

# Large deviations for stochastic generalized porous media equations<sup>☆</sup>

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## Abstract

The large deviation principle is established for the distributions of a class of generalized stochastic porous media equations for both small noise and short time.

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## 1. Introduction and main results

We first recall the existence and uniqueness results on strong solutions to the stochastic generalized porous media equations obtained recently in [9]. Let  $(E, \mathcal{M}, \mathbf{m})$  be a separable probability space and  $(L, \mathcal{D}(L))$  a negative definite self-adjoint linear operator on  $L^2(\mathbf{m})$  with spectrum contained in  $(-\infty, -\lambda_0]$  for some  $\lambda_0 > 0$ .

We assume that, for a fixed number  $r > 1$ ,  $L^{-1}$  is bounded in  $L^{r+1}(\mathbf{m})$ , which is e.g. the case if  $L$  is a Dirichlet operator (cf. e.g. [15]) since in this case the interpolation theorem or simply Jensen's inequality implies  $\|e^{tL}\|_{r+1} \leq e^{-\lambda_0 t^2/(r+1)}$  for all  $t \geq 0$ , where and in what follows,  $\|\cdot\|_p$  denotes the norm in  $L^p(\mathbf{m})$  for  $p \geq 1$ . A classical example of  $L$  is the Laplace operator on a smooth bounded domain in a complete Riemannian manifold with Dirichlet boundary condition.

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Let  $H^1 := \mathcal{D}(\sqrt{-L})$  be the real Hilbert space with inner product

$$\langle f, g \rangle_{H^1} := \langle \sqrt{-L}f, \sqrt{-L}g \rangle,$$

where  $\langle \cdot, \cdot \rangle$  is the inner product in  $L^2(\mathbf{m})$ . Then the embedding  $H^1 \subset L^2(\mathbf{m})$  is dense and continuous. Let  $H := H^{-1}$  be the dual Hilbert space of  $H^1$  realized through this embedding.

The existence and uniqueness of strong solutions to the following stochastic differential equation have been proved in [9]:

$$dX_t = (L\Psi(t, X_t) + \Phi(t, X_t))dt + QdW_t, \tag{1.1}$$

where  $Q : L^2(\mathbf{m}) \rightarrow H$  is a Hilbert–Schmidt operator with  $q := q(Q)$  the square of its Hilbert–Schmidt norm,  $W_t$  is a cylindrical Brownian motion on  $L^2(\mathbf{m})$  w.r.t. a complete filtered probability space  $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ ,

$$\Psi, \Phi : [0, \infty) \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}$$

are progressively measurable functions, i.e. for any  $t \geq 0$ , restricted on  $[0, t] \times \mathbb{R} \times \Omega$  they are measurable w.r.t.  $\mathcal{B}([0, t]) \times \mathcal{B}(\mathbb{R}) \times \mathcal{F}_t$ , and for any  $(t, \omega) \in [0, \infty) \times \Omega$ ,  $\Psi(t, \cdot)(\omega)$  and  $\Phi(t, \cdot)(\omega)$  are continuous on  $\mathbb{R}$  and satisfy certain monotonicity conditions. See [1,2] for an account of the classical (deterministic) porous media equations and [3,4,7,8] for the study of weak solutions and invariant measures for some stochastic generalized porous media equations.

To explain what is meant by strong solutions to (1.1), let us introduce the embeddings

$$V \subset H \subset V^*$$

as follows. Consider the reflexive separable Banach space  $V := L^{r+1}(\mathbf{m})$ . Then we can obtain a presentation of its dual space  $V^*$  through the embeddings  $V \subset H \equiv H' \subset V^*$ , where  $H$  is identified with its dual through the Riesz-isomorphism. In other words  $V^*$  is just the completion of  $H$  with respect to the norm

$$\|f\|_{V^*} := \sup_{\|g\|_{r+1} \leq 1} \langle f, g \rangle_H, \quad f \in H.$$

Since  $H$  is separable, so is  $V^*$ . We note that this is different from the usual representation of  $V = L^{r+1}(\mathbf{m})$  through the embedding

$$V \subset L^2(\mathbf{m}) \equiv L^2(\mathbf{m})',$$

which, of course, gives  $L^{(r+1)/r}(\mathbf{m})$  as dual. But it is easy to identify the isomorphism between  $L^{(r+1)/r}(\mathbf{m})$  and  $V^*$ . Below we simply use  $\langle \cdot, \cdot \rangle_H$  to denote  $v^*(\cdot)_V$ , i.e. the duality between  $V$  and  $V^*$ , since  $v^*(\cdot)_V = \langle \cdot, \cdot \rangle_H$  holds on  $H \times V$ . It is explained in [9] that  $L : L^{(r+1)/r}(\mathbf{m}) \rightarrow V^*$  is a densely defined bounded operator, so that it extends uniquely to a fully defined bounded operator, denoted once again by  $L$ . Likewise, the natural embedding  $L^2(\mathbf{m}) \subset H \subset V^*$  extends uniquely to a one-to-one map from  $L^{(r+1)/r}(\mathbf{m})$  to  $V^*$  (cf. [9, Corollary 1.2]). Since  $\Psi(t, v)(\omega), \Phi(t, v)(\omega) \in L^{(r+1)/r}(\mathbf{m})$ , by condition (1.2) below, the map  $b := L\Psi + \Phi : [0, \infty) \times V \times \Omega \rightarrow V^*$  is well defined.

