Discrete Fourier transform and Riemann identities for $\theta$ functions

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**Abstract**

Riemann identities on theta functions are derived using properties of eigenvectors corresponding to the discrete Fourier transform $\Phi(2)$. In particular we get various fourth order identities of classical Jacobi theta functions.

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1. Introduction

The discrete Fourier transform (DFT) is a well known computing tool for applications in engineering and physics. It is also a source of interesting mathematical problems. The trace of the DFT matrix is the well known Gauss sum up to normalization factor. This work is based on the Matveev [1] result that eigenfunctions of the DFT can be expressed as the linear combinations of theta functions. This has been used [2] to derive the basic classical identities of Jacobi theta functions, in particular extended Watson addition formula, Watson addition formula in an elementary manner. In this paper we derive an extended version of the classical Riemann identity and show that Riemann identity is a particular case of extended Riemann identity.

In Section 2 for the completion we state the basic notations and preliminary results that were given in [2]. In Section 3, we derive an extended fourth order Riemann identity for theta functions and show that the classical Riemann identity follows from it.

2. Preliminary results

The matrix $\Phi(n)$ corresponding to the DFT of size $n$ is given by

$$\Phi_{jk}(n) = \frac{1}{\sqrt{n}} q^{jk}, \quad j, k = 0, \ldots, n - 1. \quad q = e^{2\pi i n}.$$  \hspace{1cm} (1)

**Definition.** For $f = (f_0, \ldots, f_{n-1})^T \in \mathbb{C}^n$ we define the DFT $\tilde{f} \in \mathbb{C}^n$ by $\tilde{f} = \Phi f = (\tilde{f}_0, \tilde{f}_1, \ldots, \tilde{f}_{n-1})$, where

$$\tilde{f}_k = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} f_j e^{\frac{2\pi i jk}{n}}.$$  

It is clear that $\Phi^4 = I$.

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Let $\tau$ be a complex number with a positive imaginary part. The theta function $\theta_{a,b}(x, \tau)$ with characteristics $(a, b)$ is defined by (see [3]),

$$\theta_{a,b}(x, \tau) = \sum_{m=-\infty}^{\infty} \exp[\pi i\tau (m + a)^2 + 2\pi i(m + a)(x + b)].$$

$\theta_{a,b}(x, \tau)$ is connected with $\theta_{0,0}(x, \tau) = \theta(x, \tau)$ by

$$\theta_{a,b}(x, \tau) = \theta(x + a\tau + b, \tau) \exp[\pi ia^2\tau + 2\pi ia(x + b)].$$

The four classical Jacobi theta functions may be represented as $\theta_{0,0}(x, \tau), \theta_{2,0}(x, \tau), \theta_{0,2}(x, \tau), \theta_{2,2}(x, \tau)$ (see [4]). The Jacobi theta functions which are related to eigenfunctions of the DFT $\Phi(n)$ are given by

$$\theta_{h,0}(x, \tau) = \sum_{m \in \mathbb{Z}} \exp[\pi i\tau (m + j/n)^2 + 2\pi i(m + j/n)x], \quad \text{for } j = 0, 1, 2, \ldots, n - 1. \quad (2)$$

Matveev [1] has proved the following theorem which is used in the following sections.

**Theorem 2.1 (Matveev).** For any $\tau$ with positive imaginary part the vector $v(x, \tau, k)$ with components $v_j(x, \tau, k), j = 0, 1, 2, \ldots, n - 1$ given by

$$v_j(x, \tau, k) = \theta_{h,0}(x, \tau) + (-1)^k\theta_{-h,0}(x, \tau) + \frac{1}{\sqrt{n}} \left[ (-i)^k \theta \left( \frac{j + x}{n}, \tau \right) + (-i)^{3k} \theta \left( \frac{x - j}{n}, \tau \right) \right], \quad (3)$$

is an eigenvector of the DFT with an eigenvalue $i^k$:

$$\Phi(n)v(x, \tau, k) = i^k v(x, \tau, k). \quad \square$$

The proof of the above theorem follows from the fact that $\Phi^4 = I$. We use this formula for eigenvectors to explore for the DFT $\Phi(2)$ in the following section to obtain extended Riemann theta function identities and various fourth order identities.

