Discrete Mathematics 54 (1985) 241–257
North-Holland

ON THE TYPE OF PARTIAL t-SPREADS IN FINITE PROJECTIVE SPACES*

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Received 5 November 1984

A partial t-spread in a projective space $\boldsymbol{P}$ is a set of mutually skew t-dimensional subspaces of $\boldsymbol{P}$. In this paper, we deal with the question, how many elements of a partial spread $\mathcal{S}$ can be contained in a given d-dimensional subspace of $\boldsymbol{P}$. Our main results run as follows. If any d-dimensional subspace of $\boldsymbol{P}$ contains at least one element of $\mathcal{S}$, then the dimension of $\boldsymbol{P}$ has the upper bound $d-1+(d/t)$. The same conclusion holds, if no d-dimensional subspace contains precisely one element of $\mathcal{S}$. If any d-dimensional subspace has the same number $m>0$ of elements of $\mathcal{S}$, then $\mathcal{S}$ is necessarily a total t-spread. Finally, the 'type' of the so-called geometric t-spreads is determined explicitely.

1. Introduction

Denote by $\boldsymbol{P}=\mathrm{PG}(r, q)$ the finite projective space of dimension $r\geq 3$ and order q, where $q=p^h$ is a power of the prime p. A *partial t-spread* of $\boldsymbol{P}$ is a set $\mathcal{S}$ of t-dimensional subspaces of $\boldsymbol{P}$ such that any point of $\boldsymbol{P}$ is incident with at most one element of $\mathcal{S}$. The partial t-spread $\mathcal{S}$ of $\boldsymbol{P}$ is called a *t-spread*, if any point of $\boldsymbol{P}$ lies on a (unique) element of $\mathcal{S}$.

Partial t-spreads have been investigated thoroughly. In particular, one is interested in the cardinality of maximal partial t-spreads; see for instance [5, 6, 8, 15, 18].

Recently, the notion of 'type' and 'class' of a partial t-spread was introduced by Tallini [22]. Denote by $\mathcal{S}$ a partial t-spread of $\boldsymbol{P}=\mathrm{PG}(r,q)$, and let d be an integer with $t\leq d\leq r$. We say that $\mathcal{S}$ is of *type* $(T)_d$, where T is a set of non-negative integers, if the following conditions hold:

(i) For any d-dimensional subspace U of $\boldsymbol{P}$, the number of elements of $\mathcal{S}$ contained in U is a number of T.

(ii) For any number $m\in T$, there exists a d-dimensional subspace U of $\boldsymbol{P}$ such that U has exactly m elements of $\mathcal{S}$.

* Communicated at XII Congresso UMI, Perugia, September 1983.

† This paper was written while the first author was as a visiting professor at the Università degli Studi in L'Aquila, Italy. He wants to thank the C.N.R. for the support.

If $\mathcal{S}$ is a partial t-spread of type $(T)_d$, then $\mathcal{S}$ is said to be of *class* $[C]_d$, for any set C of non-negative integers with $T \subseteq C$.

If $T = \{m_1, \ldots, m_a\}$ (or $C = \{m_1, \ldots, m_b\}$), we say also that $\mathcal{S}$ is of type $(m_1, \ldots, m_a)_d$ (or class $[m_1, \ldots, m_b]_d$), if $\mathcal{S}$ is of type $(T)_d$ (or of class $[C]_d$, respectively).

Up to now, only partial t-spreads of type $(T)_{2t+1}$ (or class $[C]_{2t+1}$) have been considered. The study of these structures was initiated by Tallini [22, 23] in the case $t = 1$, and continued by de Finis and de Resmini [12], and, for $t > 1$, by Berardi [4] and Eugeni [16].

The aim of this paper is to investigate partial t-spreads of type $(T)_d$ and class $[C]_d$ for an arbitrary integer d. In Section 3 we shall prove the following surprising theorem. Denote by $\mathcal{S}$ a non-empty partial t-spread of class $[C]_d$ in PG(r, q). If $0 \notin C$ or $1 \notin C$, then $r \leq d - 1 + (d/t)$. In other words: If $r > d - 1 + (d/t)$, then $0, 1 \in C$.

In Section 4, partial t-spreads of type $(m)_d$ with $m \neq 0$ are studied. It turns out that those partial t-spreads are exactly the (total) t-spreads.

In Sections 5 and 6 we shall deal with partial t-spreads of type $(0, n)_d$ and $(1, n)_d$ in PG(r, q). We shall show that under certain assumptions it follows that $d = r - 1$.

Finally, in the last section, we shall construct many examples illustrating our theorems. In particular, we shall determine the type $(T)_d$ of a geometric 1-spread in PG(r, q) for any integer d with $1 \leq d \leq r$.

2. Preliminary definitions and results

Throughout this paper we shall use the terminology of Dembowski [13]. For any two integers d and r with $0 \leq d \leq r$ and for any prime-power q we define the following numbers, which are known as the 'q-analoguous' to the binomial coefficients (see for example [1, 10, 19]).

$$\theta_r(q^s) = \sum_{i=0}^{r} q^{si}, \quad \theta_r = \theta_r(q) \quad \theta_{-1} = 0, \tag{2.1}$$

$$\begin{bmatrix} r \\ d \end{bmatrix}_q = \frac{\theta_{r-1} \cdots \theta_0}{\theta_{d-1} \cdots \theta_0 \theta_{r-d-1} \cdots \theta_0}, \tag{2.2}$$

$$\gamma_{r,d} = \prod_{i=0}^{d} \frac{\theta_{r-i}}{\theta_{d-i}} = \begin{bmatrix} r+1 \\ d+1 \end{bmatrix}_q. \tag{2.3}$$

It is well known that $\theta_r(q)$ is the number of points in PG(r, q), and $[\begin{smallmatrix} r \\ d \end{smallmatrix}]_q$ is the number of $(d-1)$-dimensional subspaces in PG$(r-1, q)$.

Result 2.1 (Segre [20]). The projective space PG(r, q) contains a t-spread if and only if $r + 1 = (a+1)(t+1)$ for a positive integer a. For any t-spread $\mathcal{S}$ of

$PG((a+1)(t+1)-1, q)$ we have

$$|\mathcal{S}| = \theta_r/\theta_t = \theta_a(q^{t+1}). \tag{2.4}$$

For any two distinct elements V, V' of a partial spread $\mathcal{S}$, denote by $\langle V, V'\rangle$ the subspace generated by V and V'. We say that $\mathcal{S}$ *induces* a partial t-spread in $\langle V, V'\rangle$, if any element of $\mathcal{S}$ having a point in common with $\langle V, V'\rangle$ is contained in $\langle V, V'\rangle$; $\mathcal{S}$ is called *geometric* (cf. [2]), if for any two distinct elements V, V' of $\mathcal{S}$, $\mathcal{S}$ induces a partial t-spread in $\langle V, V'\rangle$. The following result is well known.