We assume that there exist two constants  $c, \alpha > 0$  such that (where the derivatives are with respect to the second variable not  $t$ )

$$\begin{aligned} |\Psi'(t, \cdot)| + |\Phi'(t, \cdot)| &\leq c(1 + |\cdot|^{r-1}), & |\Phi(t, 0)| + |\Psi(t, 0)| &\leq c, & t \in [0, T], \\ \langle u - v, b(t, u) - b(t, v) \rangle_H &\leq -\alpha\|u - v\|_{r+1}^{r+1} + c\|u - v\|_H^2, & t \in [0, T], u, v \in V \end{aligned} \tag{1.2}$$

holds on  $\Omega$ . In particular, according to [9], the last inequality in (1.2) holds for some  $\alpha, c > 0$  if there exist constants  $\theta_1 > \theta_2/\|L^{-1}\|_{r+1} \geq 0$  and  $\sigma \in \mathbb{R}$  such that

$$\begin{aligned} (a_1 - a_2)(\Psi(\cdot, a_1) - \Psi(\cdot, a_2)) &\geq \theta_1|a_1 - a_2|^{r+1}, \\ |\Phi(\cdot, a_1) - \Phi(\cdot, a_2)| &\leq \theta_2|a_1 - a_2|^r + \sigma|a_1 - a_2|, \quad a_1, a_2 \in \mathbb{R} \end{aligned} \tag{1.3}$$

holds on  $[0, T] \times \Omega$ . A simple example for  $\Psi$  and  $\Phi$  satisfying (1.2) (or (1.3)) is given by

$$\Psi(t, a) := f(t)\text{sgn}(a)|a|^r + \eta_1(t), \quad \Phi(t, a) := g(t)a + \eta_2(t), \quad t \in [0, T], a \in \mathbb{R}$$

for some strictly positive function  $f \in C([0, T])$  and some bounded adapted processes  $g, \eta_1$  and  $\eta_2$ .

According to [9] (see also [14, Theorems II.2.1 and II.2.2] for more general situations), condition (1.2) implies that Eq. (1.1) has a unique strong solution; that is, there is a unique  $H$ -valued continuous  $(\mathcal{F}_t)$ -adapted process  $X_t$  with  $X \in L^{r+1}([0, T] \times \Omega \times E, dt \times P \times \mathbf{m})$  such that for any  $e \in L^{r+1}(\mathbf{m})$ ,

$$\begin{aligned} \langle X_t, e \rangle_H &= \langle X_0, e \rangle_H - \int_0^t \mathbf{m}(\Psi(s, X_s)e + \Phi(s, X_s)L^{-1}e)ds + \langle QW_t, e \rangle_H, \\ t &\in [0, T]. \end{aligned} \tag{1.4}$$

To see that the solution defined above satisfies the equation

$$X_t = x + \int_0^t (L\Psi + \Phi)(s, X_s)ds + QW_t, \quad t \in [0, T] \tag{1.5}$$

in  $H$ , we first observe that by (1.2), the right hand side of (1.5) exists in  $V^*$  for any  $t > 0$  since  $X \in L^{r+1}([0, T] \times \Omega \times E, dt \times P \times \mathbf{m})$ . Since both  $X_t - x$  and  $QW_t$  take values in  $H$ , (1.5) does indeed hold in  $H$ .

**Remark 1.1.** In order to obtain the large deviation principle, our assumptions are indeed stronger than those used in [14] to prove existence and uniqueness of strong solutions. On the other hand, in [9] a direct proof is presented for existence, uniqueness and ergodicity of strong solutions for (1.1) under the extra assumption that the spectrum of  $L$  is discrete. Since this assumption was not really used in the proofs, it can be dropped from that paper. Furthermore, in the recent work [16], the existence and uniqueness of strong solutions have been obtained for a much more general framework so that one may take Orlicz norms instead of  $L^{r+1}(\mathbf{m})$  in applications. Our arguments for the large deviation principle presented below are, however, difficult to extend to the general situation of [16].

In this paper we study the large deviation property of the above stochastic generalized porous medium equation for both small noise and short time. Recall [11] that a sequence of probability measures  $(\mu_\varepsilon)_{\varepsilon>0}$  on some Polish space  $E$  satisfies, as  $\varepsilon \rightarrow 0$ , the *large deviation principle (LDP for short) with speed  $\lambda(\varepsilon) \rightarrow +\infty$  (as  $\varepsilon \rightarrow 0$ ) and rate function  $I : E \rightarrow [0, +\infty]$* , if  $I$  is a good rate function, i.e., the level sets  $\{I \leq r\}, r \in \mathbb{R}^+$  are compact, and for any Borel subset  $A$  of  $E$ ,

$$-\inf_{x \in A^\circ} I(x) \leq \liminf_{\varepsilon \rightarrow 0} \frac{1}{\lambda(\varepsilon)} \log \mu_\varepsilon(A) \leq \limsup_{\varepsilon \rightarrow 0} \frac{1}{\lambda(\varepsilon)} \log \mu_\varepsilon(A) \leq -\inf_{x \in \bar{A}} I(x),$$

where  $A^\circ$  and  $\bar{A}$  are respectively the closure and the interior of  $A$  in  $E$ . In that case we shall simply say that  $(\mu_\varepsilon)$  satisfies the *LDP*( $\lambda(\varepsilon), I$ ) on  $E$ , or even more simply write

$(\mu_\varepsilon) \in LDP(\lambda(\varepsilon), I)$  on  $E$ . We say that the family of  $E$ -valued random variables  $X^\varepsilon$  satisfies the  $LDP(\lambda(\varepsilon), I)$  if the family of their laws does.

Let us first consider the following stochastic differential equation with small noise:

$$dX_t^\varepsilon = (L\Psi(t, X_t^\varepsilon) + \Phi(t, X_t^\varepsilon))dt + \varepsilon QdW_t, \quad \varepsilon > 0, X_0^\varepsilon = x \in H. \tag{1.6}$$

From now on, let  $T > 0$  and  $x \in H$  be fixed. To state our main results, let us first introduce the skeleton equation associated with (1.6):

$$\frac{dz_t^\phi}{dt} = L\Psi(t, z_t^\phi) + \Phi(t, z_t^\phi) + \phi_t, \quad z_0^\phi := x, \tag{1.7}$$

where  $\phi \in L^2([0, T]; H)$ . An element  $z^\phi \in C([0, T]; H) \cap L^{r+1}([0, T] \times E, dt \times \mathbf{m})$  is called a solution to (1.7) if for any  $e \in L^{r+1}(\mathbf{m})$ ,

$$\langle z_t^\phi, e \rangle_H = \langle x, e \rangle_H - \int_0^t \{ \langle L^{-1}e, \phi_s + \Phi(s, z_s^\phi) \rangle + \langle e, \Psi(s, z_s^\phi) \rangle \} ds, \quad t \in [0, T]. \tag{1.8}$$

We shall prove that (1.2) and (1.3) imply the existence and the uniqueness of the solution to (1.7) for any  $\phi \in L^2([0, T]; H)$ , and thus, as explained above for the solution to (1.1), the solution satisfies the corresponding integral equation of (1.7) in  $H$ .

