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Three loop QCD MOM β-functions

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ABSTRACT

We present the full expressions for the QCD β-function in the MOMggg, MOMq and MOMh renormalization schemes at three loops for an arbitrary colour group in the Landau gauge. The results for all three schemes are in very good agreement with the $SU(3)$ numerical estimates provided by Chetyrkin and Seidensticker.

Renormalization schemes can be divided into various classes. For instance, they can be split into physical and unphysical or mass dependent and mass independent schemes. One of the main schemes which is used in perturbative quantum field theories is the minimal subtraction scheme, [1], or its more widely used extension which is the modified minimal subtraction scheme and abbreviated by $\overline{\text{MS}}$, [2]. A reason the latter is predominant is that it is the scheme in which one can compute to very high orders in perturbation theory. For example, the four loop QCD β-function, [3,4], represents the current state of the art for the β-function of the theory of strong interactions. Indeed computations proceed to this high order because $\overline{\text{MS}}$ is a mass independent renormalization scheme and thus one merely needs to be able to compute massless Feynman diagrams. This is possible even for vertex functions where an external momentum can be nullified to allow the simple extraction of the poles in the regulator. One consequence of this is that techniques such as integration by parts in d spacetime dimensions, [5–7], which are implemented within computer algebra programmes such as Form, [8], can be used systematically. However, a drawback of the $\overline{\text{MS}}$ scheme is that the subtraction point for the vertex renormalization, and hence the definition of the coupling constant renormalization constant, is at an unphysical point of the vertex Green's function. Therefore, despite its attractiveness with regard to computability $\overline{\text{MS}}$ is an unphysical scheme and thus one could encounter infrared inconsistencies if one were to compare with the nonperturbative structure of the vertex Green's function. By contrast it would be more appropriate to define the subtraction point of the Green's function at a physical point which avoids any potential infrared ambiguities and thus is more useful for nonperturbative measurements such as those carried out using lattice regularization. Of necessity such schemes whilst being physical introduce a mass scale within Feynman diagrams even in a theory which was originally massless in contrast to $\overline{\text{MS}}$. Moreover, nullification of an external momentum of a vertex is not permitted. As such these effectively massive Feynman integrals are more difficult to compute analytically, and thus the renormalization group functions of QCD in such schemes are not available to as high a loop order as the $\overline{\text{MS}}$ scheme.

One class of physical and mass dependent schemes which is of importance to lattice gauge theory computations is the set referred to as MOM which denotes the family of momentum subtraction schemes, [9]. Though the syntax MOM indicates a specific momentum configuration of the QCD 3-point vertices. This is the situation where the squares of all the momenta coming in through the three vertices are equal to one mass squared value. This is known as the symmetric subtraction point. The vertex renormalization is performed in such a way that after the MOM renormalization constants are set then there are no $O(a)$ corrections at the subtraction point where $a = g^2/(16\pi^2)$ and g is the coupling constant of gluon with itself or the quarks and Faddeev–Popov ghost fields. In other words if the integrals are dimensionally regularized, which we will use throughout, then not only are the poles in ϵ removed, where $d = 4 - 2\epsilon$, but also the finite part at the subtraction point. Indeed for the 2-point functions of the theory the same criterion is used so that at their subtraction point there are no $O(a)$ contributions after the wave function and gauge parameter MOM renormalization constants are set. This outline is a summary of the original MOM scheme definition of [9]. There the full one loop renormalization of QCD was carried out. However, as one has several 3-point vertices in the QCD Lagrangian when there is a linear covariant gauge fixing, it was noted in [9]

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that there are three types of MOM schemes. Each is associated with a particular 3-point vertex and are denoted by MOMggg, MOMq and MOMh. These refer respectively to the schemes derived by ensuring triple gluon, quark–gluon and ghost–gluon vertices are subtracted by the MOM criterion. Once each of these schemes is defined the structure of the remaining n-point functions of the theory are determined by the Slavnov–Taylor identities of QCD, [9]. Indeed in [9] the full one loop renormalization of QCD in each of the three MOM schemes was discussed.