### 3. Riemann identity and the DFT $\Phi(2)$

The Riemann identity is the well known fourth order identity of theta functions. The beautiful account of identities derived from Riemann identity is given in [4]. We give extended version of this identity from multiplicities of eigenvalues and of eigenvectors of the DFT $\Phi(2)$. We call it extended Riemann identity, the particular case of this is the classical Riemann identity.

**Theorem 3.1 (Extended Riemann Identity for Theta Functions).**

$$4\theta_{0,0}(x, \tau)\theta_{1,0}(y, \tau)\theta_{1,0}(u, \tau)\theta_{1,0}(v, \tau) + 4\theta_{2,0}(x, \tau)\theta_{1,0}(y, \tau)\theta(u, \tau)\theta(v, \tau)$$

$$+ 4\theta_{2,0}(u, \tau)\theta_{1,0}(v, \tau)\theta(x, \tau)\theta(y, \tau) + 4\theta_{2,0}(x, \tau)\theta(y, \tau)\theta(u, \tau)\theta(v, \tau)$$

$$= \theta \left( \frac{x}{2}, \frac{\tau}{4} \right) \theta \left( \frac{y}{2}, \frac{\tau}{4} \right) \theta \left( \frac{u}{2}, \frac{\tau}{4} \right) \theta \left( \frac{v}{2}, \frac{\tau}{4} \right) \theta \left( \frac{x + 1}{2}, \frac{\tau}{4} \right) \theta \left( \frac{y + 1}{2}, \frac{\tau}{4} \right) \theta \left( \frac{u + 1}{2}, \frac{\tau}{4} \right) \theta \left( \frac{v + 1}{2}, \frac{\tau}{4} \right)$$

$$+ \theta \left( \frac{x}{2}, \frac{\tau}{4} \right) \theta \left( \frac{y}{2}, \frac{\tau}{4} \right) \theta \left( \frac{u + 1}{2}, \frac{\tau}{4} \right) \theta \left( \frac{v + 1}{2}, \frac{\tau}{4} \right)$$

$$+ \theta \left( \frac{x + 1}{2}, \frac{\tau}{4} \right) \theta \left( \frac{y + 1}{2}, \frac{\tau}{4} \right) \theta \left( \frac{u}{2}, \frac{\tau}{4} \right) \theta \left( \frac{v}{2}, \frac{\tau}{4} \right).$$

**Proof.** We have

$$\Phi(2)[v(x, \tau, 0) + v(x, \tau, 2)] = v(x, \tau, 0) - v(x, \tau, 2).$$

This gives the following two identities

$$\theta(x, \tau) + \theta_{1,0}(x, \tau) = \theta \left( \frac{x}{2}, \frac{\tau}{2} \right), \quad (4)$$

$$\theta(x, \tau) - \theta_{1,0}(x, \tau) = \theta \left( \frac{x + 1}{2}, \frac{\tau}{2} \right). \quad (5)$$
Consider the eigenvector (3) at the arguments $x + \frac{r}{2}$ and $y + 1$, we have

$$
v\left(x + \frac{r}{2}, \tau, 0\right) = \begin{bmatrix} 2a\theta_{\frac{r}{2}, 0}(x, \tau) + a\sqrt{2}\theta\left(\frac{x + 1}{2}, \frac{\tau}{4}\right) \\ 2a\theta(x, \tau) - a\sqrt{2}\theta\left(x + 1, \frac{\tau}{2}, \frac{\tau}{4}\right) \end{bmatrix},
$$

where $a = \exp\left(-\frac{\pi i}{4} - \pi i\right)$.

$$v(y + 1, \tau, 0) = \begin{bmatrix} 2\theta(y, \tau) + \sqrt{2}\theta\left(\frac{y + 1}{2}, \frac{\tau}{4}\right) \\ -2\theta_{\frac{r}{2}, 0}(y, \tau) + \sqrt{2}\theta\left(y + 1, \frac{\tau}{2}, \frac{\tau}{4}\right) \end{bmatrix}. \tag{7}$$

Since $v(y + 1, \tau, 0), v\left(x + \frac{r}{2}, \tau, 0\right)$ are eigenvectors corresponding to the same eigenvalue 1, which has multiplicity 1 (see [5]) we have

$$\det\left(v\left(x + \frac{r}{2}, \tau, 0\right), v(y + 1, \tau, 0)\right) = 0.$$