Now, we introduce the rate function. For any  $\phi \in L^2([0, T] \times E, dt \times \mathbf{m})$ , let  $\|\phi\|_{L^2}^2 := \int_0^T dt \int_E \phi_t^2 d\mathbf{m}$ . Define

$$I(z) := \frac{1}{2} \inf \{ \|\phi\|_{L^2}^2 : z = z^{Q\phi}, \phi \in L^2([0, T] \times E, dt \times \mathbf{m}) \}, \quad z \in C([0, T]; H), \tag{1.9}$$

where we set  $\inf \emptyset = \infty$  by convention. The following result is for a Freidlin–Wentzell type estimate:

**Theorem 1.1.** *Assume (1.2). For each  $\varepsilon > 0$ , let  $X^\varepsilon = (X_t^\varepsilon)_{t \in [0, T]}$  be the solution to (1.6). Then as  $\varepsilon \rightarrow 0$ ,  $(X^\varepsilon)$  satisfies the  $LDP(\varepsilon^{-2}, I)$  on  $C([0, T]; H)$  (equipped with the sup-norm topology), where the rate function  $I$  is given by (1.9).*

Next, we consider the LDP of the solution  $X_t$  to (1.1) for short time, which in the classical finite dimensional case is the famous Varadhan large deviation estimate. Since  $\tilde{X}_t^\varepsilon := X_{\varepsilon^2 t}$  solves the equation

$$d\tilde{X}_t^\varepsilon = \varepsilon^2(L\Psi(\varepsilon^2 t, \tilde{X}_t^\varepsilon) + \Phi(\varepsilon^2 t, \tilde{X}_t^\varepsilon))dt + \varepsilon Qd\tilde{W}_t, \quad \varepsilon > 0, \tilde{X}_0^\varepsilon = x, \tag{1.10}$$

where  $(\tilde{W}_t := (1/\varepsilon)W_{\varepsilon^2 t})$  is a BM of the same law as  $(W_t)$ , it suffices to establish the LDP for the law of  $\tilde{X}^\varepsilon$ .

**Theorem 1.2.** *Assume (1.2). If  $x \in L^{r+1}(\mathbf{m})$  then  $\tilde{X}^\varepsilon = (X_{\varepsilon^2 t})$  satisfies the  $LDP(\varepsilon^{-2}, \tilde{I})$  where*

$$\tilde{I}(z) := \frac{1}{2} \inf \left\{ \|\phi\|_{L^2}^2 : z_t = x + Q \int_0^t \phi_s ds \right\}, \quad z \in C([0, T]; H).$$

Let us make some historical comments. In the finite dimensional case, under the Lipschitzian condition, the LDP of  $X_{\varepsilon^2}$  is the famous Varadhan estimate [17], and Theorem 1.1 is the well known Freidlin–Wentzell LDP [12]. For the extensions to infinite dimensional diffusions or

stochastic PDE under global Lipschitz condition on the nonlinear term, we refer the reader to Da Prato and Zabczyk [10] (also for the literature up to 1992). For the case of local Lipschitz conditions we refer the reader to [6] where also multiplicative and degenerate noise is handled. Unlike in our situation, in [6] the drift still contains a nontrivial (therefore smoothing) linear part. In many examples of SPDE, however, (local) Lipschitz conditions are rarely satisfied (such as the porous equation in this work). Without Lipschitz conditions, each type of stochastic nonlinear PDE requires specific techniques and adapted estimates. So the situation becomes much more dispersive. Here we mention only the work of Cardon-Weber [5] on the LDP for stochastic Burgers equations with small noise and the important work of Hino and Ramírez [13] for the Varadhan small time estimate of large deviations for general symmetric Markov processes, where the reader may also find other recent references.

Here are some remarks on Theorem 1.2 related with the general work of Hino and Ramírez [13]: (1) As our process  $(X_t)$  is highly non-symmetric, the result in [13] cannot be applied. (2) The extra condition on  $x \in L^{r+1}(\mathbf{m})$  (not all  $x \in H$ ) in Theorem 1.2 is also a quite general phenomenon in infinite dimension because the result of [13] holds only for  $\mu$ -a.e.  $x$  where  $\mu$  is the invariant measure, and in our case, the invariant measure is supported by  $L^{r+1}(\mathbf{m})$  [9]. (3) Furthermore the LDP in Theorem 1.2 is pathwise, unlike that in [13] which is only for the marginal law. (4) Our proof for this particular class of diffusions will be easier than the proof of the general result in [13] for two reasons: (a) our starting point is Schilder’s theorem for general Gaussian processes. This powerful tool cannot be used in the general framework of [13]; (b) unlike in [13], due to our strong enough dissipativity conditions, the Itô formula is available and yields sharp exponential estimates for the solution.

This paper is organized as follows. The next section is devoted to the study of the skeleton process  $z^\phi$ , which is crucial for identifying the rate function of our LDP. In Section 3 we give an a priori exponential estimate and recall the generalized contraction principle. The proof of Theorem 1.1 is presented in Section 4, and our strategy is based on two procedures of approximation: first for finite dimensional noise (i.e., only a finite number of directions are stochastically perturbed) we approximate the path of  $QW$  as piecewise linear; second, we approximate the whole noise  $QW$  by the finite dimensional noises. This strategy can be easily adapted for the proof of Theorem 1.2 in Section 5.

## 2. The skeleton process

**Proposition 2.1.** *Assume (1.2). Let  $\|z\| := \sup_{t \in [0, T]} \|z_t\|_H$  for  $z \in C([0, T]; H)$ . For any  $x \in H$  and any  $\phi \in L^2([0, T]; H)$  there exists a unique solution  $z^\phi$  to (1.7) and*

$$\int_0^T \mathbf{m}(|z_t^\phi - z_t^\psi|^{r+1}) dt \leq C \int_0^T \|\phi_t - \psi_t\|_H^2 dt, \tag{2.1}$$

$$\|z^\phi - z^\psi\| \leq C \int_0^T \|\phi_t - \psi_t\|_H dt \tag{2.2}$$

hold for some constant  $C > 0$  and all  $x \in H, \phi, \psi \in L^2([0, T]; H)$ .