Beyond one loop one runs into the difficulty of calculability. This is illustrated in particular in [10]. There the two loop relation between the $\overline{\mathrm{MS}}$ coupling constant and that of each of the three MOM schemes was derived *numerically*. This was partly because the basic scalar master two loop Feynman integrals for the symmetric subtraction point for the 3-point topologies were not fully available. A subset of the necessary integrals have been evaluated in [11–13] but the final master was only constructed as a corollary of the results of [14]. There the basic 3-point functions were determined for the more general configuration of all three squared external momenta being independent. Therefore in the absence of these results at the time, the approach of [10] was to approximate the 3-point integrals by using a large mass expansion with respect to one of the independent momentum squared. This allowed for the use of the Mincer algorithm [15], and application of the Exp package. With a sufficient number of terms in the expansion in the appropriate parameter an approximation was determined for the 3-point vertices to two loops. Equipped with this the mapping of the coupling constants between MOM and $\overline{\mathrm{MS}}$ schemes was established. Hence through the renormalization group equation the *three* loop MOM scheme β-functions were given numerically for $SU(3)$, [10]. This was possible as the three loop $\overline{\mathrm{MS}}$ β-function was already available [16–21]. Whilst this represents the current status of these β-functions in the MOM schemes it is the purpose of this article to report on the exact evaluation of the three loop MOM β-functions for an arbitrary colour group. This is possible with the advance not only in computer power but also because of the development of the Laporta algorithm [22]. This allows one to systematically write all Feynman integrals of a certain topology in terms of the basic scalar master integrals. As the two loop masters are available for the 3-point functions [11–14], it is merely a straightforward exercise to repeat the approach of [10] and compute the vertex functions to two loops at the symmetric subtraction point. Although the difference here is to avoid any numerical approximation to the integrals, we will still follow the ethos of constructing the mapping between the parameters of both schemes. This will allow us to use the same conversion technique to establish the precise structure of the three loop terms of the β-function in each of the three MOM schemes. We will focus specifically on the β-functions here as the limited space excludes us from reporting on the full structure of each of the three vertices at the symmetric subtraction point. However, this together with the parameter mappings and the explicit forms of all the anomalous dimensions of QCD in each of the three separate schemes for an arbitrary linear covariant gauge and colour group will appear in a longer article [23]. Finally, as background we mention that the triple gluon vertex has been examined at two loops previously in the on-shell configuration in [24].

More specifically we have computed the three Green's functions which relate to each of the vertices associated with the renormalization schemes. These are each written in terms of a set of Lorentz basis tensors built out of the two independent external momenta, p and q, together with the metric tensor $\eta_{\mu\nu}$ as well as γ-matrices in the case of the quark–gluon vertex. Denoting the basis tensors for the ith vertex by $\mathcal{P}^i_{(k)\{\mu_i\}}(p,q)$ where $\{\mu_i\}$ indicates the associated Lorentz indices for that vertex and k labels the tensor, then we have

$$\begin{aligned}
\left\langle A^a_\mu(p) A^b_\nu(q) A^c_\sigma(-p-q)\right\rangle\big|_{p^2=q^2=-\mu^2} &= f^{abc} \sum_{k=1}^{14} \mathcal{P}^{\mathrm{ggg}}_{(k)\mu\nu\sigma}(p,q)\, \Sigma^{\mathrm{ggg}}_{(k)}(p,q) \\
\left\langle \psi^i(p) \bar{\psi}^j(q) A^c_\sigma(-p-q)\right\rangle\big|_{p^2=q^2=-\mu^2} &= T^c_{ij} \sum_{k=1}^{6} \mathcal{P}^{\mathrm{qqg}}_{(k)\sigma}(p,q)\, \Sigma^{\mathrm{qqg}}_{(k)}(p,q) \\
\left\langle c^a(p) \bar{c}^b(q) A^c_\sigma(-p-q)\right\rangle\big|_{p^2=q^2=-\mu^2} &= f^{abc} \sum_{k=1}^{2} \mathcal{P}^{\mathrm{ccg}}_{(k)\sigma}(p,q)\, \Sigma^{\mathrm{ccg}}_{(k)}(p,q)
\end{aligned} \tag{1}$$

where $\Sigma^i_{(k)}(p,q)$ is the scalar amplitude associated with each basis tensor at the symmetric point. As we are focusing on the structure of the vertices at the symmetric subtraction point then the momenta satisfy