This gives

$$-4\theta_{\frac{r}{2}, 0}(x, \tau)\theta_{\frac{r}{2}, 0}(y, \tau) + 2\sqrt{2}\theta_{\frac{r}{2}, 0}(x, \tau)\theta\left(\frac{y}{2}, \frac{\tau}{4}\right) - 2\sqrt{2}\theta_{\frac{r}{2}, 0}(y, \tau)\theta\left(\frac{x}{2}, \frac{\tau}{4}\right)$$

$$+ 2\theta\left(\frac{x + 1}{2}, \frac{\tau}{2}, \frac{\tau}{4}\right)\theta\left(\frac{y}{2}, \frac{\tau}{4}\right) - 4\theta(x, \tau)\theta(y, \tau) - 2\sqrt{2}\theta(x, \tau)\theta\left(\frac{y + 1}{2}, \frac{\tau}{4}\right)$$

$$+ 2\sqrt{2}\theta(y, \tau)\theta\left(\frac{x + 1}{2}, \frac{\tau}{2}, \frac{\tau}{4}\right) + 2\theta\left(\frac{x + 1}{2}, \frac{\tau}{2}, \frac{\tau}{4}\right)\theta\left(\frac{y + 1}{2}, \frac{\tau}{2}, \frac{\tau}{4}\right) = 0. \tag{8}$$

Consider in (8) the terms with coefficients as $2\sqrt{2}$, using Eqs. (4), (5) all the terms with coefficients $2\sqrt{2}$ cancel each other out. Eq. (8) becomes

$$2\theta_{\frac{r}{2}, 0}(x, \tau)\theta_{\frac{r}{2}, 0}(y, \tau) + 2\theta(x, \tau)\theta(y, \tau) = \theta\left(\frac{x}{2}, \frac{\tau}{4}\right)\theta\left(\frac{y}{2}, \frac{\tau}{4}\right) + \theta\left(\frac{x + 1}{2}, \frac{\tau}{4}\right)\theta\left(\frac{y + 1}{2}, \frac{\tau}{4}\right). \tag{9}$$

Similarly by changing the variables $x, y$ to $u, v$ we have,

$$2\theta_{\frac{r}{2}, 0}(u, \tau)\theta_{\frac{r}{2}, 0}(v, \tau) + 2\theta(u, \tau)\theta(v, \tau) = \theta\left(\frac{u}{2}, \frac{\tau}{4}\right)\theta\left(\frac{v}{2}, \frac{\tau}{4}\right) + \theta\left(\frac{u + 1}{2}, \frac{\tau}{4}\right)\theta\left(\frac{v + 1}{2}, \frac{\tau}{4}\right). \tag{10}$$

Multiplying (9), (10) gives

$$4\theta_{\frac{r}{2}, 0}(x, \tau)\theta_{\frac{r}{2}, 0}(y, \tau)\theta_{\frac{r}{2}, 0}(u, \tau)\theta_{\frac{r}{2}, 0}(v, \tau) + 4\theta_{\frac{r}{2}, 0}(x, \tau)\theta_{\frac{r}{2}, 0}(y, \tau)\theta(u, \tau)\theta(v, \tau)$$

$$+ 4\theta_{\frac{r}{2}, 0}(u, \tau)\theta_{\frac{r}{2}, 0}(v, \tau)\theta(x, \tau)\theta(y, \tau) + 4\theta(x, \tau)\theta(y, \tau)\theta(u, \tau)\theta(v, \tau)$$

$$= \theta\left(\frac{x}{2}, \frac{\tau}{4}\right)\theta\left(\frac{y}{2}, \frac{\tau}{4}\right)\theta\left(\frac{u}{2}, \frac{\tau}{4}\right)\theta\left(\frac{v}{2}, \frac{\tau}{4}\right) + \theta\left(\frac{x + 1}{2}, \frac{\tau}{4}\right)\theta\left(\frac{y + 1}{2}, \frac{\tau}{4}\right)\theta\left(\frac{u + 1}{2}, \frac{\tau}{4}\right)\theta\left(\frac{v + 1}{2}, \frac{\tau}{4}\right)$$

$$+ \theta\left(\frac{x}{2}, \frac{\tau}{4}\right)\theta\left(\frac{y}{2}, \frac{\tau}{4}\right)\theta\left(\frac{u + 1}{2}, \frac{\tau}{4}\right)\theta\left(\frac{v}{2}, \frac{\tau}{4}\right)$$

$$+ \theta\left(\frac{x + 1}{2}, \frac{\tau}{4}\right)\theta\left(\frac{y + 1}{2}, \frac{\tau}{4}\right)\theta\left(\frac{u + 1}{2}, \frac{\tau}{4}\right)\theta\left(\frac{v + 1}{2}, \frac{\tau}{4}\right). \tag{11}$$

