A characterization of some \(\{3v_\mu + 1, 3v_\mu; k - 1, q\}\)-minihypers and some \([n, k, q^{k-1} - 3q^\mu; q]\)-codes \((k \geq 3, q \geq 5, 1 \leq \mu < k - 1)\) meeting the Griesmer bound

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Abstract

For any \([n, k, d; q]\)-code the Griesmer bound says that \(n \geq \sum_{i=0}^{k-1} \lceil \frac{d}{q^i} \rceil\). The purpose of this paper is to characterize all \([n, k, q^{k-1} - 3q^\mu; q]\)-codes meeting the Griesmer bound in the case where \(k \geq 3, q \geq 5\) and \(1 \leq \mu < k - 1\). It is shown that all such codes have a generator matrix whose columns correspond to all points in \(\text{PG}(k - 1, q)\) except for the points in a disjoint union of three \(\mu\)-flats in \(\text{PG}(k - 1, q)\).

1. Introduction

Let \(V(n, q)\) be an \(n\)-dimensional vector space consisting of row vectors over the Galois field \(\text{GF}(q)\) where \(n \geq 3\) and \(q\) is a prime power. If \(C\) is a \(k\)-dimensional subspace in \(V(n, q)\) such that every nonzero vector in \(C\) has a Hamming weight of at least \(d\), then \(C\) is called an \([n, k, d; q]\)-code. Let \(n_q(k, d)\) denote the smallest value of \(n\) for which there exists an \([n, k, d; q]\)-code. An \([n_q(k, d), k, d; q]\)-code is therefore optimal in the sense that no shorter code exists with the same \(k, d\) and \(q\). It is well known (cf. [1, 12]) that if there exists an \([n, k, d; q]\)-code, then

\[n \geq \sum_{i=0}^{k-1} \left\lceil \frac{d}{q^i} \right\rceil,\]

where \(\lceil x \rceil\) denotes the smallest integer \(\geq x\). This bound (called the Griesmer bound) shows that if an \([n, k, d; q]\)-code meeting the Griesmer bound exists, then this code is optimal. Hence, we shall consider the following problem.
Problem 1.1. (1) Find a necessary and sufficient condition for integers \( k, d, q \) such that there exists an \([n, k, d; q]\)-code meeting the Griesmer bound.

(2) Characterize up to equivalence (cf. Definition A.1) all \([n, k, d; q]\)-codes meeting the Griesmer bound for given values of \( k, d, q \) when such \([n, k, d; q]\)-codes exist.

In the case \( q = 2 \), and \( 1 \leq d \leq 2^{k-1} \), Problem 1.1 was solved completely by Helleseth [11]. Hence, we restrict ourselves to the case \( q \geq 3 \), \( k \geq 3 \), and \( 1 \leq d < q^{k-1} \). In this case \( d \) can be expressed uniquely as follows:

\[
d = q^{k-1} - \sum_{i=1}^{h} q^{\lambda_i}
\]

(1.2)

using some integers \( k, q, h \) and \( \lambda_i \) \((i = 1, 2, \ldots, h)\) such that \((a) 1 \leq h \leq (k - 1)(q - 1)\), \(0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_h < k - 1\) and \((b)\) at most \( q - 1 \) of the \( \lambda_i \)'s take the same value and the Griesmer bound (1.1) can be expressed as follows:

\[
n > \nu_k - \sum_{i=1}^{h} \nu_{\lambda_i+1},
\]

(1.3)

where \( \nu_l = (q^{l} - 1)/(q - 1) \) for any integer \( l \geq 0 \).

Let \( \text{PG}(t, q) \) be a finite projective geometry of dimension \( t \) over \( \text{GF}(q) \). An \( r \)-flat is a subspace of projective dimension \( r \) of \( \text{PG}(t, q) \).

Definition 1.1. Let \( F \) be a set of \( f \) points in \( \text{PG}(t, q) \) where \( t \geq 2 \) and \( f \geq 1 \). If \( |F \cap H| \geq m \) for every hyperplane \( H \) in \( \text{PG}(t, q) \) and \( |F \cap H| = m \) for some hyperplane \( H \) in \( \text{PG}(t, q) \), then \( F \) is called an \( \{f, m; t, q\}\)-minihyper where \( m \geq 0 \) and \( |A| \) denotes the number of elements in the set \( A \). In the special case \( t = 2 \) and \( m \geq 2 \), an \( \{f, m; 2, q\}\)-minihyper \( F \) is also called an \( m \)-blocking set if \( F \) contains no 1-flat in \( \text{PG}(2, q) \).

