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Categories of Partial Maps

E. ROBINSON AND G. ROSOLINI*

Department of Pure Mathematics and Mathematical Statistics, 16 Mill Lane, Cambridge CB2 1SB, England

This paper attempts to reconcile the various abstract notions of "category of partial maps" which appear in the literature. First a particular algebraic theory (p-categories) is introduced and a representation theorem proved. This gives the authors a coherent framework in which to place the various other definitions. Both algebraic theories and theories which make essential use of the poset-enriched structure of partial maps are discussed. Proofs of equivalence are given where possible and counterexamples where known. The paper concludes with brief sections on the representation of partial maps and on partial algebras.

INTRODUCTION

If one looks in a standard work on category theory, such as the classic "Categories for the Working Mathematician," one is hard put to it to find examples of categories where the maps represent partial functions. Almost the whole of category theory is based on the paradigm that the maps of the categories discussed represent total functions, and essential use of this is made in forming the definitions of limits, function spaces, etc. Yet, in many areas, such as topology, analysis, recursion theory, and some parts of the theory of computation, it is quite natural sometimes to consider categories in which the maps represent partial functions. These categories are likely to be, in some sense, better behaved than the corresponding categories of total functions (or there would be no need to consider them), but that sense is most unlikely to mean the possession of considerable conventional categorical structure.

In this paper we shall discuss various attempts to answer the question of what extra structure we need on a category for it to be a "well-behaved" category of partial maps. This is a wheel which has been reinvented many times, and we cannot hope to cover all of the incarnations in the detail they deserve. We have decided to concentrate our discussion at about the level of a category with products. The reader will see that this is the level at

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which we start to have some real structure to work with, and that this is reflected in the existence of comparison theorems between the different models proposed by various authors. Below this level there seems to be little of any real substance that can be said. Most definitions of an arbitrary category of partial maps seem to be there in order to introduce the language the author will want to use later, rather than for any serious discussion of their structure. Moreover, this level appears natural from the point of view of the motivating examples.

Attempts to reproduce at least some of the structures of classical category theory in the context of partial maps have come from two main sources. One of these is mainly topological (see Booth and Brown, 1978) and the other arises from the theory of computation. Most formalisms for computation (including all current programming languages, and the untyped lambda calculus, though not the first or second order typed lambda calculi) allow one to write programs which fail to terminate. In the lambda calculus this is represented by the failure of the relevant terms to have normal forms. This means that if we wish to interpret the expressions in computer programs as operations on data types, then those operations are necessarily partial. This in turn means that we can no longer describe data types via conventional categorical structure. For example, the record types in Pascal are implemented as tuples. They clearly correspond to a cartesian product. However, in the presence of non-terminating expressions, we do not expect them to be an actual cartesian product in any category that we manage to extract from the semantics of the language. Computer scientists are therefore interested in what kinds of structures of partial maps actually come out of the data types of languages with which they are working. In particular, there is a fairly extensive literature based around an attempt to use the category of sets and partial functions to give a semantics for the data types of various languages. On a rather more sophisticated level (at least technically), we have also the work of Moggi (1985) and of Plotkin (1986). Moggi considers abstract categories of partial maps, and relates them to various versions of the lambda calculus. Plotkin concentrates on the type theory and uses it to consider the effect of different disciplines of value passing. This is fascinating work, but in both cases extends well beyond the rather limited scope of this paper. Here we shall only be interested in setting out the abstract categorical ideas that can be viewed as underpinning these further investigations.

We begin with a discussion of the general categorical notion of a partial map as a map defined on a subobject. The fact that we can write a partial map in this way has frequently been used by category theorists as an excuse for not thinking about partial maps—it gives a machine for translating between properties of the category of partial maps and properties of the category of total maps. It is not, however, as good a machine as is generally supposed, and the reason for this is that we very rarely wish to consider the full category of partial maps. In the case of topological spaces we may wish to consider only those defined on open subsets, and in the case of partial maps on posets it may be only those defined on upward-closed subsets. There is no reason at all to suppose that the admissible domains should have an easy or perspicuous categorical definition. There are two relatively simple solutions to this: either we allow the admissible domains as extra information, or we can adopt a very different but extremely elegant solution due ultimately to Heller and assume that we are given the product structure on the category of total maps. Following the second route we arrive at the notion of p-category, which seems to be a convenient way of coding up the algebraic side of the theory of partial maps. We can make the latter statement precise by proving a representation theorem: any p-category can be fully and faithfully embedded in a genuine category of partial maps preserving the product structure on the category of total maps. We give a simple condition for this embedding to be an equivalence.