**Proof.** To verify the existence of the solution, we make use of [14, Theorem II.2.1]. Let  $V := L^{r+1}(\mathbf{m})$  and  $V^*$  the duality of  $V$  w.r.t.  $H$ , and let  $B := 0$  and

$$A(s, v) := L \Psi(s, v) + \Phi(s, v) + \phi_s.$$

Then, due to (1.2), it is trivial to verify assumptions  $(A_i)$  ( $i = 1, \dots, 5$ ) on page 1252 of [14] for some  $K, \alpha > 0, p := r + 1, q := \frac{r+1}{r}$ , and  $f(t) := c(1 + \|\phi_t\|_H^q)$  for some constant  $c > 0$ . Then, by [14, Theorems II.2.1 and II.2.2] (see also [19, Theorem 30.A]), (1.7) has a unique solution. Let  $z^\phi$  be the unique solution to (1.7) for  $\phi \in L^2([0, T]; H)$ .

By Itô’s formula, due to [14, Theorem I.3.2] and (1.2), we have

$$\begin{aligned} \frac{d}{dt} \|z_t^\phi - z_t^\psi\|_H^2 &= -2\langle z_t^\phi - z_t^\psi, \Psi(t, z_t^\phi) - \Psi(t, z_t^\psi) \rangle \\ &\quad - 2\langle L^{-1}(z_t^\phi - z_t^\psi), \Phi(t, z_t^\phi) - \Phi(t, z_t^\psi) + \phi_t - \psi_t \rangle \\ &\leq -2\alpha \mathbf{m}(|z_t^\phi - z_t^\psi|^{r+1}) + 2c \|z_t^\phi - z_t^\psi\|_H^2 + 2\|z_t^\phi - z_t^\psi\|_H \|\phi_t - \psi_t\|_H. \end{aligned} \tag{2.3}$$

Since (2.3) implies, for any  $\varepsilon > 0$ , that

$$\frac{d}{dt} (\varepsilon + \|z_t^\phi - z_t^\psi\|_H^2)^{1/2} \leq \|\phi_t - \psi_t\|_H + c \|z_t^\phi - z_t^\psi\|_H,$$

by Gronwall’s lemma we have

$$e^{-cT} \sqrt{\varepsilon + \|z^\phi - z^\psi\|^2} \leq \varepsilon + \int_0^T \|\phi_t - \psi_t\|_H dt.$$

This implies (2.2) for  $C := e^{cT}$  on letting  $\varepsilon \rightarrow 0$ . Finally, (2.1) follows by combining (2.2) with (2.3).  $\square$

### 3. Exponential estimates and a generalized contraction principle

The following a priori estimate will be crucial for the proof of Theorem 1.1.

**Lemma 3.1.** *Assume (1.2). Then for any  $\gamma > 0, q_0 > 0$  and  $\varepsilon_0 > 0$  there exists a constant  $c > 0$  such that for all  $Q$  with  $q(Q) \leq q_0$  and all  $\varepsilon \in (0, \varepsilon_0)$ ,*

$$\mathbb{E} \exp \left( \gamma \varepsilon^{-2} \int_0^T \|X_t^\varepsilon\|_{r+1}^{r+1} dt \right) \leq e^{c\varepsilon^{-2}}. \tag{3.1}$$

Throughout this paper we adopt the following notation: for two continuous real semimartingales  $(x_t)$  and  $(y_t)$ ,  $dx_t \leq dy_t$  means that their martingale parts are the same and  $x_t - x_s \leq y_t - y_s$  for all  $t > s \geq 0$ .

**Proof.** By the second inequality in (1.2) and noting that  $r > 1$ , we have

$$|\langle u, b(t, 0) \rangle_H| \leq c \|u\|_2 + c \|L^{-1}\|_{r+1} \|u\|_2 \leq c' \|u\|_{r+1}, \quad t \in [0, T], u \in V$$

for some constant  $c' > 0$ . Combining this with the last inequality in (1.2) we obtain

$$2\langle u, b(t, u) \rangle_H \leq -\alpha \|u\|_{r+1}^{r+1} + c_0, \quad t \in [0, T], u \in V$$

for some constant  $c_0 > 0$ . Thus, using Itô’s formula due to [14, Theorem I.3.2], there exist constants  $c_1, c_2 > 0$  such that

$$\begin{aligned} d\|X_t^\varepsilon\|_H^2 &\leq 2\langle X_t^\varepsilon, b(t, X_t^\varepsilon) \rangle_H dt + 2\varepsilon \langle X_t^\varepsilon, QdW_t \rangle_H + q\varepsilon^2 dt \\ &\leq -c_1 \mathbf{m}(|X_t^\varepsilon|^{r+1}) dt + c_2 dt + 2\varepsilon \langle X_t^\varepsilon, QdW_t \rangle_H, \quad \varepsilon \in (0, \varepsilon_0). \end{aligned}$$

Then

$$\|X_T^\varepsilon\|_H^2 - \|x\|_H^2 + c_1 \int_0^T \mathbf{m}(|X_t^\varepsilon|^{r+1}) dt \leq 2\varepsilon \int_0^T \langle X_t^\varepsilon, QdW_t \rangle_H + c_2 T. \tag{3.2}$$

Letting  $dM_t := \langle X_t^\varepsilon, QdW_t \rangle_H$  (with  $M_0 = 0$ ), since  $\forall \lambda \in \mathbb{R}$ ,  $\xi_t := \exp(\lambda M_t - \frac{\lambda^2}{2} \langle M \rangle_t)$  is a martingale and the quadratic variational process  $\langle M \rangle_t$  satisfies  $d\langle M \rangle_t \leq q_0 \|X_t^\varepsilon\|_H^2 dt$ , we obtain from (3.2) that, for  $\lambda := 8\gamma/c_1\varepsilon$ ,