Hamada [3] showed that in the case \( d = q^{k-1} - \sum_{i=1}^{h} q^{\lambda_i} \), there is a one-to-one correspondence between the set of all nonequivalent \([n, k, d; q]\)-codes meeting the Griesmer bound and the set of all \( \{\sum_{i=1}^{h} \nu_{\lambda_i+1}, \sum_{i=1}^{h} \nu_{\lambda_i}; k - 1, q\}\)-minihypers (cf. Corollary A.1). Hence, in order to solve Problem 1.1, it is sufficient to solve the following problem.

Problem 1.2. (1) Find a necessary and sufficient condition for integers \( t, q, h \) and \( \lambda_i \) \((i = 1, 2, \ldots, h)\) such that there exists a \( \{\sum_{i=1}^{h} \nu_{\lambda_i+1}, \sum_{i=1}^{h} \nu_{\lambda_i}; t, q\}\)-minihyper where \( t \geq 2, q \geq 3, 1 \leq h \leq t(q - 1), 0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_h < t \) and at most \( q - 1 \) of the \( \lambda_i \)'s take the same value.

(2) Characterize all \( \{\sum_{i=1}^{h} \nu_{\lambda_i+1}, \sum_{i=1}^{h} \nu_{\lambda_i}; t, q\}\)-minihypers when such minihypers exist.
Definition 1.2. Let $\mathcal{F}(\lambda_1, \lambda_2, \ldots, \lambda_h; t, q)$ denote the family of all unions $U_{i=1}^h V_i$ of a $\lambda_1$-flat $V_1$, a $\lambda_2$-flat $V_2$, ..., a $\lambda_h$-flat $V_h$ in $PG(t, q)$ which are mutually disjoint, where $1 \leq h \leq t(q-1)$, $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_h < t$ and at most $q-1$ of the $\lambda_i$'s take the same value.

Remark 1.1. (1) It is known (cf. [10]) that in the case $h \geq 2$, $\mathcal{F}(\lambda_1, \lambda_2, \ldots, \lambda_h; t, q) \neq \emptyset$ if and only if $t \geq \lambda_{h-1} + \lambda_h + 1$.

(2) It is known (cf. [3]) that if $F \in \mathcal{F}(\lambda_1, \lambda_2, \ldots, \lambda_h; t, q)$ in the case $h \geq 2$ and $t \geq \lambda_{h-1} + \lambda_h + 1$, then $F$ is a $\{v_{\lambda_1+1}, v_{\lambda_2}; t, q\}$-minihyper.

For the case $h = 1$, Problem 1.2 was solved by Tamari [13, 14]. For the case $h = 2$, Problem 1.2 was solved by Hamada [2, 5]. For the case $h = 3$, Problem 1.2 was solved by Hamada et al. [2-9] except for the case $1 \leq \lambda_1 = \lambda_2 < \lambda_3 < t$ and $q \geq 5$. The purpose of this paper is to give a solution of Problem 1.2 for this remaining case, i.e., to prove the following theorem.

Theorem 1.1. Let $t$, $\mu$ and $q$ be any integers such that $1 < \mu < t$ and $q \geq 5$.

1. There exists a $\{3v_{\mu+1}, 3v_{\mu}; t, q\}$-minihyper if and only if $t \geq 2\mu + 1$.

2. In the case $t \geq 2\mu + 1$, $F$ is a $\{3v_{\mu+1}, 3v_{\mu}; t, q\}$-minihyper if and only if $F \in \mathcal{F}(\mu, \mu; t, q)$, i.e., $F$ is a union of three $\mu$-flats in $PG(t, q)$ which are mutually disjoint.

From Theorem 1.1 and Corollary A.1, we have the following corollary.

Corollary 1.1. Let $k$, $\mu$ and $q$ be any integers such that $1 \leq \mu < k - 1$ and $q \geq 5$.

1. There exists a $[v_{k-1} - 3v_{\mu+1}, k, q^{k-1} - 3q^\mu; q]$-code meeting the Griesmer bound (1.3) if and only if $k \geq 2\mu + 2$.