Result 2.2 (Serge [20]). The projective space $PG(r, q)$ contains a geometric t-spread if and only if $t+1$ divides $r+1$.

For a geometric t-spread $\mathcal{S}$ of $\boldsymbol{P} = PG(r, q)$, let $\boldsymbol{P}(\mathcal{S}) = (\mathcal{S}, \mathcal{B})$ be the following incidence structure. The *points* are the elements of $\mathcal{S}$, the *blocks* are the sets of elements of $\mathcal{S}$ belonging to the subspaces $\langle V, V'\rangle$ for any two distinct elements V, V' of $\mathcal{S}$. Then the following result is true.

Result 2.3 (Serge [20]). If $\mathcal{S}$ is a geometric t-spread of $PG(r, q)$ with $r+1=(a+1)(t+1)$, then the incidence structure $\boldsymbol{P}(\mathcal{S})$ is a projective space of dimension a and order q^{t+1}.

For a generalization of this result see Theorem 5.1 in [6]. We present now some easy lemmas.

Lemma 2.4. *Denote by $\mathcal{S}$ a partial t-spread of $\boldsymbol{P} = PG(r, q)$ with $r+1=(a+1)(t+1)$. Then $\mathcal{S}$ is a t-spread if and only if any hyperplane of $\boldsymbol{P}$ contains exactly $\theta_{a-1}(q^{t+1})$ elements of $\mathcal{S}$.*

Proof. Consider a hyperplane H and denote by s the number of elements of $\mathcal{S}$ which are subspaces of H. Since any element of $\mathcal{S}$ which is not contained in H intersects H in a subspace of dimension $t-1$, the number n of points of H which are incident with an element of $\mathcal{S}$ equals

$$n = s\cdot\theta_t + (|\mathcal{S}| - s)\theta_{t-1} = s\cdot q^t + |\mathcal{S}|\,\theta_{t-1}.$$

Since $|\mathcal{S}| \leq \theta_a(q^{t+1})$, it follows

$$n \leq s\cdot q^t + \theta_{r-1} - q^t\cdot\theta_{a-1}(q^{t+1}).$$

Therefore, $s = \theta_{a-1}(q^{t+1})$ if and only if $n = \theta_{r-1}(q)$, i.e. if and only if any point of H is incident with an element of $\mathcal{S}$.

Now the assertion of our lemma follows easily. If any hyperplane has $\theta_{a-1}(q^{t+1})$ elements of $\mathcal{S}$, then any point is incident with an element of $\mathcal{S}$, and so, $\mathcal{S}$ is a t-spread. On the other hand, if $\mathcal{S}$ is a t-spread, then $n = \theta_{r-1}(q)$ for any hyperplane H; hence H contains exactly $\theta_{a-1}(q^{t+1})$ elements of $\mathcal{S}$. □

Denote by $\gcd(s, t)$ the greatest common divisor of the positive integers s and t.

Lemma 2.5. *Let q be a prime-power, and let s and t be two positive integers. Moreover, define γ by $\gamma+1=\gcd(s+1, t+1)$. Then $\gcd(\theta_s, \theta_t)=\theta_\gamma$.*

Proof. The following fact is well known (see for example [7, p. 105]):

$$\theta_u \mid \theta_v \text{ if and only if } u+1 \mid v+1.$$

A repeated application of this assertion proves the lemma. □

3. Non-existence theorems

In this section we shall prove a rather restrictive non-existence theorem for partial t-spreads of type $(T)_d$. We shall apply this result to partial t-spreads of class $[\geqslant 1]_d$ and $[0, \geqslant 2]_d$. This result says in particular that for any partial t-spread $\mathcal{S}$ in $\mathrm{PG}(r, q)$ with $r>d-1+(d/t)$, there is a d-dimensional subspace which contains no element of $\mathcal{S}$ and a d-dimensional subspace which contains exactly one element of $\mathcal{S}$.

Lemma 3.1. *Denote by $\mathcal{S}$ a partial t-spread of type $(T)_d$ in $\mathbf{P}=\mathrm{PG}(r, q)$. Suppose that there exists a u-dimensional subspace U of $\mathbf{P}$ with $u \leqslant d-t-1$ in which $\mathcal{S}$ induces a partial spread $\mathcal{S}_U$ having exactly s elements. If $s \notin T$, then $r \leqslant d-1+(d/t)$.*

Proof. If h denotes the smallest number in T which is greater than s, then $m:=h-s>0$. In particular, any d-dimensional subspace through U contains at least m elements of $\mathcal{S}-\mathcal{S}_U$. We claim: Any subspace of dimension $d+i$ through U contains at least $mq^{(t+1)i}$ elements of $\mathcal{S}-\mathcal{S}_U$.

Namely: We have already observed that the assertion holds for $i=0$. Suppose now $i \geqslant 1$ and assume that the assertion is true for $i-1$. Consider a subspace W of dimension $d+i$ through U. By induction, any hyperplane of W through U has at least $s_{i-1}:=mq^{(t+1)(i-1)}$ elements of $\mathcal{S}-\mathcal{S}_U$. Denote by s_i the number of elements of $\mathcal{S}-\mathcal{S}_U$ in W. Counting the number of pairs (V, H), where V is an element of $\mathcal{S}-\mathcal{S}_U$ in W and H is a hyperplane of W through U and V we obtain

$$s_i \cdot \theta_{d+i-u-t-2} \geqslant \theta_{d+i-u-1} \cdot mq^{(t+1)(i-1)},$$

hence

$$s_i \geqslant mq^{(t+1)(i-1)} \cdot q^{t+1} = m \cdot q^{(t+1)i}.$$

It follows in particular $|\mathcal{S}-\mathcal{S}_U| \geqslant m \cdot q^{(t+1)(r-d)}$. Since $|\mathcal{S}| \leqslant \theta_r/\theta_t < q^{r+1}/q^t$, we have

$$q^{(t+1)(r-d)} \cdot q^t \leqslant m \cdot q^{(t+1)(r-d)} \cdot q^t < q^{r+1},$$

thus, $(t+1)(r-d)+t<r+1$. □

A partial t-spread $\mathcal{S}$ of $\boldsymbol{P}$ is said to be of *class* $[\geqslant 1]_d$, (or, of *class* $[0, \geqslant 2]_d$) if no d-dimensional subspace of $\boldsymbol{P}$ contains no element of $\mathcal{S}$ (or, exactly one element of $\mathcal{S}$, respectively). As corollaries of Lemma 3.1 we have the following theorems which generalize the results given in [16] in the cases $d=2t+1$ and $d=2t+2$.