$$p^2 = q^2 = (p+q)^2 = -\mu^2 \tag{2}$$

which imply

$$pq = \frac{1}{2}\mu^2 \tag{3}$$

where μ is the mass scale introduced in dimensional regularization to ensure the coupling constant remains massless in d dimensions. It ought to be noted that away from this point the tensor basis will be larger than that indicated in the decomposition. However, for the purpose of the construction of the MOM β-functions the only tensor of interest, and hence its associated amplitude, is that which has the same Lorentz structure as the original vertex in the QCD Lagrangian. By renormalizability it is this channel which contains the poles in ϵ which have to be removed in the scheme of interest. Indeed the definition of the MOM type schemes is to render the $O(a)$ corrections absent after subtraction in the channel, or channels in the case of the triple gluon vertex, with the divergences. The amplitudes of the remaining channels do not necessarily have no $O(a)$ contributions. As ultimately they do not affect the β-functions but instead will be useful for lattice measurements of these Green's functions we will record them in a longer article, [23], where the full tensor basis will be provided.

To evaluate each of the amplitudes we use the method of projection where we determine the linear combinations of the basis tensors which in turn determine the amplitude of interest. This combination is then applied to the vertex and the resulting scalar integral is evaluated by using the Laporta algorithm, [22]. The major working tool in this respect was the use of the Reduze package, [25], which uses the symbolic manipulation formalism of Ginac, [26], which itself is written in C++. In order to handle the tedious algebra which arises

with the rearrangement of the expressions resulting from each projection to a form in which the Laporta algorithm is applied we used the symbolic manipulation language Form [8]. The underlying Feynman diagrams were generated using the Qgraf package, [27], before the indices were appended and the Feynman rules substituted. For each of the vertices in turn, triple gluon, quark–gluon and ghost–gluon, there were 8, 2 and 2 one loop diagrams. At two loops there were respectively 106, 33 and 33 graphs to determine. To perform the renormalization itself in the various schemes we followed the method of [21] where all the diagrams are determined in terms of bare parameters which here are the coupling constant and gauge parameter of the linear gauge fixing. Then the renormalized variables are introduced by rescaling with the renormalization constants. The advantage of this approach is that one needs only to generate the results for each of the Green's functions once and then the renormalization constants for each scheme are deduced as simple corollaries without having to regenerate the expressions for each individual diagram.

Once each of the Green's functions of (1) has been renormalized in their respective schemes then the β-function in each scheme is determined from, [10],

$$\beta^{\mathrm{MOMi}}(a_{\mathrm{MOMi}},\alpha_{\mathrm{MOMi}}) = \left[\beta^{\overline{\mathrm{MS}}}(a_{\overline{\mathrm{MS}}})\frac{\partial a_{\mathrm{MOMi}}}{\partial a_{\overline{\mathrm{MS}}}} + \alpha_{\overline{\mathrm{MS}}}\gamma_\alpha^{\overline{\mathrm{MS}}}(a_{\overline{\mathrm{MS}}},\alpha_{\overline{\mathrm{MS}}})\frac{\partial a_{\mathrm{MOMi}}}{\partial \alpha_{\overline{\mathrm{MS}}}}\right]_{\overline{\mathrm{MS}}\to\mathrm{MOMi}} \tag{4}$$

where i represents ggg, q or h and $\gamma_\alpha(a,\alpha)$ is the anomalous dimension of the linear covariant gauge parameter. The scheme the renormalization group functions and variables are in are denoted by the labels. Each of these β-functions requires the relation of the parameters a and α in each MOM scheme to the same parameters in the $\overline{\mathrm{MS}}$ scheme. The identification mapping indicated on the right-hand side is a reminder that the variables are initially in the $\overline{\mathrm{MS}}$ scheme and must then be mapped back to their MOMi counterparts. To achieve this we define the relations between both parameters by