2. In the case $k \geq 2\mu + 2$, $C$ is a $[v_{k-1} - 3v_{\mu+1}, k, q^{k-1} - 3q^\mu; q]$-code meeting the Griesmer bound (1.3) if and only if $C$ has a generator matrix whose columns correspond to all points in $PG(k-1, q)$ except for the points of a disjoint union of three $\mu$-flats in $PG(k-1, q)$.

2. Preliminary results

In order to prove Theorem 1.1, we shall prepare several results in this section. Since $1 < q/(q+1-h) < 2$ in the case $h = 3$ and $q \geq 5$, we have the following lemma from Theorem A.2.

Lemma 2.1. There is no $\{3v_2, 3v_1; 2, q\}$-minihyper in the case $q \geq 5$.

Lemma 2.2. Let $F$ be any $\{3v_2, 3v_1; t, q\}$-minihyper where $t \geq 3$ and $q \geq 5$. 
(1) If $H$ is a hyperplane in $\text{PG}(t, q)$ such that $m(q + 1) \leq |F \cap H| < (m + 1)(q + 1)$ for some integer $m$ in $\{0, 1, 2, 3\}$, then $F \cap H$ is an $\{f, m; t, q\}$-minihyper in $H$ where $f = |F \cap H|$.

(2) $|F \cap H| = 3, q + 3, 2q + 3$ or $3q + 3$ for any hyperplane $H$ in $\text{PG}(t, q)$.

**Proof.** (1) Let $H$ be a hyperplane in $\text{PG}(t, q)$ such that $m(q + 1) \leq |F \cap H| < (m + 1)(q + 1)$. In the case $m = 0$, it is obvious that (1) holds.

In the case $1 \leq m \leq 3$, suppose there exists a $(t-2)$-flat $G$ in $H$ such that $|F^* \cap G| \leq m - 1$ where $F^* = F \cap H$. Let $H_i (i = 1, 2, \ldots, q)$ be $q$ hyperplanes in $\text{PG}(t, q)$, except for $H$, which contain $G$. Then we have

\[ |F| = |F \cap H| + \sum_{i=1}^{q} (|F \cap H_i| - |F^* \cap G_i|) \geq m(q + 1) + q(2 - (m - 1)) > 3(q + 1) = |F|, \]

which is a contradiction. Hence $|F^* \cap G| \geq m$ for every $(t-2)$-flat $G$ in $H$. If $|F^* \cap G| = m$ for some $(t-2)$-flat $G$ in $H$, it follows that $F^*$ is an $\{f, m; t, q\}$-minihyper in $H$.

Suppose $|F^* \cap G| \geq m + 1$ for any $(t-2)$-flat $G$ in $H$. Then there exists a subset $K$ of the set $F^*$ such that $|K \cap G| \geq m + 1$ for every $(t-2)$-flat $G$ in $H$ and $|K \cap G_0| = m + 1$ for some $(t-2)$-flat $G_0$ in $H$. Since $t \geq 3$ and $m + 1 \leq q + 1$, there exists a $t$-flat $A$ in $G_0$ such that $K \cap A = \emptyset$. Let $G_i (i = 1, 2, \ldots, q)$ be $q$ $(t-2)$-flats in the $(t-1)$-flat $H$, except for $G_0$, which contain $A$. Then

\[ |F^*| \geq |K| = \sum_{i=0}^{q} |K \cap G_i| \geq (m + 1)(q + 1) > |F^*|, \]

which is a contradiction. Hence, $|F^* \cap G| = m$ for some $(t-2)$-flat $G$ in $H$.

(2) Suppose there exists a hyperplane $H$ in $\text{PG}(t, q)$ such that $mq + 3 < |F \cap H| < (m + 1)(q + 1)$ for some integer $m$ in $\{0, 1, 2\}$. Then it follows from (1) that there exists a $(t-2)$-flat $G$ in $H$ such that $|F \cap G| = m$. Let $H_i (i = 1, 2, \ldots, q)$ be $q$ hyperplanes in $\text{PG}(t, q)$, except for $H$, which contain $G$. Then

\[ |F| = |F \cap H| + \sum_{i=1}^{q} (|F \cap H_i| - |F \cap G_i|) > 3(q + 1) = |F|, \]

which is a contradiction. Hence, $|F \cap H| = 3, q + 1, q + 2, q + 3, 2q + 2, 2q + 3$ or $3q + 3$ for any hyperplane $H$ in $\text{PG}(t, q)$.