$$\begin{aligned} \mathbb{E} \exp \left( \gamma \varepsilon^{-2} \int_0^T \|X_t^\varepsilon\|_{r+1}^{r+1} dt \right) &= \mathbb{E} \exp \left( 2\gamma \varepsilon^{-2} \int_0^T \|X_t^\varepsilon\|_{r+1}^{r+1} dt - \gamma \varepsilon^{-2} \int_0^T \|X_t^\varepsilon\|_{r+1}^{r+1} dt \right) \\ &\leq \mathbb{E} \exp \left( \frac{4\gamma}{c_1\varepsilon} M_T + \frac{2c_2\gamma}{c_1\varepsilon^2} T + \frac{2\gamma \|x\|_H^2}{c_1\varepsilon^2} - \gamma \varepsilon^2 \int_0^T \|X_t^\varepsilon\|_{r+1}^{r+1} dt \right) \\ &\leq \mathbb{E} \exp \left( \frac{\lambda}{2} M_T - \frac{\lambda^2}{4} \langle M \rangle_T + \int_0^T \left( \frac{q_0\lambda^2}{4} \|X_t^\varepsilon\|_H^2 - \gamma \varepsilon^{-2} \|X_t^\varepsilon\|_{r+1}^{r+1} \right) dt \right. \\ &\quad \left. + \frac{2c_2\gamma}{c_1\varepsilon^2} T + \frac{2\gamma \|x\|_H^2}{c_1\varepsilon^2} \right) \\ &\leq \{\mathbb{E} \xi_T\}^{1/2} \left\{ \mathbb{E} \exp \left( \int_0^T \left( \frac{q_0\lambda^2}{2} \|X_t^\varepsilon\|_H^2 - 2\gamma \varepsilon^{-2} \|X_t^\varepsilon\|_{r+1}^{r+1} \right) dt \right. \right. \\ &\quad \left. \left. + \frac{4c_2\gamma}{c_1\varepsilon^2} T + \frac{4\gamma \|x\|_H^2}{c_1\varepsilon^2} \right) \right\}^{1/2} \\ &\leq \exp(c\varepsilon^{-2}T) \end{aligned}$$

for some constant  $c > 0$  and all  $\varepsilon \in (0, \varepsilon_0)$ , where the last step is due to the martingale property of  $\xi_t$  and that  $\|\cdot\|_{r+1} \geq c\|\cdot\|_H$  for some  $c > 0$  and that  $r > 1$ .  $\square$

In large deviation theory, when  $(\mu_\varepsilon)$  satisfies the  $LDP(\lambda(\varepsilon), I)$  on a Polish space  $\mathbf{E}$  and if  $f : \mathbf{E} \rightarrow \mathbf{F}$  is continuous where  $\mathbf{F}$  is another Polish space, then  $(\mu_\varepsilon \circ f^{-1}) \in LDP(\lambda(\varepsilon), I_f)$ , where

$$I_f(z) := \inf_{f^{-1}(z)} I, \quad z \in \mathbf{F}.$$

That is the so called contraction principle. The following generalization is taken from [18] (some preceding weaker versions can be found in [11, Theorems 4.2.16 and 4.2.23]).

**Theorem 3.2 (Generalized Contraction Principle).** *Let  $\mathbf{E}, \mathbf{F}$  be two Polish spaces and  $(\mu_\varepsilon)$  a family of probability measures on  $E$ . If  $(\mu_\varepsilon) \in LDP(\lambda(\varepsilon), I)$  and there exists a sequence of continuous mappings  $f^N : \mathbf{E} \rightarrow \mathbf{F}$  such that*

$$\lim_{N \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \frac{1}{\lambda(\varepsilon)} \log \mu_\varepsilon \left( \rho_F \left( f^N, f \right) > \delta \right) = -\infty, \quad \delta > 0, \tag{3.3}$$

where  $\rho_F$  is some compatible metric on  $\mathbf{F}$  and  $f : \mathbf{E} \rightarrow \mathbf{F}$  is a measurable mapping, then there exists a function  $\tilde{f} : \{I < +\infty\} \rightarrow \mathbf{F}$  which is continuous on every level set  $[I \leq L]$  ( $\forall L \geq 0$ )

such that

$$\lim_{N \rightarrow \infty} \sup_{I \leq r} \rho_F(f^N, \tilde{f}) = 0, \quad r > 0; \tag{3.4}$$

and  $(\mu_\varepsilon(f \in \cdot)) \in LDP(\lambda(\varepsilon), I_{\tilde{f}})$ , where

$$I_{\tilde{f}}(z) := \inf_{\tilde{f}^{-1}(z)} I, \quad z \in \mathbf{F}. \tag{3.5}$$

**Proof.** The proof, taken from [18], is very easy. First  $\mu_\varepsilon(\rho_F(f^{N_1}, f^{N_2}) \in \cdot)$  satisfies the LDP by the contraction principle. Then by (3.3) we have for any  $L > 0$  and  $\delta > 0$ , there is some  $N \in \mathbb{N}$  such that for all  $N_1, N_2 \geq N$ ,

$$-\inf\{I(x) : \rho_F(f^{N_1}(x), f^{N_2}(x)) > \delta\} < -L.$$

Therefore,  $[I \leq L] \subset [\rho_F(f^{N_1}, f^{N_2}) \leq \delta]$ . This proves (3.4) and the proof is then finished by [11, Theorem 4.2.23].  $\square$

#### 4. Proof of Theorem 1.1

We shall prove Theorem 1.1 by two procedures of approximation. Let  $\{e_i : i \geq 1\}$  be dense in  $L^{r+1}$  and hence, also dense in  $H$ . For any fixed  $n \geq 1$ , let  $H_n := \text{span}\{e_i : 1 \leq i \leq n\}$  and  $P_n : H \rightarrow H_n$  be the orthogonal projection. Let  $X_t^{\varepsilon,n}$  be the solution of

$$dX_t^{\varepsilon,n} = (L\Psi + \Phi)(t, X_t^{\varepsilon,n})dt + \varepsilon P_n Q dW_t, \quad X_0^{\varepsilon,n} = x. \tag{4.1}$$

Next for each  $N \in \mathbb{N}$  and for any path  $w \in C([0, T]; H)$ , let  $t_i := iT/N$  for  $0 \leq i \leq N$  and define the ( $N$ -times) piecewise linear approximation of  $w$  by

$$w_t^{(N)} := \frac{N}{T} \sum_{i=0}^{N-1} 1_{(t_i, t_{i+1}]}(t)((t - t_i)w_{t_{i+1}} + (t_{i+1} - t)w_{t_i}), \quad t \in [0, T].$$