$$\begin{aligned}
a_{\mathrm{MOMggg}}(\mu) &= a_{\overline{\mathrm{MS}}}(\mu)\left[\frac{\Pi_g^{\mathrm{MOMggg}}(p)}{\Pi_g^{\overline{\mathrm{MS}}}(p)}\right]^3\Bigg|_{p^2=-\mu^2}\left[\frac{\Sigma^{\mathrm{ggg}}_{(1)\overline{\mathrm{MS}}}(-\mu^2,-\mu^2)}{\Sigma^{\mathrm{ggg}}_{(1)\mathrm{MOMggg}}(-\mu^2,-\mu^2)}\right]^2 \\
a_{\mathrm{MOMq}}(\mu) &= a_{\overline{\mathrm{MS}}}(\mu)\left[\frac{\Pi_g^{\mathrm{MOMq}}(p)(\Sigma_q^{\mathrm{MOMq}}(p))^2}{\Pi_g^{\overline{\mathrm{MS}}}(p)(\Sigma_q^{\overline{\mathrm{MS}}}(p))^2}\right]\Bigg|_{p^2=-\mu^2}\left[\frac{\Sigma^{\mathrm{qqg}}_{(1)\overline{\mathrm{MS}}}(-\mu^2,-\mu^2)}{\Sigma^{\mathrm{qqg}}_{(1)\mathrm{MOMq}}(-\mu^2,-\mu^2)}\right]^2 \\
a_{\mathrm{MOMh}}(\mu) &= a_{\overline{\mathrm{MS}}}(\mu)\left[\frac{\Pi_g^{\mathrm{MOMh}}(p)(\Sigma_c^{\mathrm{MOMh}}(p))^2}{\Pi_g^{\overline{\mathrm{MS}}}(p)(\Sigma_c^{\overline{\mathrm{MS}}}(p))^2}\right]\Bigg|_{p^2=-\mu^2}\left[\frac{\Sigma^{\mathrm{gcc}}_{(1)\overline{\mathrm{MS}}}(-\mu^2,-\mu^2)}{\Sigma^{\mathrm{gcc}}_{(1)\mathrm{MOMh}}(-\mu^2,-\mu^2)}\right]^2
\end{aligned} \tag{5}$$

where ggg, qqg and gcc indicate the respective triple gluon, quark–gluon and ghost–gluon vertices and

$$\alpha_{\mathrm{MOMi}}(\mu) = \frac{Z^{\mathrm{MOMi}}}{Z^{\overline{\mathrm{MS}}}}\alpha_{\overline{\mathrm{MS}}}(\mu). \tag{6}$$

In the coupling constant definitions the contributions from the quark and Faddeev–Popov ghost 2-point functions are $\Sigma_q(p)$ and $\Sigma_c(p)$ respectively whilst $\Pi_g(p)$ is the contribution to the transverse part of the gluon 2-point function. In (5) we have included the contribution from the MOMi scheme Green's functions in order that the overall normalization is consistent even though there are no $O(a)$ corrections in their expansion. Also in constructing the mapping between the parameters a and α between schemes, (5) and (6) are solved iteratively.

As a consequence of this we can now record our main result which is the three loop expressions for the β-functions, $\beta^{\mathrm{MOMi}}(a,\alpha)$, in each of the three MOM schemes. Though due to space considerations we will only record the Landau gauge expressions. The full gauge dependent results will be provided in [23]. With the convention that the scheme given in the superscript on the left-hand side is the scheme the variables are defined in via (5) and (6), then we have

$$\begin{aligned}
\beta^{\mathrm{MOMggg}}(a,0) = &-\left[\frac{11}{3}C_A - \frac{4}{3}T_F N_f\right]a^2 - \left[\frac{34}{3}C_A^2 - 4C_F T_F N_f - \frac{20}{3}C_A T_F N_f\right]a^3 \\
&+\left[\left[209484\left(\psi'\left(\frac{1}{3}\right)\right)^2 - 279312\pi^2\psi'\left(\frac{1}{3}\right) + 37087200\psi'\left(\frac{1}{3}\right)\right.\right. \\
&+ 368874\psi'''\left(\frac{1}{3}\right) + 266359104 s_2\left(\frac{\pi}{6}\right) - 532718208 s_2\left(\frac{\pi}{2}\right) \\
&- 443931840 s_3\left(\frac{\pi}{6}\right) + 355145472 s_3\left(\frac{\pi}{2}\right) - 890560\pi^4 \\
&- 24724800\pi^2 + 416988\Sigma - 30440124\zeta(3) - 51650217 \\
&\left.\left.+ 1849716\frac{\ln^2(3)\pi}{\sqrt{3}} - 22196592\frac{\ln(3)\pi}{\sqrt{3}} - 1986732\frac{\pi^3}{\sqrt{3}}\right]C_A^3\right. \\
&+\left[1656000\pi^2\psi'\left(\frac{1}{3}\right) - 1242000\left(\psi'\left(\frac{1}{3}\right)\right)^2 - 38988864\psi'\left(\frac{1}{3}\right)\right. \\
&- 134136\psi'''\left(\frac{1}{3}\right) - 220029696 s_2\left(\frac{\pi}{6}\right) + 440059392 s_2\left(\frac{\pi}{2}\right) \\
&\left.+ 366716160 s_3\left(\frac{\pi}{6}\right) - 293372928 s_3\left(\frac{\pi}{2}\right) - 194304\pi^4\right.
\end{aligned}$$

