓ-th powers of the relators generating K. With these preparations, the following result is evidence for the Conjecture 5.1:

Proposition 7.3. *If $\ell \in \{3, 5\}$, then $[e_1, e_2] \in [F, K]K^\ell$ for all e_1, e_2 in $\{z, u_1, \dots, u_r\}$.*

The proof of this proposition is given next based on GAP [9]. We remark that the case $\ell = 3$ is known [1] but the proof given here and the case $\ell = 5$ are new.

The Case $\ell = 3$. The free group F is given by

```
F:=FreeGroup(8)
z:=F.1; u1:=F.2; a:=F.3; b:=F.4;
b0:=F.5; b1:=F.6; b2:=F.7; w:=F.8;
```

The relators generating K are given in Theorem 6.2 for $\ell = 3$ by the list

```
k:=[b0^-1*b*a,
b1^-1*z*b*z*a,
b2^-1*z^2*b*z^2*a,
w^-1*z*u1,
z^3, z*u1*z^-1*u1^-1,
a^4, a^2*z*a^-2*z^-1,
a^2*u1*a^-2*u1^-1,
z*a*z*a^-1,
u1*a*u1*a^-1,
b0*b1*b0^-1*b1^-1,
b0*b2*b0^-1*b2^-1,
b1*b2*b1^-1*b2^-1,
b^3*a^-2, b0*b1*b2*a^-2,
b0^-3*w^-1*b0^-1*w,
b1^-3*w^-1*b1^-1*w,
b2^-3*w^-1*b2^-1*w,
(b0*b1^-1*a^-1*u1)^3,
a^2*b^-1*u1*b*z^2*b^-1*b0^-1*z*b*z^2*u1];
```

The relators generating K^3 are given by the list

```
k3:=List(k,x->x^3);
```

The relators generating $[F, K]$ are given by the following algorithm

```
c:=function(i,j) return Comm(i,j);end;;
f:=GeneratorsOfGroup(F);
fk:=ListX(f,k,c);
```

The only commutator of the form $[e_1, e_2]$ in Proposition 7.3 for $\ell = 3$ is the word “k[6]”, i.e. the sixth on the list “k”. To check that “k[6]” belongs to $[F, K]K^3$ we use the following algorithm:

```
H:=F/Concatenation(fk,k3);
RequirePackage("kbmag");
RH:=KBMAGRewritingSystem(H);
OR:=OptionsRecordOfKBMAGRewritingSystem(RH);
OR.maxeqns:=500000;
OR.tidyint:=1000;
OR.confnum:=100;
MakeConfluent(RH);
ReducedWord(RH,k[6]);
<identity...>
```

The Case $\ell = 5$. The free group F is given by

```
F:=FreeGroup(11);
z:=F.1; u1:=F.2; u2:=F.3; a:=F.4; b:=F.5;
b0:=F.6; b1:=F.7; b2:=F.8; b3:=F.9; b4:=F.10; w:=F.11;
```