By Proposition 2.1, the following equation has a unique solution  $X_{t,N}^{\varepsilon,n}$  in  $H$ :

$$\dot{X}_{t,N}^{\varepsilon,n} := \frac{dX_{t,N}^{\varepsilon,n}}{dt} = L\Psi(t, X_{t,N}^{\varepsilon,n}) + \Phi(t, X_{t,N}^{\varepsilon,n}) + \varepsilon \frac{d}{dt}(P_n Q W)_t^{(N)}, \quad X_{0,N}^{\varepsilon,n} = x. \tag{4.2}$$

We claim that it is enough to establish

$$\limsup_{N \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \varepsilon^2 \log \mathbb{P}(\|X^{\varepsilon,n} - X_{\cdot,N}^{\varepsilon,n}\| > \delta) = -\infty, \quad \forall \delta > 0 \tag{4.3}$$

and

$$\limsup_{n \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \varepsilon^2 \log \mathbb{P}(\|X^{\varepsilon,n} - X^\varepsilon\| > \delta) = -\infty, \quad \forall \delta > 0. \tag{4.4}$$

In fact, by Schilder’s theorem, the law of  $\varepsilon QW$  satisfies the LDP on  $C([0, T]; H)$  with speed  $\lambda(\varepsilon) = \varepsilon^{-2}$  and with rate function given by

$$J(\tilde{\phi}) = \inf \left\{ \frac{1}{2} \|\phi\|_{L^2}^2 : \tilde{Q}\phi = \tilde{\phi}(\cdot) \right\}$$



where  $\tilde{Q} : \phi \mapsto \int_0^t Q\phi(s)ds$  is a continuous linear mapping from  $L^2([0, T] \times E, dt \times \mathbf{m})$  to  $C([0, T]; H)$  (with the convention that  $\inf \emptyset := +\infty$ ). Next, let  $f_{n,N}$  denote the map which associates each path  $\omega \in C([0, T]; H)$  of  $\varepsilon QW$  with the solution  $X_{t,N}^{\varepsilon,n}$  of (4.2), i.e.,  $\gamma := f_{n,N}(\omega)$  is the unique solution of

$$\gamma_t = x + \int_0^t [L\Psi(s, \gamma_s) + \Phi(s, \gamma_s)]ds + (P_n w)_t^{(N)},$$

where  $(P_n w)_t^{(N)}$  is the ( $N$ -times) piecewise linear approximation of  $P_n w$ . Applying Proposition 2.1 with  $Q$  replaced by  $P_n Q$  and noting that  $(P_n w)_t^{(N)} = P_n(P_n w)_t^{(N)}$  and that all norms on  $H_n$  are equivalent, we see that  $f_{n,N} : C([0, T]; H) \rightarrow C([0, T]; H)$  is continuous. Furthermore, by (4.3) and (4.4), for each  $n$ , there is some  $N(n)$  such that

$$\limsup_{n \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \varepsilon^2 \log \mathbb{P}(\|X^\varepsilon - f_n(\varepsilon QW)\| > \delta) = -\infty, \quad \forall \delta > 0$$

where  $f_n := f_{n,N(n)}$ . Hence by Theorem 3.2,  $X^\varepsilon$  satisfies the LDP on  $C([0, T]; H)$  with rate function given by

$$I(z) = \inf\{J(\tilde{\phi}) : \tilde{f}(\tilde{\phi}) = z\} = \inf\left\{\frac{1}{2}\|\phi\|_{L^2}^2 : \tilde{f}(\tilde{Q}\phi) = z\right\}$$

and  $\tilde{f}(\tilde{Q}\phi) = \lim_{n \rightarrow \infty} f_{n,N(n)}(\tilde{Q}\phi)$  by (3.4). But by Proposition 2.1 and the following Lemma 4.1,  $f_{n,N}(\tilde{Q}\phi) \rightarrow z^{Q\phi}$  as  $n, N$  goes to infinity. Thus  $\tilde{f}(\tilde{Q}\phi) = z^{Q\phi}$ , which yields the claimed rate function.

**Lemma 4.1.** For any  $\phi \in L^2([0, T] \times E, dt \times \mathbf{m})$ , let  $h_t := \int_0^t Q\phi_s ds, t \in [0, T]$ . For any sequence  $N(n) \rightarrow \infty$  as  $n \rightarrow \infty$  we have

$$\lim_{n \rightarrow \infty} \int_0^T \left\| \frac{d}{dt} (P_n h)_t^{N(n)} - Q\phi_t \right\|_H^2 dt = 0.$$

**Proof.** Let  $t_i := Ti/N(n)$ . Since

$$\frac{d}{dt} h^{(N(n))} = \sum_{i=1}^{N(n)} 1_{[t_{i-1}, t_i)} \frac{N(n)}{T} \int_{t_{i-1}}^{t_i} Q\phi_s ds$$

which converges to  $Q\phi$  in  $L^2([0, T]; H)$  as  $n \rightarrow \infty$ , it suffices to prove that

$$I_n := \int_0^T \left\| \frac{d}{dt} (P_n h)_t^{(N(n))} - \frac{d}{dt} h_t^{(N(n))} \right\|_H^2 dt \rightarrow 0 \tag{4.5}$$

as  $n \rightarrow \infty$ . Note that for any  $\psi \in L^2([0, T]; H)$  we have

$$\begin{aligned} \int_0^T \left\| \frac{d}{dt} \left( \int_0^t \psi_s ds \right)^{(N(n))} \right\|_H^2 dt &= \sum_{i=1}^{N(n)} \frac{T}{N(n)} \left\| \frac{N(n)}{T} \int_{t_{i-1}}^{t_i} \psi_t dt \right\|_H^2 \\ &\leq \sum_{i=1}^{N(n)} \int_{t_{i-1}}^{t_i} \|\psi_t\|_H^2 dt = \int_0^T \|\psi_t\|_H^2 dt. \end{aligned}$$

Then

$$\lim_{n \rightarrow \infty} I_n \leq \lim_{n \rightarrow \infty} \int_0^T \|P_n Q\phi_t - Q\phi_t\|_H^2 dt = 0. \quad \square$$

So, to finish the proof of **Theorem 1.1**, we have to prove (4.3) and (4.4) which will be done in the following two subsections.