Suppose there exists a hyperplane $H$ in $\text{PG}(t, q)$ such that $|F \cap H| = q + 1, q + 2$, or $2q + 2$.

**Case 1:** $(|F \cap H| = q + 1)$. It follows from (1), $v_1 = 1$ and $v_2 = q + 1 + 1 = F \cap H$ is a $\{v_2, v_1; t, q\}$-minihyper in $H$. Hence, it follows from Theorem A.3 ($\lambda = 1$) that $F \cap H$ is a 1-flat (denoted by $L$) in $H$. Let $G$ be any $(t-2)$-flat in $H$ such that $|G \cap L| = 1$ (i.e., $|G \cap F| = 1$) and let $H_i (i = 1, 2, \ldots, q)$ be $q$ hyperplanes in $\text{PG}(t, q)$, except for $H$,
which contain \( G \). Without loss of generality, we can assume that \(|F \cap H_1| \geq \cdot \cdots \geq |F \cap H_q|\).

Since \( \sum_{i=1}^q |F \cap (H_i \setminus G)| = |F| - |F \cap H| = 2q + 2 \) and \(|F \cap (H_i \setminus G)| = |F \cap H_i| - |F \cap G| \geq 2\) for \( i = 1, 2, \ldots, q \), it follows that \(|F \cap (H_i \setminus G)| = 3\) or \(4\). Since \(|F \cap H_i| = |F \cap G| + |F \cap (H_i \setminus G)|\) and \(|F \cap G| = 1\), this implies that \(|F \cap H_i| = 4\) or \(5\), i.e., \(3 < |F \cap H_i| < q + 1\) in the case \(q \geq 5\), which is a contradiction. Hence, there is no hyperplane \( H \) in \( \text{PG}(t, q) \) such that \(|F \cap H| = q + 1\).

Case II: \((|F \cap H| = q + 2)\). It follows from (1) and Theorem A.4 \((\lambda_1 = 0, \lambda_2 = 1)\) that \((F \cap H = L \cup \{P\})\) for some \(1\)-flat \(L\) and some point \(P\) in \(H\). Let \(G\) be any \((t-2)\)-flat in \(H\) such that \(|G \cap L| = 1\) and \(P \not\in G\) (i.e., \(|G \cap F| = 1\)) and let \(H_i\) \((i = 1, 2, \ldots, q)\) be \(q\) hyperplanes in \(\text{PG}(t, q)\), except for \(H\), which contain \(G\) where \(|F \cap H_i| \geq |F \cap H_2| \geq \cdots \geq |F \cap H_q|\). Then \(|F \cap H_1| = 4\), which is a contradiction.

Case III: \((|F \cap H| = 2q + 2)\). It follows from (1) and Theorem A.4 \((\lambda_1 = \lambda_2 = 1)\) that (a) in the case \(t = 3\) (i.e., \(t - 1 = 2\)), there is no \(2\)-flat \(H\) in \(\text{PG}(3, q)\) such that \(|F \cap H| = 2(q + 1)|\) and (b) in the case \(t \geq 4, F \cap H = L_1 \cup L_2\) for some \(1\)-flats \(L_1\) and \(L_2\) in \(H\) which are mutually disjoint.

In the case (b), let \(G\) be a \((t-2)\)-flat in \(H\) such that \(|G \cap L_1| = 1\) and \(|G \cap L_2| = 1\) (i.e., \(|G \cap F| = 2\)) and let \(H_i\) \((i = 1, 2, \ldots, q)\) be \(q\) hyperplanes in \(\text{PG}(t, q)\), except for \(H\), which contain \(G\) where \(|F \cap H_i| \geq |F \cap H_2| \geq \cdots \geq |F \cap H_q|\). Then \(|F \cap H_1| = 4\), which is a contradiction. This completes the proof.

Lemma 2.3. If \(F\) is a \(\{3v_2, 3v_1; t, q\}\)-minihyper in the case \(t \geq 3\) and \(q \geq 5\), then \(F\) is a union of three \(1\)-flats in \(\text{PG}(t, q)\) which are mutually disjoint.