Theorem 3.2. *Denote by d and r two positive integers with $d<r$. If $\mathcal{S}$ is a partial t-spread of class $[\geqslant 1]_d$ in $\boldsymbol{P}=\mathrm{PG}(r,q)$, then $r \leqslant d-1+(d/t)$.*

Proof. Put $U=\emptyset$. □

Theorem 3.3. *Denote by $\mathcal{S}$ a partial t-spread of $\boldsymbol{P}=\mathrm{PG}(r,q)$ of class $[0, \geqslant 2]_d$ with $2t+1 \leqslant d < r$. If $\mathcal{S} \neq \emptyset$, then $r \leqslant d-1+(d/t)$.*

Proof. Let U be an element of $\mathcal{S}$. □

Corollary 3.4. *Denote by $\mathcal{S}$ a non-empty partial t-spread in $\boldsymbol{P}=\mathrm{PG}(r,q)$ of class $[\geqslant 1]_{2t+1}$ or of class $[0, \geqslant 2]_{2t+1}$ with $r>2t+1$.*
(a) *If $t=1$, then $r \leqslant 5$.*
(b) *If $t>1$, then $r=2t+2$.*

Proof. If $d=2t+1$, Theorems 3.2 and 3.3 reduce to $r \leqslant d-1+(d/t)=2t+2+(1/t)$. □

The remainder of this section is devoted to examples which show that the above bounds are best possible. In Section 7 we shall prove that a geometric 1-spread in $\mathrm{PG}(2d-1,q)$ is always of class $[\geqslant 1]_d$ (see Corollary 7.4).

Proposition 3.5. *For any positive integer t and any prime-power q there exists a partial t-spread of class $[\geqslant 1]_d$ in $\mathrm{PG}(2t+2,q)$.*

Proof. *Example 1.* Denote by H a hyperplane of $\boldsymbol{P}$, and let $\mathcal{S}$ be a t-spread of H. Then, by Result 2.1, $\mathcal{S}$ is a partial t-spread of type $(1, q^{t+1}+1)_{2t+1}$ of $\boldsymbol{P}$. (Any hyperplane $H' \neq H$ intersects H in a $2t$-dimensional subspace which contains exactly one element of $\mathcal{S}$ (see Lemma 2.4).)

Example 2. Embed $\boldsymbol{P}$ as a hyperplane in $\Sigma=\mathrm{PG}(2t+3,q)$ and consider a $(t+1)$-spread $\mathcal{F}$ of Σ. Denote by F_0 the element of $\mathcal{F}$ in $\boldsymbol{P}$. We define

$$\mathcal{S}=\{F \cap \boldsymbol{P} \mid F \in \mathcal{F}-\{F_0\}\}$$

and $\mathcal{S}'=\mathcal{S} \cup \{S_0\}$, where S_0 is an arbitrary t-dimensional subspace of F_0. Clearly, $\mathcal{S}'$ is a partial t-spread of $\boldsymbol{P}$ (cf. [6, Theorem 4.2]; see also [16]). We claim that $\mathcal{S}'$ is of type $(1, q, q+1)_{2t+1}$.

In order to show this, consider a hyperplane W of $\boldsymbol{P}$. There are exactly q

hyperplanes $H_1, \ldots, H_q$ of Σ through W with $H_i \neq \boldsymbol{P}$. Each of these hyperplanes contains exactly one element of $\mathcal{F}$. If $F_0 \subseteq W$, then this element is in any case F_0. On the other hand, if $F_0 \nsubseteq W$, then these q elements of $\mathcal{F}$ are mutually distinct. Thus, W contains exactly q elements of $\mathcal{S}$. It may be that W contains also S_0. Therefore, $\mathcal{S}'$ is of type $(1, q, q+1)_{2t+1}$. □

If we consider the partial spread $\mathcal{S}$ of $\boldsymbol{P}$ again, we see immediately

Proposition 3.6. *For any positive integer t and any prime-power q there exists a partial t-spread of type $(0, q)_{2t+1}$ in* PG$(2t+2, q)$.

We remark that this was proved in [4] in the case $t=1$ and in [16].

The authors do not know an example of a non-empty partial 1-spread of class $[0, \geq 2]_d$ in PG$(2d-1, q)$. However, in Section 7 we shall see (cf. Theorem 7.3) that for any integer $d \geq 3$ and any prime-power q, a geometric 1-spread in PG$(2d-3, q)$ is of class $[0, \geq 2]_d$.

4. Partial t-spreads of singular type

Throughout this section, we denote by $\mathcal{S}$ a partial t-spread in $\boldsymbol{P} = \text{PG}(r, q)$ of type $(m)_d$ with $2t+1 \leq d < r$. If $m = 0$, then $\mathcal{S} = \emptyset$. So, we may also suppose that $m \geq 1$. The main result of this section is a precise description of all partial t-spreads of $\boldsymbol{P}$ of type $(m)_d$ (see Theorem 4.7).

Lemma 4.1. *Let h be a positive integer with $d+h<r$, and denote by U a $(d+h)$-dimensional subspace of $\boldsymbol{P}$. If m_h denotes the number of elements of $\mathcal{S}$ in U, then*

$$m_h \cdot \begin{bmatrix} d+h-t \\ h \end{bmatrix}_q = m \cdot \begin{bmatrix} d+h+1 \\ d+1 \end{bmatrix}_q. \tag{4.1}$$

In other words, $\mathcal{S}$ is of type $(m_h)_{d+h}$.

Proof. Counting the pairs (S, W), where S is an element of $\mathcal{S}$ in U and W is a d-dimensional subspace of U containing S we obtain (4.1). It follows in particular that m_h is independent from the choice of the $(d+h)$-dimensional subspace U. □

Corollary 4.2. *Suppose $d \leq r-1$. Then $\mathcal{S}$ is of type $(n)_{r-1}$ with*

$$n = m\theta_{r-1} \cdots \theta_{r-t-1}/\theta_d \cdots \theta_{d-t-1} = |\mathcal{S}| \cdot \theta_{r-t-1}/\theta_r. \tag{4.2}$$

Proof. By Lemma 4.1, $\mathcal{S}$ is of type $(m_{r-d-1})_{r-1}$ with

$$n := m_{r-d-1} = m \cdot \begin{bmatrix} r \\ d+1 \end{bmatrix}_q \Big/ \begin{bmatrix} r-1-t \\ r-1-d \end{bmatrix}_q.$$

Counting the pairs (S, W), where $S \in \mathcal{S}$ and W is a hyperplane of $\boldsymbol{P}$ with $S \subseteq W$ we get $\theta_r \cdot n = |\mathcal{S}|\, \theta_{r-t-1}$. □

Denote by a and b the uniquely defined positive integers with

$$r = a(t+1) + b \quad \text{and} \quad 1 \leqslant b \leqslant t+1.$$

Lemma 4.3. *Define the integer* g *by* $g+1 = \gcd(t+1, b+1)$. *Then* θ_g *divides* θ_{r-t-1}, *and* θ_{r-t-1}/θ_g *is a divisor of the above defined number* n.

Proof. Since $g+1$ divides $(a-1)(t+1)+b+1$ $(=r-t)$, by Lemma 2.4, θ_{r-t-1} is a multiple of θ_g. Moreover,

$$\gcd(t+1, r-t) = \gcd(r+1, t+1) = \gcd(b+1, t+1) = g+1.$$

Again using Lemma 2.4 we have $\gcd(\theta_t, \theta_{r-t-1}) = \theta_g$. In view of (4.2), the assertion follows. □

Corollary 4.4. *We have* $g > 0$ *and* $b \leqslant t$.