$$
\begin{aligned}
&+ 25992576\pi^2 - 8363088\Sigma + 43914960\zeta(3) + 49845132 \\
&- 1527984\frac{\ln^2(3)\pi}{\sqrt{3}} + 18335808\frac{\ln(3)\pi}{\sqrt{3}} + 1641168\frac{\pi^3}{\sqrt{3}}\Bigg] C_A^2 T_F N_f \\
&+ \Bigg[2045952\left(\psi'\left(\frac{1}{3}\right)\right)^2 - 2727936\pi^2\psi'\left(\frac{1}{3}\right) + 11591424\psi'\left(\frac{1}{3}\right) \\
&+ 44789760 s_2\left(\frac{\pi}{6}\right) - 89579520 s_2\left(\frac{\pi}{2}\right) - 74649600 s_3\left(\frac{\pi}{6}\right) \\
&+ 59719680 s_3\left(\frac{\pi}{2}\right) + 909312\pi^4 - 7727616\pi^2 \\
&+ 2985984\Sigma - 11943936\zeta(3) - 8460288 \\
&+ 311040\frac{\ln^2(3)\pi}{\sqrt{3}} - 3732480\frac{\ln(3)\pi}{\sqrt{3}} - 334080\frac{\pi^3}{\sqrt{3}}\Bigg] C_A T_F^2 N_f^2 \\
&+ \Bigg[786432\pi^2\psi'\left(\frac{1}{3}\right) - 589824\left(\psi'\left(\frac{1}{3}\right)\right)^2 - 442368\psi'\left(\frac{1}{3}\right) \\
&- 262144\pi^4 + 294912\pi^2 - 82944\Bigg] T_F^3 N_f^3 \\
&+ \Bigg[4758912\psi'\left(\frac{1}{3}\right) - 456192\psi'''\left(\frac{1}{3}\right) + 1216512\pi^4 - 3172608\pi^2 \\
&+ 16422912\Sigma - 24634368\zeta(3) + 23421312\Bigg] C_A C_F T_F N_f \\
&+ \Bigg[165888\psi'''\left(\frac{1}{3}\right) - 442368\pi^4 - 5971968\Sigma + 8957952\zeta(3) \\
&- 7091712\Bigg] C_F T_F^2 N_f^2 - 839808 C_F^2 T_F N_f \Bigg] \frac{a^4}{419904} + O\left(a^5\right)
\end{aligned} \tag{7}
$$