Proof. Let \(F\) be any \(\{3v_2, 3v_1; t, q\}\)-minihyper. There exists a hyperplane \(H\) in \(\text{PG}(t, q)\) such that \(|F \cap H| = 3\), i.e., \(F \cap H = \{P_1, P_2, P_3\}\) for some points \(P_1, P_2\) and \(P_3\) in \(H\). Since \(q + 1 > 2\), there exists a \((t-2)\)-flat \(A_i\) in \(H\) such that \(\{P_1, P_2, P_3\} \cap A_i = \{P_i\}\) for each integer \(i\) in \(\{1, 2, 3\}\).

Let \(H_i\) \((i = 1, 2, \ldots, q)\) be \(q\) hyperplanes in \(\text{PG}(t, q)\), except for \(H\), which contain \(A_i\), where \(|F \cap H_1| \geq |F \cap H_2| \geq \cdots \geq |F \cap H_q|\). Since \(\sum_{i=1}^q |F \cap (H_i \setminus A_i)| = |F| - |F \cap H| = 3q\) and \(|F \cap (H_i \setminus A_i)| = |F \cap H_i| - |F \cap H| \geq 2\) for \(i = 1, 2, \ldots, q\), it follows that \(3 \leq |F \cap (H_i \setminus A_i)| \leq 3q - 2(q - 1) = q + 2\), i.e., \(4 \leq |F \cap H_1| = q + 3\). Hence, it follows from Lemma 2.2 and Theorem A.5 that \(|F \cap H_1| = q + 3\) and \(F \cap H_1 = L_1 \cup \{Q_1, Q_2\}\) for some \(1\)-flat \(L_1\) and some points \(Q_1\) and \(Q_2\) in \(H_1\). Since \(H \cap H_1 = A_i\) and \(L_1 \cap A_i = \{P_i\}\), this implies that there exists a \(1\)-flat \(L_1\) in \(F\) such that \(\{P_1, P_2, P_3\} \cap L_1 = \{P_i\}\).

Similarly, it can be shown that there exists a \(1\)-flat \(L_i\) in \(F\) such that \(\{P_1, P_2, P_3\} \cap L_i = \{P_i\}\) for \(i = 2, 3\). This implies that \(F = L_1 \cup L_2 \cup L_3 \cup S\) for some set \(S\) in \(\text{PG}(t, q)\) such that \(|S| = 3(q + 1) - |L_1 \cup L_2 \cup L_3|\). If \(L_1, L_2\) and \(L_3\) are mutually disjoint, then \(S = \emptyset\) and Lemma 2.3 holds.

Suppose \(L_1, L_2\) and \(L_3\) are not mutually disjoint. Without loss of generality, we can assume that \(L_1 \cap L_2 \neq \emptyset\) (i.e., \(L_1 \cap L_2 = \{Q\}\)).
Case I: \((t = 3)\). Let \(\Pi\) be the hyperplane (i.e., 2-flat) in PG(3, q) which contains \(L_1\) and \(L_2\). Then \(|F \cap \Pi| \geq 2q + 1\). Hence, it follows from Lemma 2.2 that \(|F \cap \Pi| = 2q + 3\) or \(3q + 3\).

In the case \(|F \cap \Pi| = 2q + 3\), it follows from Lemma 2.2 that \(F \cap \Pi\) is a \((2q + 3, 2; 3, q)\)-minihyper in \(\Pi\). Since \(\Pi\) is a 2-flat, this implies that there exists a \(\{v_1 + 2v_2, v_0 + 2v_1; 2, q\}\)-minihyper where \(v_0 = 0, v_1 = 1\) and \(v_2 = q + 1\). Hence, we have a contradiction from Theorem A.6.

In the case \(|F \cap \Pi| = 3q + 3\), it follows from \(|F| = 3q + 3\) that \(F \subset \Pi\). This implies that there exists a \(\{3v_2, 3v_1; 2, q\}\)-minihyper which is contradictory to Lemma 2.1. Hence, \(L_1, L_2\) and \(L_3\) must be mutually disjoint.

Case II: \((t \geq 4)\). Let \(\Pi\) be a hyperplane in PG(t, q) which contains \(L_1\) and \(L_2\). Since \(|F \cap \Pi| \geq 2q + 1\), it follows from Lemma 2.2 that \(|F \cap \Pi| = 2q + 3\) or \(3q + 3\).