Proof. Assume $g = 0$. Then Lemma 4.3 implies that θ_{r-t-1} divides n. In particular, $\theta_{r-t-1} \leqslant n$. On the other hand, $n \leqslant \theta_{r-1}/\theta_t$. Thus,

$$\theta_{r-t-1} \cdot \theta_t \leqslant \theta_{r-1},$$

a contradiction.

If $b = t+1$, then $g = 0$. But we have already shown that this is impossible. □

Theorem 4.5. *We have that* $\mathcal{S}$ *is a (total)* t*-spread of* $\boldsymbol{P}$.

Proof. Since θ_g divides θ_{r-t-1}, θ_r/θ_g is a divisor of θ_r/θ_{r-t-1}. Therefore, Corollary 4.2 implies that θ_r/θ_g divides $|\mathcal{S}|$. In particular we have

$$\theta_r/\theta_g \leqslant |\mathcal{S}|.$$

On the other hand, clearly, $|\mathcal{S}| \leqslant \theta_r/\theta_t$. Since $g \leqslant b$ and $b \leqslant t$ it follows

$$\theta_r/\theta_b \leqslant \theta_r/\theta_g \leqslant |\mathcal{S}| \leqslant \theta_r/\theta_t.$$

Therefore, $|\mathcal{S}| = \theta_r/\theta_t$. This means that $\mathcal{S}$ is a (total) t-spread of $\boldsymbol{P}$. □

Lemma 4.6. *Denote by* $\mathcal{F}$ *a* t*-spread in* $\boldsymbol{P} = \mathrm{PG}(r, q)$. *Then* $\mathcal{F}$ *is not of type* $(m)_{r-2}$.

Proof. Assume to the contrary that $\mathcal{F}$ is of type $(m)_{r-2}$ for a positive integer m.

Counting the pairs (S, W), where $S \in \mathcal{F}$ and W is an $(r-2)$-dimensional subspace of $\boldsymbol{P}$ through S, we get

$$\theta_r \cdot \theta_{r-1} \cdot m = |\mathcal{F}| \cdot \theta_{r-t-1} \cdot \theta_{r-t-2}.$$

Since $\mathcal{F}$ is a t-spread, we have $|\mathcal{F}| = \theta_r/\theta_t$, and so

$$\theta_t \cdot \theta_{r-1} \cdot m = \theta_{r-t-1} \cdot \theta_{r-t-2}.$$

Since $r = a(t+1)-1$, we have $\gcd(r, r-t-1) = 1$. Therefore, θ_{r-1} divides θ_{r-t-1}, a contradiction. □

In the following theorem, we determine all partial t-spreads of singular type.

Theorem 4.7. *A partial t-spread in* $\boldsymbol{P} = \mathrm{PG}(r, q)$ *is of type* $(m)_d$ $(m \neq 0)$ *if and only if it is a (total) t-spread of* $\boldsymbol{P}$ *and we have* $d = r-1$. *Moreover, in this situation,* $m = \theta_{r-t-1}/\theta_t$.

Proof. By Lemma 2.4, any t-spread of $\boldsymbol{P}$ is of type $(\theta_{r-t-1}/\theta_t)_{r-1}$.

Suppose on the other hand that $\mathcal{S}$ is a partial t-spread of type $(m)_d$ in $\boldsymbol{P}$. Then, by Corollary 4.2, $\mathcal{S}$ is of type $(n)_{r-1}$. Now, Theorem 4.5 implies that $\mathcal{S}$ is a t-spread of $\boldsymbol{P}$. Finally, Lemma 4.6 in connection with Lemma 4.1 shows $d = r-1$. □

5. Partial spreads of type $(0, m)_d$

Throughout this section, we denote by $\mathcal{S}$ a partial t-spread of type $(0, m)_d$ in $\boldsymbol{P} = \mathrm{PG}(r, q)$ with $2t+1 \leq d < r$. Without loss in generality, we can suppose $|\mathcal{S}| \geq 2$. We shall prove that under these hypotheses we have necessarily $d = r-1$.

Consider a subspace U of $\boldsymbol{P}$ with $d+1 \leq \dim(U) < r$. Then the elements of $\mathcal{S}$ in U form a partial t-spread of type $(0, m)_d$ of U. Therefore, it suffices to show that the assumption $r = d+2$ yields a contradiction. We shall work under this assumption.

Lemma 5.1. *Under the above assumptions, we have*

$$(|\mathcal{S}|-1)\theta_{d-2t} \cdot \theta_{d-2t-1} = \theta_{d-t+1} \cdot \theta_{d-t} \cdot (m-1). \tag{5.1}$$

Proof. Fix $V_0 \in \mathcal{S}$. We count the pairs (V, H), where $V \in \mathcal{S} - \{V_0\}$ and H is a d-dimensional subspace through V and V_0. Since any of these d-dimensional subspaces H contains precisely m elements of $\mathcal{S}$, the assertion follows. □

Denote by a and b the uniquely defined integers with

$$d+2 = r = a(t+1)+b \quad \text{and} \quad 1 \leq b \leq t+1. \tag{5.2}$$

Lemma 5.2. *If we define*

$$\gamma_1+1=\gcd(t+1, b+1), \qquad \gamma_2+1=\gcd(t, a+b-1),$$
$$\gamma_3+1=\gcd(b+2-a, t+2), \qquad \gamma_4+1=\gcd(t+1, b),$$

then

$$g_1:=\gcd(\theta_{d-2t}, \theta_{d-t+1})=\theta_{\gamma_1}, \qquad g_2:=\gcd(\theta_{d-2t}, \theta_{d-t})=\theta_{\gamma_2},$$
$$g_3:=\gcd(\theta_{d-2t-1}, \theta_{d-t+1})=\theta_{\gamma_3}, \qquad g_4:=\gcd(\theta_{d-2t-1}, \theta_{d-t})=\theta_{\gamma_4}.$$

Moreover, $\gcd(g_1, g_2)=1$.

Proof. Using Lemma 2.4, this follows by elementary calculations. □

Corollary 5.3. *Either* $\theta_{d-t+1}\cdot\theta_{d-t}/g_1g_2g_3$ *or* $\theta_{d-t+1}\cdot\theta_{d-t}/g_2g_3g_4$ *is a divisor of* $|\mathcal{S}|-1$.

Proof. In view of Lemma 5.2, this follows by Lemma 5.1. □

Corollary 5.4. $d\leq t+3+\gamma_2+\gamma_3+\max\{\gamma_1, \gamma_4\}$.