4.1. Proof of (4.3)

Let  $b := L\psi + \Phi$ , and

$$\hat{X}_t := X_t^{\varepsilon,n} - X_{t,N}^{\varepsilon,n}, \quad \hat{\gamma}_t := \varepsilon P_n Q(W_t - W_t^{(N)}).$$

By (4.1) and (4.2) we have

$$\begin{aligned} \frac{d\|\hat{X}_t - \hat{\gamma}_t\|_H^2}{dt} &= 2\langle \hat{X}_t - \hat{\gamma}_t, b(t, X_t^{\varepsilon,n} - \hat{\gamma}_t) - b(t, X_{t,N}^{\varepsilon,n}) \rangle_H \\ &\quad + 2\langle \hat{X}_t - \hat{\gamma}_t, b(t, X_t^{\varepsilon,n}) - b(t, X_t^{\varepsilon,n} - \hat{\gamma}_t) \rangle_H. \end{aligned}$$

Combining this with the last inequality in (1.2) and using Young’s inequality  $xy \leq x^{r+1}/(r+1) + [r/(r+1)]y^{(r+1)/r}$ ,  $\forall x, y \geq 0$ , we conclude that there exist  $\lambda, c > 0$  and  $c(\lambda) > 0$  such that

$$\begin{aligned} \frac{d\|\hat{X}_t - \hat{\gamma}_t\|_H^2 e^{-ct}}{dt} &\leq -\lambda \|\hat{X}_t - \hat{\gamma}_t\|_{r+1}^{r+1} + 2\|\hat{X}_t - \hat{\gamma}_t\|_{r+1} \|L^{-1}(b(t, X_t^{\varepsilon,n}) - b(t, X_t^{\varepsilon,n} - \hat{\gamma}_t))\|_{(r+1)/r} \\ &\leq c(\lambda) \|L^{-1}(b(t, X_t^{\varepsilon,n}) - b(t, X_t^{\varepsilon,n} - \hat{\gamma}_t))\|_{(r+1)/r}^{(r+1)/r}. \end{aligned} \tag{4.6}$$

Since  $|\psi'(s)| + |\Phi'(s)| \leq c(1 + |s|^{r-1})$  and  $L^{-1}$  is bounded in  $L^{(r+1)/r}(\mathbf{m})$ , there exist  $c_1, c_2 > 0$  such that

$$\begin{aligned} &\|L^{-1}(b(t, X_t^{\varepsilon,n}) - b(t, X_t^{\varepsilon,n} - \hat{\gamma}_t))\|_{(r+1)/r}^{(r+1)/r} \\ &\leq c_1 \|\hat{\gamma}_t\| (1 + |\hat{\gamma}_t|^{r-1} + |X_t^{\varepsilon,n}|^{r-1}) \|X_t^{\varepsilon,n}\|_{(r+1)/r}^{(r+1)/r} \\ &\leq c_2 \int_E (|\hat{\gamma}_t|^{r+1} + |\hat{\gamma}_t|^{(r+1)/r} + |\hat{\gamma}_t|^{(r+1)/r} |X_t^{\varepsilon,n}|^{(r^2-1)/r}) d\mathbf{m}. \end{aligned} \tag{4.7}$$

From (4.6) and (4.7) and Young’s inequality we obtain that for each  $R > 1$ ,

$$\begin{aligned} \frac{d\|\hat{X}_t - \hat{\gamma}_t\|_H^2 e^{-ct}}{dt} &\leq c(\lambda)c_2 \left\{ (1 + R) \|\hat{\gamma}_t\|_{r+1}^{r+1} + \|\hat{\gamma}_t\|_{(r+1)/r}^{(r+1)/r} \right. \\ &\quad \left. + C(r)R^{-1/(r-1)} \|X_t^{\varepsilon,n}\|_{r+1}^{r+1} \right\} \end{aligned} \tag{4.8}$$

for some  $C(r) > 0$ . Since all  $L^p$ -norms ( $1 \leq p \leq r+1$ ) on  $H_n$  are equivalent, for any norm  $\|\cdot\|_p$  on  $H_n$ , by the LDP of  $\varepsilon P_n QW_t$  on  $C([0, T]; H_n)$ , whose good rate function is denoted by  $I_n$ , and recalling that  $\hat{\gamma}_t := \varepsilon P_n[(QW)_t - (QW)_t^{(N)}]$ , we have

$$\limsup_{N \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \varepsilon^2 \log \mathbb{P} \left( \sup_{t \in [0, T]} \|\hat{\gamma}_t\|_p > \delta \right)$$

$$\begin{aligned} &\leq \limsup_{N \rightarrow \infty} - \inf \left\{ I_n(w) : w \in C([0, T]; H_n), \sup_{t \in [0, T]} \|w_t - w_{t,N}\|_p \geq \delta \right\} \\ &= -\infty, \quad \forall \delta > 0, \end{aligned}$$

where the equality follows from the fact that  $\inf_{F_N} I_n \rightarrow +\infty$  ( $N \rightarrow \infty$ ) for any sequence of closed subsets decreasing to  $\emptyset$  (an elementary property of a good rate function).

Combining this with (4.8), we see that for any  $\delta \in (0, 1)$ , there exists  $c_3 > 0$  such that the l.h.s. of (4.3) is less than

$$\limsup_{n \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \varepsilon^2 \log \mathbb{P} \left( c_3 R^{-1/(r-1)} \int_0^T \|X_t^{\varepsilon,n}\|_{r+1}^{r+1} dt > \delta \right)$$

which goes to  $-\infty$  when  $R \rightarrow +\infty$  by Chebyshev’s inequality and the a priori exponential estimate in Lemma 3.1.