In the case \(|F \cap \Pi| = 2q + 3\), it follows from Lemma 2.2 and Theorem A.6 that \(F \cap \Pi\) is a union of one point and two 1-flats in \(\Pi\) which are mutually disjoint. Since \(L_1 \cup L_2 \cap F \cap \Pi\) and \(L_1 \cap L_2 \neq \emptyset\), this is a contradiction.

In the case \(|F \cap \Pi| = 3q + 3\), it follows from \(|F| = 3q + 3\) that \(F\) is a \(\{3v_2, 3v_1; t, q\}\)-minihyper in the \((t - 1)\)-flat \(\Pi\). Since Lemma 2.3 holds in the case \(t = 3\) (cf. case I), it follows by induction on \(t\) that \(F\) is a union of three 1-flats in \(\Pi\) which are mutually disjoint. This completes the proof.

3. The proof of Theorem 1.1

In order to prove Theorem 1.1, we shall use the following lemma due to Hamada [3].

**Lemma 3.1.** Let \(G\) be a \((t - 2)\)-flat in PG(t, q) and let \(W_1, W_2\) and \(W_3\) be three \((\mu - 2)\)-flats in \(G\) which are mutually disjoint where \(4 \leq 2\mu \leq t\) and \(q \geq 2\). Let \(H_i\) \((i = 0, 1, \ldots, q)\) be \(q + 1\) hyperplanes in PG(t, q) which contain \(G\). Let \(V_{ij}\) \((i = 0, 1, \ldots, q, j = 1, 2, 3)\) be \((\mu - 1)\)-flats in \(H_i\) such that (a) \(G \cap V_{ij} = W_j\) and (b) \(V_{i1}, V_{i2}\) and \(V_{i3}\) are mutually disjoint. Let \(Y_j = \bigcup_{i=0}^{q} V_{ij}\) for \(j = 1, 2, 3\). Then \(Y_1 \cap Y_2 \cap Y_3\) is a \(\{3v_{\mu+1}, 3v_{\mu}; t, q\}\)-minihyper if and only if \(Y_1, Y_2\) and \(Y_3\) are three \(\mu\)-flats in PG(t, q) which are mutually disjoint.

**Proof of Theorem 1.1.** We shall prove Theorem 1.1 by induction on \(\mu\).

**Case I:** \((\mu = 1)\). It follows from Remark 1.1 and Lemmas 2.1 and 2.3 that Theorem 1.1 holds.

**Case II:** \((\mu \geq 2)\). Suppose there exists a \(\{3v_{\mu+1}, 3v_{\mu}; t, q\}\)-minihyper \(F\) for some integer \(t > \mu\). Using a method similar to the proof of Lemma 2.2, it can be shown that \(|F \cap G^*| \geq 3v_{\mu-1}\) for any \((t - 2)\)-flat \(G^*\) in PG(t, q) and \(|F \cap G| = 3v_{\mu-1}\) for some \((t - 2)\)-flat \(G\) in PG(t, q).
Let $H_i$ $(i = 0, 1, \ldots, q)$ be $q + 1$ hyperplanes in $\text{PG}(t, q)$ which contain $G$. Since $|F \cap H_0| + \sum_{i=1}^{q} |F \cap H_i| - |F \cap G| = |F| = 3v_{\mu + 1}$, it follows from $|F \cap H_i| \geq 3v_{\mu}$ $(i = 0, 1, \ldots, q)$ that $|F \cap H_i| = 3v_{\mu}$ for $i = 0, 1, \ldots, q$. Since $|F \cap G^*| \geq 3v_{\mu - 1}$ for any $(t - 2)$-flat $G^*$ in $H_i$ and $|F \cap G| = 3v_{\mu - 1}$ for the $(t - 2)$-flat $G$ in $H_i$, this implies that $F \cap H_i$ is a $(3v_{\mu}, 3v_{\mu - 1}; t, q)$-minihyper in the $(t - 1)$-flat $H_i$ for $i = 0, 1, \ldots, q$.