Proof. If we put $M:=\max\{g_1g_2g_3, g_2g_3g_4\}$, it follows by Corollary 5.3

$$\theta_{d-t+1}\cdot\theta_{d-t}\leq(|\mathcal{S}|-1)M.$$

Since

$$|\mathcal{S}|-1\leq(\theta_{d+2}-\theta_t)/\theta_t=q^{t+1}\cdot\theta_{d+1-t}/\theta_t,$$

we have

$$\theta_{d-t}\cdot\theta_t\leq q^{t+1}\cdot M.$$

Therefore,

$$q^{d-t}\cdot q^t<q^{t+1}\cdot\max\{q^{\gamma_1+1}q^{\gamma_2+1}q^{\gamma_3+1}, q^{\gamma_2+1}q^{\gamma_3+1}q^{\gamma_4+1}\},$$

and so

$$d-t+t<t+1+\gamma_2+\gamma_3+\max\{\gamma_1, \gamma_4\}. \quad \square$$

Corollary 5.5. $d\leq 4t+2$.

Proof. By Lemma 5.2, $\max\{\gamma_1, \gamma_4\}\leq t$. Clearly, $\gamma_2\leq t-1$ and $\gamma_3\leq t+1$. Thus, by the preceeding corollary, it follows $d\leq 4t+3$.

Assume $d=4t+3$. Then $a=4$ and $b=1$. Consequently, $\gamma_1\leq 1$, $\gamma_2\leq 3$, $\gamma_3\leq t+1$, $\gamma_4=0$, and so $4t+3=d\leq t+3+3+t+1+1$. Thus, $2t\leq 5$. But in the cases $t=1$ and $t=2$, a contradiction follows easily. □

The following assertion turns out to be very useful.

Proposition 5.6. $a=3$.

Proof. Since $d \leqslant 4t+2$, we have $a \leqslant 3$. Clearly, $a \geqslant 2$. (Otherwise, $d=1+t+1+b-2 \leqslant 2t-1$.) Assume $a=2$. Then

$$\gamma_1+1=\gcd(t+1, b+1), \qquad \gamma_2+1=\gcd(t, b+1),$$
$$\gamma_3+1=\gcd(b, t+2), \qquad \gamma_4+1=\gcd(t+1, b).$$

So, $\gamma_1+\gamma_2 \leqslant b$ and $\gamma_3+\gamma_4 \leqslant b-1$. Therefore, Corollary 5.4 implies

$$2(t+1)+b-2=d \leqslant t+3+b+b-1,$$

and so $b \geqslant t-2$. Assume first $b=t+1$. Then (5.1) yields

$$(|\mathcal{S}|-1)(q^{t+1}+\cdots+1)=(q^{2t+2}+\cdots+1)(q^{t+1}+1)(m-1).$$

Therefore, $(q^{2t+2}+\cdots+1)(q^{t+1}+1)$ divides $|\mathcal{S}|-1$ contradicting

$$|\mathcal{S}|-1 \leqslant q^{t+1}(q^{2t+2}+\cdots+1).$$

In the cases $b=t$, $b=t-1$ and $b=t-2$ we get similarly

$$(|\mathcal{S}|-1)(q^{t-1}+\cdots+1)=(q^{t+1}+1)(q^{2t}+\cdots+1)(m-1),$$
$$(|\mathcal{S}|-1)(q^{t-2}+\cdots+1)=(q^{2t}+\cdots+1)(q^{t}+1)(m-1),$$

and

$$(|\mathcal{S}|-1)(q^{t-2}+\cdots+1)(q^{t-3}+\cdots+1)$$
$$=(q^{2t-1}+\cdots+1)(q^{2t-2}+\cdots+1)(m-1).$$

In any case, a contradiction follows. □

Lemma 5.7. $b \neq t+1$, t.

Proof. If $b=t+1$, then $\gamma_1=0$, $\gamma_2=\gcd(t, t+3)-1 \leqslant 2$, $\gamma_3=\gcd(t, t+2)-1 \leqslant 1$, $\gamma_4=t$. Thus,

$$4t+2=3(t+1)+b-2 \leqslant t+3+2+1+t,$$

hence $2t \leqslant 4$, i.e. $t \leqslant 2$. If $t=2$, then $\gamma=0$, $\gamma_3=1$; if $t=1$, then $\gamma_2=0=\gamma_3$. In both cases we get a contradiction.

Suppose now $b=t$. It follows $\gamma_1=t$, $\gamma_2=\gcd(t, t+2)-1 \leqslant 1$, $\gamma_3=\gcd(t-1, t+2)-1 \leqslant 2$, $\gamma_4=0$. So, $3t+1+t \leqslant t+3+t+1+2$, therefore $t \leqslant 2$. But this contradicts (5.1). □

In a similar way, one can prove the following

Lemma 5.8. $b \neq t-1$, $t-2$.

Proof. One has to note that in the case $b=t-1$ we have $1 \leqslant b=t-1$, so $t \geqslant 2$. Similarly, if $b=t-2$, then $t \geqslant 3$. □

We can now get a final contradiction.

Theorem 5.9. *Denote by $\mathcal{S}$ a partial t-spread of class $[0, m]_d$ in $\boldsymbol{P}=\mathrm{PG}(r, q)$ with $2t+1 \leqslant d<r$. If $|\mathcal{S}| \geqslant 2$, then $d=r-1$.*

Proof. Lemmas 5.7 and 5.8 imply in particular $\gamma_1 \neq t$; so $\gamma_1 \leqslant[(t+1)/2]-1$. Similarly, $\gamma_2 \leqslant[\frac{1}{2}t]-1$, $\gamma_3 \leqslant[\frac{1}{2}(t+2)]-1$, $\gamma_4 \leqslant[\frac{1}{2}(t+1)]-1$. By Corollary 5.4 we have

$$d \leqslant t+3+\frac{t}{2}-1+\frac{t+2}{2}-1+\frac{t+1}{2}-1=\frac{5t}{2}.$$

But, by Proposition 5.6 we have $d \geqslant 3(t+1)-1=3t+2$. Together, we get a contradiction. □

In Section 7 we shall construct a class of partial t-spreads of type $(0, m)_d$. Another class of examples can be found in [16].

Proposition 5.10. *Suppose that in $\boldsymbol{P}=\mathrm{PG}(r, q)$ there exists a partial t-spread $\mathcal{S}$ of type $(0, m)_{r-1}$ with $|\mathcal{S}| \geqslant 2$. Then*

$$\theta_{r-2t-2} \mid (m-1)\theta_{r-t-1} \quad \textit{and} \quad |\mathcal{S}|=1+(m-1)\theta_{r-t-1}/\theta_{r-2t-2}.$$

Moreover,

$$m \leqslant \theta_{r-t-1}/\theta_t \quad \textit{and} \quad m \cdot \theta_{r-2t-2} \mid q^{r-2t-1} \cdot \theta_t(mq^{r-2t-1} \cdot \theta_t-\theta_{r-t-1}).$$

Proof. If we count the incident pairs (S, H), where $S \in \mathcal{S}$ and H is a hyperplane of $\boldsymbol{P}$, we get

$$(|\mathcal{S}|-1)\theta_{r-2t-2}=\theta_{r-t-1} \cdot (m-1). \quad \square$$

6. On partial spreads of class $[1, n]_d$

In this section we shall prove that—under certain assumptions—the existence of a partial t-spread of type $(1, n)_d$ in $\mathrm{PG}(r, q)$ implies $r=d+1$. By similar methods as in the last section, we can prove a little more, namely the following theorem, which applies in particular to partial t-spreads of type $(1, n)_d$.