This proves extended Riemann identity. \(\square\)

**Corollary 3.2** \((\text{Riemann Identity})\).

$$\theta_{\frac{r}{2}, 0}(x, \tau)\theta_{\frac{r}{2}, 0}(y, \tau)\theta_{\frac{r}{2}, 0}(u, \tau)\theta_{\frac{r}{2}, 0}(v, \tau)$$

$$\times \theta_{\frac{r}{2}, 0}(u, \tau)\theta_{\frac{r}{2}, 0}(v, \tau)\theta_{\frac{r}{2}, 0}(x, \tau)\theta_{\frac{r}{2}, 0}(y, \tau)\theta(x, \tau)\theta(y, \tau)\theta(u, \tau)\theta(v, \tau)$$

$$= 2 \sum_{m,n,p,q \in \mathbb{Z}} \exp\left[\pi i (m^2 + n^2 + p^2 + q^2) \tau + 2\pi i (mx + ny + px + qy)\right]$$

$$m, n, p, q \text{ are either integers or } m, n, p, q \in \frac{1}{2} + \mathbb{Z} \text{ and } \sum m = m + n + p + q \in 2\mathbb{Z}. \tag{12}$$
Proof. We have from (11)\[\theta_{\frac{1}{2},0}(x,\tau)\theta_{\frac{1}{2},0}(y,\tau)\theta_{\frac{1}{2},0}(u,\tau)\theta_{\frac{1}{2},0}(v,\tau) + \theta_{\frac{1}{2},0}(x,\tau)\theta_{\frac{1}{2},0}(y,\tau)\theta(u,\tau)\theta(v,\tau)\]
\[\times \theta_{\frac{1}{2},0}(u,\tau)\theta_{\frac{1}{2},0}(v,\tau)(x,\tau)\theta(y,\tau) + \theta(x,\tau)\theta(y,\tau)\theta(u,\tau)\theta(v,\tau)\]
\[= \sum_{m+n,p+q=0} \exp \left[ \pi i \left( \sum m^2 \right) \frac{\tau}{4} + 2\pi i \left( \sum \frac{mx}{2} \right) \right]. \tag{13}\]

In (13) replacing x, y, z, u by x + \frac{1}{2}, y + \frac{1}{2}, u + \frac{1}{2}, v + \frac{1}{2} we get,
\[\theta_{\frac{1}{2},0}(x,\tau)\theta_{\frac{1}{2},0}(y,\tau)\theta_{\frac{1}{2},0}(u,\tau)\theta_{\frac{1}{2},0}(v,\tau) + \theta_{\frac{1}{2},0}(x,\tau)\theta_{\frac{1}{2},0}(y,\tau)\theta_{\frac{1}{2},0}(u,\tau)\theta_{\frac{1}{2},0}(v,\tau)\]
\[\times \theta_{\frac{1}{2},0}(u,\tau)\theta_{\frac{1}{2},0}(v,\tau)(x,\tau)\theta_{\frac{1}{2},0}(y,\tau) + \theta_{\frac{1}{2},0}(x,\tau)\theta_{\frac{1}{2},0}(y,\tau)\theta_{\frac{1}{2},0}(u,\tau)\theta_{\frac{1}{2},0}(v,\tau)\]
\[= \sum_{m+n,p+q=0} \exp \left[ \pi i \left( \sum m^2 \right) \frac{\tau}{4} + 2\pi i \left( \sum \frac{m(x + \frac{1}{2})}{2} \right) \right]. \tag{14}\]