By induction on $\mu$, it follows that (i) in the case $t - 1 \leq 2(\mu - 1)$ (i.e., $t \leq 2\mu - 1$), there is no $(3v_{\mu}, 3v_{\mu - 1}; t, q)$-minihyper in $H$ for any $(t - 1)$-flat $H$, which is a contradiction, and (ii) in the case $t - 1 \geq 2(\mu - 1) + 1$ (i.e., $t \geq 2\mu \geq 4$), $F \cap H_i$ is a union of three $(\mu - 1)$-flats (denoted by $V_{i1}$, $V_{i2}$ and $V_{i3}$) in $H_i$ which are mutually disjoint. Hence, (i) in the case $t \leq 2\mu - 1$, there is no $(3v_{\mu}, 3v_{\mu - 1}; t, q)$-minihyper and (ii) in the case $t \geq 2\mu$, $F \cap H_i = V_{i1} \cup V_{i2} \cup V_{i3}$ for some $(\mu - 1)$-flats $V_{i1}$, $V_{i2}$ and $V_{i3}$ in $H_i$ which are mutually disjoint.

Let $W_i = G \cap V_{ij}$ for $j = 1, 2, 3$. Then $W_i$ is a $(\mu - 2)$-flat or a $(\mu - 1)$-flat in $G$. If $W_i$ is a $(\mu - 1)$-flat for some integer $j$, then $|W_i| = v_{\mu}$ and $|F \cap G| = |W_{i1}| + |W_{i2}| + |W_{i3}| \geq v_{\mu} + 2v_{\mu - 1} > 3v_{\mu - 1} = |F \cap G|$, a contradiction. Hence, $W_1$, $W_2$, and $W_3$ must be $(\mu - 2)$-flats in $G$ which are mutually disjoint.

Similarly, it can be shown that $G \cap V_{ij}$ is a $(\mu - 2)$-flat in $G$ for $i = 1, 2, \ldots, q$ and $j = 1, 2, 3$. Since $G \cap V_{i1} = W_{i\alpha}$, $G \cap V_{i2} = W_{i\beta}$ and $G \cap V_{i3} = W_{i\gamma}$ for some integers $\alpha$, $\beta$ and $\gamma$ such that $\{\alpha, \beta, \gamma\} = \{1, 2, 3\}$, we can assume, without loss of generality, that $G \cap V_{ij} = W_j$ for $i = 1, 2, \ldots, q$ and $j = 1, 2, 3$.

Let $Y_j = \bigcup_{i=0}^{q} V_{ij}$ for $j = 1, 2, 3$. Then $F = Y_1 \cup Y_2 \cup Y_3$. Hence, it follows from Lemma 3.1 that $F$ is a union of three $\mu$-flats $Y_1$, $Y_2$ and $Y_3$ in $\text{PG}(t, q)$ which are mutually disjoint. Since $\mathcal{F}(\mu, \mu, \mu; t, q) \neq \emptyset$ if and only if $t \geq 2\mu + 1$ (cf. Remark 1.1), it follows from Remark 1.1 that Theorem 1.1 holds in the case $\mu \geq 2$. This completes the proof. \[ \square \]

Appendix: Connections between minihypers and codes

Let $S(k, q)$ be the set of all column vectors $c$, $c = (c_1, c_2, \ldots, c_k)^T$, in $W(k, q)$ such that either $c_1 = 1$ or $c_1 = c_2 = \cdots = c_{i-1} = 0$, $c_i = 1$ for some integer $i$ in $\{2, 3, \ldots, k\}$ where $k \geq 3$ and $W(k, q)$ denotes a $k$-dimensional vector space consisting of column vectors over GF$(q)$. Then $S(k, q)$ consists of all $(q^k - 1)/(q - 1)$ projectively distinct nonzero vectors in $W(k, q)$ which may be regarded as $(q^k - 1)/(q - 1)$ points in $\text{PG}(k - 1, q)$.

**Theorem A.1** (Hamada [3]). Let $F$ be a set of vectors in $S(k, q)$ and let $C$ be the subspace of $V(n, q)$ generated by a $k \times n$ matrix (denoted by $G$) whose column vectors are all the vectors in $S(k, q) \setminus F$ where $n = v_k - f$, $1 \leq f < v_k - 1$ and $v_i = (q^i - 1)/(q - 1)$ for any integer $i \geq 0$.