Theorem 6.1. *Denote by $\mathcal{S}$ a partial t-spread in $\boldsymbol{P}=\mathrm{PG}(r, q)$. Suppose that there exists a subspace U of dimension $2t+1$ in which $\mathcal{S}$ induces a partial t-spread $\mathcal{S}_U$ with u elements, such that any subspace of dimension d through U contains exactly $u+k>u$ elements of $\mathcal{S}$. Suppose moreover $d>4t+3$. Then either $r=d+1$, or one of the following cases occurs:*

$$(t, d)=(1,7),\ (2,11),\ (3,15),\ (5,25),\ (8,40).$$

Proof. Suppose $r \geq d+2$. Then, without loss in generality, $r=d+2$. Counting the incident pairs (V, W) with $V \in \mathcal{S}-\mathcal{S}_U$, where W is a d-dimensional subspace through U and V, we get

$$(|\mathcal{S}|-u)\cdot\theta_{d-3t-1}\cdot\theta_{d-3t-2}=\theta_{d-2t}\cdot\theta_{d-2t-1}\cdot k. \tag{6.1}$$

Put $d+2=a(t+1)+b$ with $1 \leq b \leq t+1$. Then

$$\gamma_1+1:=\gcd(d-3t, d-2t+1)=\gcd(t+1, b+1),$$

$$\gamma_2+1:=\gcd(d-3t, d-2t)=\gcd(t, a+b-2),$$

$$\gamma_3+1:=\gcd(d-3t-1, d-2t+1)=\gcd(t+2, b+3-a),$$

$$\gamma_4+1:=\gcd(d-3t-1, d-2t)=\gcd(t+1, b).$$

In particular, $\gamma_1+\gamma_4=\max\{\gamma_1, \gamma_4\} \leq \max\{b, t\}$. If we define $g_i:=\theta_{\gamma_i}$ $(i=1,2,3,4)$, it follows by (6.1) that either $\theta_{d-2t}\theta_{d-2t-1}/g_1g_2g_3$ or $\theta_{d-2t}\theta_{d-2t-1}/g_2g_3g_4$ is a divisor of $|\mathcal{S}|-u$. Since $\mathcal{S}-\mathcal{S}_U$ is a partial t-spread of $\boldsymbol{P}-U$, we have

$$|\mathcal{S}|-u \leq |\boldsymbol{P}-U|/\theta_t=q^{2t+2}\cdot\theta_{d-2t}/\theta_t.$$

Hence, if μ denotes the maximum of $g_1g_2g_3$ and $g_2g_3g_4$, then

$$\theta_{d-2t}\cdot\theta_{d-2t-1}/\mu \leq |\mathcal{S}|-u \leq q^{2t+2}\cdot\theta_{d-2t}/\theta_t,$$

and therefore

$$\theta_{d-2t-1}\cdot\theta_t \leq q^{2t+2}\cdot\max\{g_1g_2g_3, g_2g_3g_4\}.$$

Hence, $d-2t-1+t<2t+2+3+\gamma_2+\gamma_3+\max\{\gamma_1, \gamma_4\}$, i.e. $d \leq 3t+5+\gamma_2+\gamma_3+\max\{\gamma_1, \gamma_4\}$.

Since $\max\{\gamma_1, \gamma_4\} \leq b$, $\gamma_2 \leq t-1$ and $\gamma_3 \leq t+1$, it follows in particular $a(t+1)+b-2=d \leq 3t+5+(t-1)+(t+1)+b$, and so $a(t+1) \leq 5t+7=5(t+1)+2$.

We claim $a \leq 5$. (Assume $a \geq 6$. Then $a=6$ and $2=b=t+1$. But then we have $\gamma_3=0$, a contradiction.) On the other hand,

$$a(t+1)+b-2 \geq 4t+3, \tag{6.2}$$

which means $a \geq 4$. Thus, $4 \leq a \leq 5$. We consider first the case $a=5$. In this situation we have

$$\gamma_2 \leq b+2, \qquad \gamma_3 \leq t+1, \qquad \max\{\gamma_1, \gamma_4\} \leq b-1,$$

and so $(b+2)+(t+1)+(b-1) \geq 2t+2+b$, i.e. $b \geq t$. But in these cases we have $\gamma_2+\gamma_3 \leq 5$, which implies $t=1$. This yields a contradiction.

Thus, $a=4$. Therefore,

$$t+b+3 \leq \gamma_2+\gamma_3+\max\{\gamma_1, \gamma_4\}.$$

Moreover,

$$\gamma_1=\gcd(t+1, b+1)-1, \qquad \gamma_2=\gcd(t, b+2)-1 \leq b+1,$$

$$\gamma_3=\gcd(t+2, b-1)-1, \qquad \gamma_4=\gcd(t+1, b)-1.$$

Let us first consider the cases $b=t+1$, t, $t-2$. If $b\geq t$, then $\gamma_2+\gamma_3\leq 3$ and $\max\{\gamma_1,\gamma_4\}=t$, So, $t+b-3\leq 3+t$, i.e. $b\leq 6$. But these values of b (and t) are impossible by (6.1).

Consider now the case $b=t-2$. Then $t=b+2\geq 3$ and $\gamma_1\leq 1$, $\gamma_2=t-1$, $\gamma_3\in\{0,4\}$ and $\gamma_4\leq 2$. Therefore, $2t-5=t+b-3\leq(t-1)+4+2$, and so $t\leq 10$. The values $t=10$, 9, 7, 6, 4 can be excluded immediately, but the cases $(t,d)=(1,7)$, $(2,11)$, $(3,15)$, $(5,25)$, $(8,40)$ yield no contradiction.