$$
\begin{aligned}
\beta^{\text{MOMq}}(a,0) = &-\left[\frac{11}{3}C_A - \frac{4}{3}T_F N_f\right]a^2 - \left[\frac{34}{3}C_A^2 - 4C_F T_F N_f - \frac{20}{3}C_A T_F N_f\right]a^3 \\
&+ \Bigg[\Bigg[203148\pi\sqrt{3}\ln(3) - 16929\pi\sqrt{3}\ln^2(3) + 18183\pi^3\sqrt{3} + 214434\left(\psi'\left(\frac{1}{3}\right)\right)^2 \\
&- 285912\pi^2\psi'\left(\frac{1}{3}\right) - 120798\psi'\left(\frac{1}{3}\right) - 23463\psi'''\left(\frac{1}{3}\right) - 7313328 s_2\left(\frac{\pi}{6}\right) \\
&+ 14626656 s_2\left(\frac{\pi}{2}\right) + 12188880 s_3\left(\frac{\pi}{6}\right) - 9751104 s_3\left(\frac{\pi}{2}\right) + 157872\pi^4 \\
&+ 80532\pi^2 + 598752\Sigma + 1812294\zeta(3) - 10781910\Bigg] C_A^3 \\
&+ \Bigg[49896\pi\sqrt{3}\ln^2(3) - 598752\pi\sqrt{3}\ln(3) - 53592\pi^3\sqrt{3} - 465696\left(\psi'\left(\frac{1}{3}\right)\right)^2 \\
&+ 620928\pi^2\psi'\left(\frac{1}{3}\right) + 5855328\psi'\left(\frac{1}{3}\right) - 14256\psi'''\left(\frac{1}{3}\right) + 21555072 s_2\left(\frac{\pi}{6}\right) \\
&- 43110144 s_2\left(\frac{\pi}{2}\right) - 35925120 s_3\left(\frac{\pi}{6}\right) + 28740096 s_3\left(\frac{\pi}{2}\right) - 168960\pi^4 \\
&- 3903552\pi^2 - 513216\zeta(3) - 478224\Bigg] C_A^2 C_F \\
&+ \Bigg[1404\pi\sqrt{3}\ln^2(3) - 16848\pi\sqrt{3}\ln(3) - 1508\pi^3\sqrt{3} - 77976\left(\psi'\left(\frac{1}{3}\right)\right)^2 \\
&+ 103968\pi^2\psi'\left(\frac{1}{3}\right) - 1176984\psi'\left(\frac{1}{3}\right) + 18036\psi'''\left(\frac{1}{3}\right) + 606528 s_2\left(\frac{\pi}{6}\right) \\
&- 1213056 s_2\left(\frac{\pi}{2}\right) - 1010880 s_3\left(\frac{\pi}{6}\right) + 808704 s_3\left(\frac{\pi}{2}\right) - 82752\pi^4 \\
&+ 784656\pi^2 - 217728\Sigma + 1735992\zeta(3) + 9399240\Bigg] C_A^2 T_F N_f
\end{aligned}
$$

$$
\begin{aligned}
&+\left[9504\pi\sqrt{3}\ln^2(3)-114048\pi\sqrt{3}\ln(3)-10208\pi^3\sqrt{3}+266112\left(\psi'\left(\frac{1}{3}\right)\right)^2\right.\\
&-354816\pi^2\psi'\left(\frac{1}{3}\right)-4162752\psi'\left(\frac{1}{3}\right)+85536\psi'''\left(\frac{1}{3}\right)+4105728s_2\left(\frac{\pi}{6}\right)\\
&-8211456s_2\left(\frac{\pi}{2}\right)-6842880s_3\left(\frac{\pi}{6}\right)+5474304s_3\left(\frac{\pi}{2}\right)-109824\pi^4\\
&\left.+2775168\pi^2-342144\Sigma-4790016\zeta(3)+1283040\right]C_AC_F^2\\
&+\left[217728\pi\sqrt{3}\ln(3)-18144\pi\sqrt{3}\ln^2(3)+19488\pi^3\sqrt{3}+169344\left(\psi'\left(\frac{1}{3}\right)\right)^2\right.\\
&-225792\pi^2\psi'\left(\frac{1}{3}\right)-1778112\psi'\left(\frac{1}{3}\right)+5184\psi'''\left(\frac{1}{3}\right)-7838208s_2\left(\frac{\pi}{6}\right)\\
&+15676416s_2\left(\frac{\pi}{2}\right)+13063680s_3\left(\frac{\pi}{6}\right)-10450944s_3\left(\frac{\pi}{2}\right)+61440\pi^4\\
&\left.+1185408\pi^2-3919104\zeta(3)+3584736\right]C_AC_FT_FN_f\\
&+\left[1728\pi\sqrt{3}\ln^2(3)-20736\pi\sqrt{3}\ln(3)-1856\pi^3\sqrt{3}\right.\\
&+490752\psi'\left(\frac{1}{3}\right)-3456\psi'''\left(\frac{1}{3}\right)+746496s_2\left(\frac{\pi}{6}\right)\\
&-1492992s_2\left(\frac{\pi}{2}\right)-1244160s_3\left(\frac{\pi}{6}\right)+995328s_3\left(\frac{\pi}{2}\right)\\
&\left.+9216\pi^4-327168\pi^2-870912\zeta(3)-1757376\right]C_AT_F^2N_f^2\\
&+\left[41472\pi\sqrt{3}\ln(3)-3456\pi\sqrt{3}\ln^2(3)+3712\pi^3\sqrt{3}-96768\left(\psi'\left(\frac{1}{3}\right)\right)^2\right.\\
&+129024\pi^2\psi'\left(\frac{1}{3}\right)+1389312\psi'\left(\frac{1}{3}\right)-31104\psi'''\left(\frac{1}{3}\right)-1492992s_2\left(\frac{\pi}{6}\right)\\
&+2985984s_2\left(\frac{\pi}{2}\right)+2488320s_3\left(\frac{\pi}{6}\right)-1990656s_3\left(\frac{\pi}{2}\right)+39936\pi^4\\
&\left.-926208\pi^2+124416\Sigma+1741824\zeta(3)+513216\right]C_F^2T_FN_f\\
&+\left[55296\pi^2-82944\psi'\left(\frac{1}{3}\right)+1492992\zeta(3)-1057536\right]C_FT_F^2N_f^2N_f\Big]\frac{a^4}{69984}\\
&+O\left(a^5\right)
\end{aligned}
\tag{8}
$$