By adding (13), (14) we get
\[\theta_{\frac{1}{2},0}(x,\tau)\theta_{\frac{1}{2},0}(y,\tau)\theta_{\frac{1}{2},0}(u,\tau)\theta_{\frac{1}{2},0}(v,\tau) + \theta_{\frac{1}{2},0}(x,\tau)\theta_{\frac{1}{2},0}(y,\tau)\theta_{\frac{1}{2},0}(u,\tau)\theta_{\frac{1}{2},0}(v,\tau)\]
\[\times \theta_{\frac{1}{2},0}(u,\tau)\theta_{\frac{1}{2},0}(v,\tau)(x,\tau)\theta_{\frac{1}{2},0}(y,\tau) + \theta_{\frac{1}{2},0}(x,\tau)\theta_{\frac{1}{2},0}(y,\tau)\theta_{\frac{1}{2},0}(u,\tau)\theta_{\frac{1}{2},0}(v,\tau)\]
\[= \sum_{m+n,p+q=0} \exp \left[ \pi i \left( \sum m^2 \right) \frac{\tau}{4} + 2\pi i \left( \sum \frac{m(x + \frac{1}{2})}{2} \right) \right] \tag{15}\]
\[+ \sum_{m+n,p+q=0} \exp \left[ \pi i \left( \sum m^2 \right) \frac{\tau}{4} + 2\pi i \left( \sum \frac{m(x + \frac{1}{2})}{2} \right) \right].\]

It is clear that (15) for m, n, p, q are either integers or m, n, p, q are \(\frac{1}{2} + Z\) and \(\sum m = m + n + p + q \in 2Z\) becomes
\[\theta_{\frac{1}{2},0}(x,\tau)\theta_{\frac{1}{2},0}(y,\tau)\theta_{\frac{1}{2},0}(u,\tau)\theta_{\frac{1}{2},0}(v,\tau) + \theta_{\frac{1}{2},0}(x,\tau)\theta_{\frac{1}{2},0}(y,\tau)\theta_{\frac{1}{2},0}(u,\tau)\theta_{\frac{1}{2},0}(v,\tau)\]
\[\times \theta_{\frac{1}{2},0}(u,\tau)\theta_{\frac{1}{2},0}(v,\tau)(x,\tau)\theta_{\frac{1}{2},0}(y,\tau) + \theta_{\frac{1}{2},0}(x,\tau)\theta_{\frac{1}{2},0}(y,\tau)\theta_{\frac{1}{2},0}(u,\tau)\theta_{\frac{1}{2},0}(v,\tau)\]
\[= 2 \sum_{m,n,p,q=\frac{1}{2}} \exp \left[ \pi i \left( \sum m^2 \right) + 2\pi i \left( \sum \frac{m(x + \frac{1}{2})}{2} \right) \right]. \tag{16}\]

Label the summands on the left hand side of (16) serially as
\[\sum_{k=1}^{k=8} A_k. \]

Since \(\sum_{m,n,p,q} m \in 2Z\), we have \(A_2 + A_7 = 0\) and \(A_3 + A_6 = 0\). Hence the Riemann identity follows from (15) and (16). (see [4]). □

For simplicity if we do a change of the variable as follows
\[n_1 = \frac{1}{2}(n + m + p + q), \quad x_1 = \frac{1}{2}(x + y + u + v)\]
\[m_1 = \frac{1}{2}(n + m - p - q), \quad y_1 = \frac{1}{2}(x + y - u - v)\]
\[p_1 = \frac{1}{2}(n - m + p - q), \quad u_1 = \frac{1}{2}(x - y + u - v)\]
\[q_1 = \frac{1}{2}(n - m - p + q), \quad v_1 = \frac{1}{2}(x - y + u + v).\]
Then the particular restrictions on parameters $n, m, p, q$ of the summation above exactly means that the resulting $n_1, m_1, p_1, q_1$ are integers. Also observe that we have the identities: $\sum n^2 = \sum n_1^2$ and $\sum x n = \sum x_n n_1$. (12) becomes

$$\theta_{1/2, 1/2}(x, \tau) \theta_{1/2, 1/2}(y, \tau) \theta_{1/2, 1/2}(u, \tau) \theta_{1/2, 1/2}(v, \tau) + \theta_{1/2, 0}(x, \tau) \theta_{1/2, 0}(y, \tau) \theta_{1/2, 0}(u, \tau) \theta_{1/2, 0}(v, \tau)$$

$$\times \theta_{0, 1/2}(u, \tau) \theta_{0, 1/2}(v, \tau) \theta_{0, 1/2}(x, \tau) \theta_{0, 1/2}(y, \tau) + \theta(x, \tau) \theta(y, \tau) \theta(u, \tau) \theta(v, \tau)$$

$$= 2 \theta(x_1, \tau) \theta(y_1, \tau) \theta(u_1, \tau) \theta(v_1, \tau).$$

The beautiful account of identities derived from Riemann identity are given in [4]. This includes all the fourth order identities of theta functions. The method presented above indicates that all these identities can be derived from the formula for eigenvectors of DFT.

References