1. Let $H_2 = \{y \in S(k, q) : \exists \cdot y = 0 \text{ over GF}(q)\}$ for a nonzero vector $z$ in $W(k, q)$. Then $H_2$ is a hyperplane in $\text{PG}(k - 1, q)$ and the weight of the code vector $z^T G$ in $C$ is equal to $|F \cap H_2| - v_k - 1 + n$ where $z^T$ denotes the transpose of the vector $z$. 
(2) In the case $k \geq 3$ and $1 \leq d < q^{k-1}$, $C$ is an $[n, k, d; q]$-code meeting the Griesmer bound if and only if $F$ is a \{v_k - n, v_{k-1} - n + d; k-1, q\}-minihyper.

**Definition A.1.** Two $[n, k, d; q]$-codes $C_1$ and $C_2$ are said to be equivalent if there exists a $k \times n$ generator matrix $G_2$ of the code $C_2$ such that $G_2 = G_1PD$ (or $G_2 = G_1DP$) for some permutation matrix $P$ and some nonsingular diagonal matrix $D$ with entries from $GF(q)$, where $G_1$ is a $k \times n$ generator matrix of $C_1$.

From Theorem A.1 and Definition A.1, we have the following corollary.

**Corollary A.1.** In the case $k \geq 3$ and $d = q^{k-1} - \sum_{i=1}^{h} q^{h_i}$, there is a one-to-one correspondence between the set of all nonequivalent $[n, k, d; q]$-codes meeting the Griesmer bound and the set of all $\{\sum_{i=1}^{h} v_{\lambda_i+1}, \sum_{i=1}^{h} v_{\lambda_i}; k-1, q\}$-minihypers where $n = v_k - \sum_{i=1}^{h} v_{\lambda_i+1}$.

It is well known that in the special case $t = 2$, $2 \leq h < q$ and $\lambda_1 = \lambda_2 = \ldots = \lambda_h = 1$, the following theorem holds.

**Theorem A.2.** If there exists an $\{hv_2, hv_1; 2, q\}$-minihyper $F$ for some prime power $q$ and some positive integer $h (< q)$, then $|F \cap H| = h$ or $q + 1$ for any 1-flat $H$ in $PG(2, q)$ and $q/(q + 1 - h)$ must be an integer where $v_1 = 1$ and $v_2 = q + 1$.

The following four theorems due to Tamari and Hamada play an important role in proving Theorem 1.1.

**Theorem A.3** (Tamari [13, 14]). In the case $1 \leq \lambda < t$ and $t \geq 2$, $F$ is a $\{v_{\lambda+1}, v_{\lambda}; t, q\}$-minihyper if and only if $F$ is a $\lambda$-flat in $PG(t, q)$.

**Theorem A.4** (Hamada [2, 5]). Let $t$, $\lambda_1$ and $\lambda_2$ be integers such that $0 \leq \lambda_1 \leq \lambda_2 < t$ and let $q$ be a prime power $\geq 3$.

1. In the case $t \geq \lambda_1 + \lambda_2 + 1$, $F$ is a $\{v_{\lambda_1+1} + v_{\lambda_2+1}, v_{\lambda_1} + v_{\lambda_2}; t, q\}$-minihyper if and only if $F$ is a union of a $\lambda_1$-flat and a $\lambda_2$-flat in $PG(t, q)$ which are mutually disjoint.

2. In the case $t \leq \lambda_1 + \lambda_2$, there is no $\{v_{\lambda_1+1} + v_{\lambda_2+1}, v_{\lambda_1} + v_{\lambda_2}; t, q\}$-minihyper.

**Theorem A.5** (Hamada [4]). In the case $t \geq 2$ and $q \geq 5$, $F$ is a $\{2v_1 + v_2, 2v_0 + v_1; t, q\}$-minihyper if and only if $F$ is a union of two points and one 1-flat in $PG(t, q)$ which are mutually disjoint.

**Theorem A.6** (Hamada [5]). Let $q$ be a prime power $\geq 4$.

1. In the case $t \geq 3$, $F$ is a $\{v_1 + 2v_2, v_0 + 2v_1; t, q\}$-minihyper if and only if $F$ is a union of one point and two 1-flats in $PG(t, q)$ which are mutually disjoint.

2. In the case $t = 2$, there is no $\{v_1 + 2v_2, v_0 + 2v_1; t, q\}$-minihyper.
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