and

$$
\begin{aligned}
\beta^{\mathrm{MOMh}}(a,0)=&-\left[\frac{11}{3}C_A-\frac{4}{3}T_FN_f\right]a^2-\left[\frac{34}{3}C_A^2-4C_FT_FN_f-\frac{20}{3}C_AT_FN_f\right]a^3\\
&+\left[\left[97416\pi\sqrt{3}\ln^2(3)-1168992\pi\sqrt{3}\ln(3)-104632\pi^3\sqrt{3}+14850\left(\psi'\left(\frac{1}{3}\right)\right)^2\right.\right.\\
&-19800\pi^2\psi'\left(\frac{1}{3}\right)+7112448\psi'\left(\frac{1}{3}\right)+35343\psi'''\left(\frac{1}{3}\right)+42083712s_2\left(\frac{\pi}{6}\right)\\
&-84167424s_2\left(\frac{\pi}{2}\right)-70139520s_3\left(\frac{\pi}{6}\right)+56111616s_3\left(\frac{\pi}{2}\right)-87648\pi^4\\
&\left.-4741632\pi^2-85536\Sigma-1689336\zeta(3)-35200008\right]C_A^3\\
&+\left[881280\pi\sqrt{3}\ln(3)-73440\pi\sqrt{3}\ln^2(3)+78880\pi^3\sqrt{3}-5400\left(\psi'\left(\frac{1}{3}\right)\right)^2\right.\\
&+7200\pi^2\psi'\left(\frac{1}{3}\right)-5593536\psi'\left(\frac{1}{3}\right)-12852\psi'''\left(\frac{1}{3}\right)-31726080s_2\left(\frac{\pi}{6}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ 63452160 s_2\left(\frac{\pi}{2}\right) + 52876800 s_3\left(\frac{\pi}{6}\right) - 42301440 s_3\left(\frac{\pi}{2}\right) + 31872\pi^4 \\
&+ 3729024\pi^2 + 31104\Sigma + 11562912\zeta(3) + 29167776\Big] C_A^2 T_F N_f \\
&+ \Big[13824\pi\sqrt{3}\ln^2(3) - 165888\pi\sqrt{3}\ln(3) - 14848\pi^3\sqrt{3} \\
&+ 1057536\psi'\left(\frac{1}{3}\right) + 5971968 s_2\left(\frac{\pi}{6}\right) - 11943936 s_2\left(\frac{\pi}{2}\right) - 9953280 s_3\left(\frac{\pi}{6}\right) \\
&+ 7962624 s_3\left(\frac{\pi}{2}\right) - 705024\pi^2 - 3981312\zeta(3) - 4105728\Big] C_A T_F^2 N_f^2 \\
&+ \Big[103680\pi^2 - 155520\psi'\left(\frac{1}{3}\right) - 16422912\zeta(3) + 18817920\Big] C_A C_F T_F N_f \\
&+ \big[5971968\zeta(3) - 5723136\big] C_F T_F^2 N_f^2 - 559872 C_F^2 T_F N_f \Big] \frac{a^4}{279936} \\
&+ O\left(a^5\right)
\end{aligned}
\tag{9}
$$

where $\psi(z)$ is the derivative of the logarithm of the Euler Γ-function, N_f is the number of massless quarks and C_F, C_A and T_F are the usual colour group Casimirs. Aside from the usual class of numbers which appear in the renormalization group functions such as rationals and the Riemann zeta function, $\zeta(z)$, numbers deriving from the basic one and two loop scalar master diagrams computed in [11–14] also occur which are

$$
s_n(z) = \frac{1}{\sqrt{3}} \Im\left[\mathrm{Li}_n\left(\frac{e^{iz}}{\sqrt{3}} \right) \right], \qquad \Sigma = \mathcal{H}^{(2)}_{31} + \mathcal{H}^{(2)}_{43} \tag{10}
$$

where $\mathrm{Li}_n(z)$ is the polylogarithm function and Σ is a particular combination of harmonic polylogarithms [14,28].