Let us now suppose $b\neq t+1, t, t-2$. Then

$$\gamma_1\leq\frac{t+1}{2}-1=\frac{t-1}{2},\qquad \gamma_2\leq\frac{t}{2}-1=\frac{t-2}{2},\qquad \gamma_3\leq\frac{t+1}{2}-1=\frac{t-1}{2}.$$

Since $d>4t+3$, we have $b>1$. Therefore, $\gamma_3\leq b-2$. It follows

$$t+b-3\leq\frac{t-2}{2}+b-2+\frac{t-1}{2},$$

a final contradiction. □

Corollary 6.2. *Let $\mathcal{S}$ be a partial t-spread of type $(1,m)_d$ in $\boldsymbol{P}=\mathrm{PG}(r,q)$ with $4t+3<d<r$. Suppose that there is a $(2t+1)$-dimensional subspace U of $\boldsymbol{P}$ such that $\mathcal{S}$ induces a partial t-spread in U. Then $r=d+1$ or one of the following cases occurs:*

$$(t,d)=(1,7),\ (2,11),\ (3,15),\ (5,25),\ (8,40).$$

Proof. Since there exists a d-dimensional subspace of $\boldsymbol{P}$ which does not contain all elements of $\mathcal{S}$ in U, there exists an element of $\mathcal{S}$ outside U. Consequently, any d-dimensional subspace through U contains exactly $m-u$ elements of $\mathcal{S}$ outside U, where u is the number of elements of $\mathcal{S}$ in U. Now, the assertion follows by the above theorem. □

Remarks. (1) In Proposition 7.6 we shall construct partial 1-spreads $\mathcal{S}'$ in $\boldsymbol{P}=\mathrm{PG}(7,q)$ with the following property: There is a 3-dimensional subspace U of $\boldsymbol{P}$ such that any subspace of dimension 5 through U has exactly $q(q-1)$ elements of $\mathcal{S}'$. This example shows that the assumption '$d\geq 4t+3$' of Theorem 6.1 cannot be weakened very much.

(2) For any prime-power q, there exists a partial 1-spread of type $(1,q+1)_3$ in $\mathrm{PG}(4,q^2)$. Cf. de Finis and de Resmini [12].

7. Examples. The type of a geometric spread

Denote by $\mathcal{S}$ a geometric t-spread in $\boldsymbol{P}=\mathrm{PG}(r,q)$, where $r+1=(a+1)(t+1)$, $a\geq 2$. By Lemma 2.4, any hyperplane of $\boldsymbol{P}$ contains exactly $\theta_{a-1}(q^{t+1})$ elements of $\mathcal{S}$, i.e. $\theta_{a-1}(q^{t+1})$ points of a hyperplane of the associated projective space $\boldsymbol{P}(\mathcal{S})$ defined in Section 2.

Proposition 7.1. *In* $\boldsymbol{P}=\mathrm{PG}((a+1)(t+1)-1, q)$, *any geometric t-spread* $\mathcal{S}$ *has type* $(\theta_{a-2}(q^{t+1}), \theta_{a-1}(q^{t+1}))_{r-2}$.

Proof. Denote by W a subspace of dimension $r-2$, and let H be a hyperplane through W. Since H intersects $\mathcal{S}$ in the points of a hyperplane of $\boldsymbol{P}(\mathcal{S})$, there is a subspace V of dimension $a(t+1)-1$ of H such that any element of $\mathcal{S}$ in H is in V.

If W contains V, then W has exactly $\theta_{a-1}(q^{t+1})$ elements of $\mathcal{S}$. If W does not contain V, then the hyperplane W of H intersects V in a hyperplane U of V. By Lemma 2.4, u contains exactly $\theta_{a-2}(q^{t+1})$ elements of $\mathcal{S}$. □

A *maximal* $\{k; n\}$*-arc* (cf. Barlotti [3]) in a projective plane $\boldsymbol{P}$ of order q is a non-empty set $\mathcal{K}$ of points of $\boldsymbol{P}$ such that any line of $\boldsymbol{P}$ intersects $\mathcal{K}$ in 0 or exactly n points. Any maximal $\{k; n\}$-arc has precisely $k=(q+1)(n-1)+1$ points. These structures have been investigated in detail; see for example [3, 9, 11, 14, 17, 24–26].

Proposition 7.2. (a) *In* $\boldsymbol{P}=\mathrm{PG}((a+1)(t+1)-1, q)$ *there exists a partial t-spread of type* $(0, q^{(a-1)(t+1)})_{r-1}$.

(b) *Suppose that the desarguesian projective plane of order* q^{t+1} *contains a maximal* $\{(q^{t+1}+1)(n-1)+1, n\}$*-arc. Then, in* $\mathrm{PG}(3t+2, q)$, *there exists a partial t-spread of type* $(0, n)_{3t+1}$.

Proof. (a) Consider a geometric t-spread $\mathcal{S}$ of $\boldsymbol{P}$. Remove from $\mathcal{S}$ the points of a hyperplane of $\boldsymbol{P}(\mathcal{S})$. Since any hyperplane of $\boldsymbol{P}$ intersects $\mathcal{S}$ in the points of a hyperplane of $\boldsymbol{P}(\mathcal{S})$, the assertion follows.

(b) Consider a maximal $\{(q^{t+1}+1)(n-1)+1; n\}$-arc in the projective plane $\boldsymbol{P}(\mathcal{S})$, where $\mathcal{S}$ is a geometric t-spread of $\mathrm{PG}(3t+2, q)$. □

The most important result of this section is the following.

Theorem 7.3. *Denote by* $\mathcal{S}$ *a geometric* 1*-spread in* $\boldsymbol{P}=\mathrm{PG}(2a+1, q)$. *Moreover, let s be an integer with* $0\leqslant s\leqslant a$. *Then any* $(a+s)$*-dimensional subspace of* $\boldsymbol{P}$ *contains at least* $\theta_{s-1}(q^2)$ *elements of* $\mathcal{S}$.

Proof. By induction on s. The case $s=0$ is trivial. Suppose $s\geqslant 1$ and suppose moreover that the assertion is true for $s-1$.

Denote by U a subspace of dimension $a+s$ and assume that U has fewer than $\theta_{s-1}(q^2)$ elements of $\mathcal{S}$. Since $\mathcal{S}$ is geometric, by induction, U has exactly $\theta_{s-2}(q^2)$ elements of $\mathcal{S}$. Moreover, the elements of $\mathcal{S}$ in U form a spread $\mathcal{S}_0$ of a $(2s-3)$-dimensional subspace U_0 of U. (Note that $\mathcal{S}_0$ is the point set of an $(s-2)$-dimensional subspace of $\boldsymbol{P}(\mathcal{S})$.)

Consider now the $\theta_{a+s}-\theta_{2s-3}=q^{2(s-1)}\cdot\theta_{a-s+2}$ elements of $\mathcal{S}$ which intersect U

in exactly one point. Each of these lines generates together with $\mathcal{S}_0$ an $(s-1)$-dimensional subspace of $\mathbf{P}(\mathcal{S})$. Consider the corresponding $(2s-1)$-dimensional subspaces $V_1, \ldots, V_b$ of $\mathbf{P}$. Each of these subspaces V_i intersects U in a $(2s-2)$-dimensional subspace. So, the number b of these subspaces V_i equals

$$b = q^{2(s-1)} \cdot \theta_{a-s+2}/q^{2s-2} = \theta_{a-s+2}.$$

We claim that for any two distinct subspaces V_i, V_j we have $\langle U, V_i\rangle \neq \langle U, V_j\rangle$. (*Otherwise*, $\langle V_i, V_j\rangle$ would be contained in the subspace $X = \langle U, V_i\rangle = \langle U, V_j\rangle$ of dimension $a+s+1$. Since $\mathcal{S}$ is geometric, it induces a spread $\mathcal{S}'$ in $\langle V_i, V_j\rangle$ with $|\mathcal{S}'| = \theta_s(q^2)$. Therefore, the hyperplane $\langle V_1, V_j\rangle \cap U$ of $\langle V_i, V_j\rangle$ would contain exactly $\theta_{s-1}(q^2)$ elements of $\mathcal{S}'$, a contradiction to our assumption.)