The relators generating K are given in Theorem 6.2 for $\ell = 5$ by the list

```
k:=[b0^-1*b*a,
b1^-1*z^2*b*z^2*a,
b2^-1*z^4*b*z^4*a,
b3^-1*z*b*z*a,
b4^-1*z^3*b*z^3*a,
w^-1*z*u1*u2,
z^5, z*u1*z^-1*u1^-1,
z*u2*z^-1*u2^-1,
u1*u2*u1^-1*u2^-1,
a^4, a^2*z*a^-2*z^-1,
a^2*u1*a^-2*u1^-1,
a^2*u2*a^-2*u2^-1,
z*a*z*a^-1,
u1*a*u1*a^-1,
u2*a*u2*a^-1,
b0*b1*b0^-1*b1^-1,
b0*b2*b0^-1*b2^-1,
b0*b3*b0^-1*b3^-1,
b0*b4*b0^-1*b4^-1,
b1*b2*b1^-1*b2^-1,
b1*b3*b1^-1*b3^-1,
b1*b4*b1^-1*b4^-1,
b2*b3*b2^-1*b3^-1,
b2*b4*b2^-1*b4^-1,
b3*b4*b3^-1*b4^-1,
b^3*a^-2, b0*b1*b2*b3*b4*a^-2,
b0^-5*w^-1*b0*w,
b1^-5*w^-1*b1*w,
b2^-5*w^-1*b2*w,
b3^-5*w^-1*b3*w,
b4^-5*w^-1*b4*w,
(b0*b1^-1*a^-1*u1)^3,
(b0*b2^-1*a^-1*u2)^3,
(b0*b1^-1*b2^-1*b3*a^-1*u1*u2)^3,
a^2*b^-1*u1*b*z^3*b^-1*b0^-1*z^2*b*z^4*u1,
a^2*b^-1*u2*b*z*b^-1*b0^-1*z^4*b*z^3*u2];
```

The relators generating K^5 are given by the list

```
k5:=List(k,x->x^5);
```

The relators generating $[F, K]$ are given by a list "fk" via the same algorithm as for the case $\ell = 3$ but applied to the new "f" and "k". The only commutators of the form $[e_1, e_2]$ in Proposition 7.3 are the words "k[8]", "k[9]", and "k[10]" but the algorithm used in the case $\ell = 3$ is inconclusive in the case $\ell = 5$ due to its increased complexity. For this reason, we show that these words belong to $[F, K]K^5$ by proving the following

Lemma 7.4. $[F, F] \cap K \subset [F, K]K^5$.

Proof. By trial and error we find a sublist "e⊂ k" of 11 elements

```
e:=k{[5,6,15,16,17,30,31,32,33,34,37]};
```

such that the complementary sublist "n⊂ k"

```
n:=k{[1,2,3,4,7,8,9,10,11,12,13,14,18,19,20,
21,22,23,24,25,26,27,28,29,35,36,39,38]};
```

consists of elements vanishing "mod e" i.e. represent zero in the group

```
t:=F/Concatenation(fk,k5,e);
```

according to the following algorithm:

```
RequirePackage("kbmag");
Rt:=KBMAGRewritingSystem(t);;
OR:=OptionsRecordOfKBMAGRewritingSystem(Rt);
OR.maxeqns:=500000;
OR.tidyint:=1000;
OR.confnum:=100;
MakeConfluent(Rt);
nt:=List([1..Length(n)],i->ReducedWord(Rt,n[i]));
<identity...>
```

This means that the group $K/[F, K]K^5$ is generated by the elements in "e". The commutator group $[F, F]$ is generated by the list of relators

```
ff:=ListX(f,f,\<,c);
```

The "reduced" group $F/[F, F]K^5$ is given by

```
h:=F/Concatenation(ff,k5);
```

Observe that "h" is a vector space of dimension 11 over $\mathbb{F}_5$ by using

```
typeh:=AbelianInvariants(h);
```

Moreover the elements in the list "e" generate "h" since $s = 1$ where

```
s:=Size(F/Concatenation(ff,k5,e));
```

Putting these facts together and using formula (7.4) we conclude that there is a short exact sequence

$$0 \to H_2(SE_2; \mathbb{F}_5) \to \frac{K}{[F, K]K^5} \to \frac{F}{[F, F]K^5} \to 0$$

where the last term is a vector space of dimension 11 while the middle term is a vector space of dimension at most 11 being generated by the elements in the list "e". So that $H_2(SE_2; \mathbb{F}_5) = 0$. □

By [6, p. 7], the canonical homomorphism $\pi : SE_2 \to SL_2$ is a group isomorphism if the ring R is Euclidean and by [13] the ring R is indeed Euclidean for $\ell = 5$. Hence, we deduce the following

Corollary 7.5. $H_2(SL_2; \mathbb{F}_5) = 0$.

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