#### 4.2. Proof of (4.4)

By (1.2) and using Itô’s formula in [14, Theorem I.3.2], we have

$$d\|X_t^\varepsilon - X_t^{\varepsilon,n}\|_H^2 \leq (a_n \varepsilon^2 + c\|X_t^\varepsilon - X_t^{\varepsilon,n}\|_H^2)dt + 2\varepsilon dM_t^{(n)},$$

where  $c > 0$  is a constant,  $a_n := q(P_n Q - Q)$  is the square of the Hilbert–Schmidt norm of  $P_n Q - Q$  from  $L^2(\mathbf{m})$  to  $H$ , and  $dM_t^{(n)} := \langle X_t^\varepsilon - X_t^{\varepsilon,n}, (I - P_n)QdW_t \rangle_H$ . The quadratic variation process of the local martingale  $M^{(n)}$  verifies

$$d\langle M^{(n)} \rangle_t \leq \|X_t^\varepsilon - X_t^{\varepsilon,n}\|_H^2 a_n dt.$$

For any constant  $\alpha > 0$ , let  $\xi_t := \exp[\alpha \varepsilon^{-2} \|X_t^\varepsilon - X_t^{\varepsilon,n}\|_H^2 e^{-(1+c)t}] =: \exp[\alpha \varepsilon^{-2} Y_t]$ . We have by Itô’s formula in [14, Theorem I.3.2] that

$$\begin{aligned} d\xi_t &\leq 2\alpha \varepsilon^{-1} e^{-(1+c)t} \xi_t dM_t^{(n)} \\ &\quad + \varepsilon^{-2} \alpha e^{-(1+c)t} \xi_t \{a_n \varepsilon^2 - \|X_t^\varepsilon - X_t^{\varepsilon,n}\|_H^2 + 2\alpha \varepsilon^{-t} a_n \|X_t^\varepsilon - X_t^{\varepsilon,n}\|_H^2\} dt \\ &\leq 2\alpha \varepsilon^{-1} e^{-(1+c)t} \xi_t dM_t^{(n)} + \alpha a_n \xi_t dt, \end{aligned}$$

once  $1 \geq 2\alpha a_n$  which holds for all sufficiently large  $n$  for  $a_n \rightarrow 0$  as  $n \rightarrow \infty$ . So  $N_t := \xi_t \exp[-\alpha a_n t]$  is a supermartingale. Therefore, for all  $n$  large enough,

$$\begin{aligned} \mathbb{P}(\|X^\varepsilon - X^{\varepsilon,n}\| > \delta) &\leq \mathbb{P} \left( \sup_{t \in [0, T]} N_t > \exp[\delta^2 \alpha \varepsilon^{-2} e^{-(1+c)T} - \alpha a_n T] \right) \\ &\leq \exp[-\alpha \delta^2 \varepsilon^{-2} e^{-(1+c)T} + \alpha a_n T]. \end{aligned}$$

This implies (4.4) since  $\alpha > 0$  was arbitrary.

### 5. Proof of Theorem 1.2

**Proof of Theorem 1.2.** (a) Let  $q_{ij} := \langle Qe_j, e_i \rangle$ ,  $i, j \geq 1$ . We first assume that there exists  $n \in \mathbb{N}$  such that  $q_{ij} = 0$  for  $i > n$ . In this case the law of  $\varepsilon QW_t + x$  satisfies the large deviation

principle with the given rate function of compact level sets. Thus by the approximation lemma in large deviations (see [11, Theorem 4.2.13]), it suffices to show that

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon^2 \log \mathbb{P}(\|\tilde{X}^\varepsilon - x - \varepsilon Q \tilde{W}\| > \delta) = -\infty, \quad \delta > 0. \tag{5.1}$$

By (1.2), there exist  $\lambda, c, c_0 > 0$  such that

$$\begin{aligned} \frac{d\|\tilde{X}_t^\varepsilon - \varepsilon Q \tilde{W}_t - x\|_H^2 e^{-ct}}{dt} &\leq -\lambda \varepsilon^2 \|\tilde{X}_t^\varepsilon - \varepsilon Q \tilde{W}_t - x\|_{r+1}^{r+1} \\ &\quad + 2\varepsilon^2 e^{-ct} \langle \tilde{X}_t^\varepsilon - \varepsilon Q \tilde{W}_t - x, (L\Psi + \Phi)(\varepsilon Q \tilde{W}_t + x) \rangle_H \\ &\leq c_0 \varepsilon^2 \|(\Psi + L^{-1}\Phi)(x + \varepsilon Q \tilde{W}_t)\|_{(r+1)/r}^{(r+1)/r}, \quad t \in [0, T]. \end{aligned}$$

Since  $L^{-1}$  is bounded in  $L^{(r+1)/r}(E, \mathbf{m})$ ,  $|\Psi'(s)| + |\Phi'(s)| \leq c(1 + |s|^{r-1})$  for some  $c > 0$ , and  $x \in L^{r+1}$ , there exists  $c_1 > 0$  such that

$$\frac{d\|\tilde{X}_t^\varepsilon - \varepsilon Q \tilde{W}_t - x\|_H^2 e^{-ct}}{dt} \leq c_1 \varepsilon^2 (\|\varepsilon Q \tilde{W}_t\|_{r+1}^{r+1} + 1).$$

This immediately implies (5.1) by the LDP of  $\varepsilon Q \tilde{W}$  in  $C([0, T]; H_n)$ . Note that on  $H_n$  the norms  $\|\cdot\|_H$  and  $\|\cdot\|_{r+1}$  are equivalent.

(b) In general, for any  $n \geq 1$  let  $Q^{(n)} := P_n Q$ . By (a), the law of  $\tilde{X}_t^{\varepsilon, n}$ , the solution to (1.10) for  $Q^{(n)}$  in place of  $Q$ , satisfies the LDP with the good rate function

$$\tilde{I}_n(z) := \frac{1}{2} \inf \left\{ \|\phi\|_2^2 : z_t = x + \int_0^t Q^{(n)} \phi_s ds \right\}, \quad z \in C([0, T]; H).$$

Similarly to in the proof of (4.3) in Section 4.2 we have

$$\limsup_{n \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \varepsilon^2 \log \mathbb{P}(\|\tilde{X}^{\varepsilon, n} - \tilde{X}^\varepsilon\| > \delta) = -\infty, \quad \delta > 0.$$

Moreover, since  $\delta_n := \|L^{-1/2}(Q - Q^{(n)})\|_{2 \rightarrow 2} \rightarrow 0$  as  $n \rightarrow \infty$  and since

$$\int_0^T \|(Q - Q^{(n)})\phi_t\|_H dt \leq a_n \int_0^T \|\phi_t\|_2 dt \leq a_n \sqrt{T} \|\phi\|_2,$$

we conclude that the law of  $\tilde{X}^\varepsilon$  satisfies the LDP with the claimed rate function  $I$  by the approximation lemma (see [11, Theorem 4.2.13]).  $\square$

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