Consequently, there are at least θ_{a-s+2} subspaces of dimension $a+s+1$ through U. But the exact number of these subspaces is θ_{a-s}. This is a contradiction. □

In view of Theorem 3.2, the above theorem implies in particular

Corollary 7.4. *Let $\mathcal{S}$ be a geometric* 1*-spread in* PG$(2a+1, q)$. *Then $\mathcal{S}$ is of class* $[\geq 1]_d$ *if and only if* $d \geq a+1$.

Corollary 7.5. *Let $\mathcal{S}$ be a geometric* 1*-spread in* $\mathbf{P} = \mathrm{PG}(2a+1, q)$, *and denote by U an a-dimensional subspace of $\mathbf{P}$ containing no element of $\mathcal{S}$. Define $\mathcal{S}'$ to be the set of lines of $\mathcal{S}$ which do not intersect U. If s is an integer with $1 \leq s \leq a$, then any subspace of dimension $a+s$ through U contains exactly $\theta_{s-1}(q^2) - \theta_{s-1}$ elements of $\mathcal{S}'$.*

Proof. Denote by V a subspace of dimension $a+s$ through U. Then, by Theorem 7.3, V contains at least $\theta_{s-1}(q^2)$ elements of $\mathcal{S}$.

Step 1. V contains exactly $\theta_{s-1}(q^2)$ elements of $\mathcal{S}$.

Assume to the contrary that V has more than $\theta_{s-1}(q^2)$ elements in common with $\mathcal{S}$. Then V contains at least $\theta_s(q^2)$ elements of $\mathcal{S}$. Since $\mathcal{S}$ is geometric, there exists a $(2s+1)$-dimensional subspace Y of V in which $\mathcal{S}$ induces a geometric spread $\mathcal{S}_Y$. Since

$$\dim(Y \cap U) \geq 2s+1+a-(a+s) = s+1,$$

by Corollary 7.4, $Y \cap U$ contains at least one element of $\mathcal{S}_Y$, a contradiction.

Step 2. V contains exactly $\theta_{s-1}(q^2) - \theta_{s-1}$ elements of $\mathcal{S}'$.

For: By Step 1, there is a $(2s-1)$-dimensional subspace W of V such that any element of $\mathcal{S}$ in V is in W. Moreover,

$$\dim(W \cap U) \geq 2s-1+a-(a+s) = s-1.$$

But $\dim(W \cap U) \geq s$ is impossible, since otherwise (by Theorem 7.3), $V \cap U$ would contain an element of $\mathcal{S}$. Thus, $\dim(W \cap U) = s-1$. Consequently, V contains exactly $\theta_{s-1}(q^2) - \theta_{s-1}$ elements of $\mathcal{S}'$. □

Clearly, a geometric 1-spread of $\mathrm{PG}(2a+1,q)$ is of class $[0,1,\theta_1(q^2),\ldots,\theta_a(q^2)]_d$. In the remainder of this section we shall determine the type of $\mathcal{S}$ for any d with $0\leqslant d\leqslant 2a+1$.

Proposition 7.6. *Let $\mathcal{S}$ be a geometric 1-spread in $\boldsymbol{P}=\mathrm{PG}(2a+1,q)$. Denote by s an integer with $0\leqslant s\leqslant a-1$. Then for any $i\in\{-1,0,1,\ldots,a-1-s\}$ there is a subspace U of dimension $2s+2+i$ such that U has exactly $\theta_s(q^2)$ elements in common with $\mathcal{S}$.*

Proof. Let $\mathcal{S}'$ be the point set of an s-dimensional subspace of $\boldsymbol{P}(\mathcal{S})$, and denote by W' the subspace of dimension $2s+1$ in which $\mathcal{S}$ induces the spread $\mathcal{S}'$.

Let $\mathcal{S}''$ be the point set of a complement of $\boldsymbol{P}(\mathcal{S}')$ in $\boldsymbol{P}(\mathcal{S})$. This means that $\boldsymbol{P}(\mathcal{S}'')$ has dimension $a-s-1$ and that $\mathcal{S}''$ has no element in common with $\mathcal{S}'$. If W'' denotes the subspace of dimension $2(a-s-1)+1$ of $\boldsymbol{P}$ in which $\mathcal{S}$ induces the spread $\mathcal{S}''$, then W' and W'' are complementary subspaces of $\boldsymbol{P}$.

By Corollary 7.4, for any integer $i\in\{-1,0,1,\ldots,a-s-1\}$, there is an i-dimensional subspace V of W'' which has no element in common with $\mathcal{S}''$.

Then $U:=\langle V,W'\rangle$ is a subspace of dimension $2s+2+i$ of $\boldsymbol{P}$. It remains to show that the only elements of $\mathcal{S}$ in U are the elements of $\mathcal{S}'$. Indeed, if U would contain a line $l\in\mathcal{S}-\mathcal{S}'$, then $\mathcal{S}$ would induce a spread in $\langle l,W'\rangle$, and $\langle l,W'\rangle$ would intersect W'' non-trivially, a contradiction. □

Theorem 7.7. *Let $\mathcal{S}$ be a geometric 1-spread in $\boldsymbol{P}=\mathrm{PG}(2a+1,q)$.*

(a) *If h is an integer with $0\leqslant h\leqslant a$, then $\mathcal{S}$ has type $(0,1,\theta_1(q^2),\ldots,\theta_u(q^2))_h$, where u is defined by $u=[\frac{1}{2}(h-1)]$.*

(b) *If h is an integer with $1\leqslant h\leqslant a$, then $\mathcal{S}$ has type $(\theta_{h-1}(q^2),\ldots,\theta_u(q^2))_{a+h}$, where u is defined by $u=[\frac{1}{2}(a+h-1)]$.*

Proof. (a) Fix a number s with $0\leqslant s\leqslant u$, and define $i=h-2s-2$. It follows $-1\leqslant i\leqslant a-2s-2\leqslant a-s-1$. So, by the above proposition, there is a subspace of dimension $2s+2+i=h$ which has exactly $\theta_s(q^2)$ elements in common with $\mathcal{S}$.

(b) Fix a number s with $h-1\leqslant s\leqslant u$ and define $i=a+h-2s-2$. Since $s\geqslant h-1$, we have $i\leqslant a+h-(h-1)-s-2=a-s-1$. Moreover, $s\leqslant u=[\frac{1}{2}(a+h-1)]$ implies that $i\geqslant -1$. Now, the assertion follows in view of Proposition 7.6. □

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