Having computed the three loop β-functions it is worthwhile evaluating them numerically in order to compare with [10]. We find

$$
\begin{aligned}
\beta^{\mathrm{MOMggg}}(a,0) &= -(11.0000000 - 0.6666667 N_f)a^2 - (102.0000000 - 12.6666667 N_f)a^3 \\
&\quad - \left(1570.9843804 + 0.5659290 N_f - 67.0895364 N_f^2 + 2.6581155 N_f^3\right)a^4 \\
&\quad + O\left(a^5\right) \\
\beta^{\mathrm{MOMq}}(a,0) &= -(11.0000000 - 0.6666667 N_f)a^2 - (102.0000000 - 12.6666667 N_f)a^3 \\
&\quad - \left(1843.6527285 - 588.6548455 N_f + 22.5878118 N_f^2\right)a^4 + O\left(a^5\right) \\
\beta^{\mathrm{MOMh}}(a,0) &= -(11.0000000 - 0.6666667 N_f)a^2 - (102.0000000 - 12.6666667 N_f)a^3 \\
&\quad - \left(2813.4929484 - 617.6471542 N_f + 21.5028181 N_f^2\right)a^4 + O\left(a^5\right)
\end{aligned}
\tag{11}
$$

for $SU(3)$. Clearly the expressions have the same structure from the point of view of the N_f dependence as [10]. As the term in the polynomial in N_f with the largest error estimate in [10] was the N_f independent term, then we can use that term for comparison. Therefore, taking the ratio of the central value of the estimates in [10] to those of the exact expressions, (7), (8) and (9), we find that the percentage errors are 2.227%, 0.184% and 1.954% for MOMggg, MOMq and MOMh respectively. Clearly MOMq is the most accurate whilst MOMggg was the worst. Though it is worth noting that whilst the N_f independent coefficient of the MOMggg three loop term had a relatively large 8% error estimate the exact result was comfortably within this. Also there was a large uncertainty for the N_f coefficient in this scheme but we find the same sign where the actual coefficient turns out to be very small. Indeed this probably reflects the accidental cancellation noted in [10] for this term. We obtain the precise coefficient for the cubic term in N_f in the MOMggg case. Despite the larger discrepancy for this scheme it is still a testimony to the ingenuity of the authors of [10] and the software and hardware technology of over a decade ago that results were produced which were very close to the exact answer for all three schemes. As [10] also commented on a lattice based operator product expansion prediction, [29], of our benchmark coefficient for the MOMggg scheme in relation to the corresponding $\widetilde{\mathrm{MOM}}$gg scheme coefficient, we note that for comparison we have

$$
\beta_2^{\mathrm{MOMggg}}\big|_{N_f=0} = 0.6512787 \beta_2^{\widetilde{\mathrm{MOMgg}}}\big|_{N_f=0} \tag{12}
$$

which is very close and well within the error of the estimate of 0.64(5) of [10] than the 1.5(3) of [29].

We conclude with brief remarks. We have provided the explicit forms of the QCD β-function in the MOMggg, MOMq and MOMh renormalization schemes in the Landau gauge where those for an arbitrary linear covariant gauge will be provided in [23]. The results for all three schemes are consistent to within a few percent of the numerical estimates of [10] which involved a resummation based on the 2-point function substructure at the symmetric subtraction point. Given this the method of [10] could actually prove useful now for determining a numerical estimate of the *four* loop MOM scheme β-functions. This is partly due to the fact that [10] used Mincer [15], which can handle the three loop computations required to extend the coupling constant mapping to the next order, but also because computer power has increased significantly since the appearance of [10]. Moreover, with analytic results now available for the three MOM schemes, there is independent information which will provide intermediate checks on such numerical computations.

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