Having set up this basic algebraic machinery we can then compare the various other essentially algebraic approaches, such as those of DiPaola and Heller (1986) and Longo and Moggi (1984), before moving on to a discussion of some more sophisticated approaches which take as basic the extension ordering on partial functions (which, incidentically, is also definable in terms of the "product" structure). Looking at things from this side, we are concerned with certain categories in which the hom-sets carry a partial order, for which composition is continuous, or to put it more technically, with certain categories enriched over the category of partially ordered sets, and so with 2-categories. The theory of 2-categories is a large and well-developed area, but one which is often concerned with a vastly more general situation than ours, and which, coming mainly from Australia, is a culture often inaccessible to Europeans. Nevertheless, we feel that we should not ignore this general theory out of an idea that partial maps are simple and 2-categories are not. Thus we have tried to include the briefest possible idiots guide to some of the terminology of 2-categories in the case of categories enriched over posets, and to indicate how the approaches of Hoehnke and his collaborators (cf. Hoehnke, 1977a) and Curien and Obtułowicz (1986) fit into this general theory. This is not necessary in the case of Carboni (1986), who knows vastly more about 2-categories than either of the authors.

We conclude the paper with a discussion of the representations of partial maps and a brief note on algebra in categories of partial maps. Although by this time we will have seen many inequivalent approaches to categories with less structure, all authors agree on what it means to have partial function spaces. There is, however, no obviously best form for the rather inelegant definition that is necessary. We demonstrate the equivalence of three forms that together cover the range of possibility.

This paper then is very much a survey of and comparison between other people's work on categories of partial maps. The original part of the paper is the use of a coherent categorical framework and the demonstration of the connections between the various approaches discussed. The authors feel that they, in common with most others writers in this area, have not yet exploited the full power of this abstraction.

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1. An Overview

The first step in the categorical abstraction of partial functions is to consider a category **A** and to define a partial map between objects A and B of **A** as a pair $(m\colon D \rightarrowtail A,\ \phi\colon D \to B)$ consisting of a monomorphism and a map in **A**. However, one immediately recognizes that if the map $j\colon E \to D$ is an isomorphism, then essentially the same information is supplied by the pair $(mj\colon E \rightarrowtail A,\ \phi j\colon E \to B)$ (we have an isomorphism over both A and B). Hence we define officially a *partial map* $[m, \phi]\colon A \rightharpoonup B$ between objects of **A** as an equivalence class of pairs $(m\colon D \rightarrowtail A,\ \phi\colon D \to B)$ with respect to isomorphic variations of D. It follows that given a partial map $[m, \phi]\colon A \rightharpoonup B$, its *domain of definition* can only be determined up to isomorphism, and so is the subobject $[m\colon D \rightarrowtail A]$ represented by m.

Next one aims to define a "category" Ptl(**A**) of partial maps on **A** by letting the composition of $[m, \phi]\colon A \rightharpoonup B$ and $[n, \psi]\colon B \rightharpoonup C$ be defined as the equivalence classes determined by the outmost sides in

$$\begin{tikzcd}
\phi^{-1}E \arrow[r] \arrow[d, tail] & E \arrow[r, "\psi"] \arrow[d, tail, "n"] & C \\
D \arrow[r, "\phi"] \arrow[d, tail, "m"'] & B & \\
A & &
\end{tikzcd}$$

where the square is a pullback. To ensure that this exists, **A** must have inverse images (= pullbacks of monos); but given that, it is easy to prove that composition is independent of the choice of representatives and that the identity maps are $[\mathrm{id}, \mathrm{id}]\colon A \rightharpoonup A$.

A small foundational problem arises when considering the size of the hom-sets of Ptl(**A**). One can easily check that the hom-sets of Ptl(**A**) are indeed sets if and only if the collections Sub(A) are sets for every object A in **A**. This is the case in most examples met with in practice, but is not always true (consider for example $\mathbf{On}^{\mathrm{op}}$, the opposite of the linearly ordered collection of ordinals). We shall, however, ignore this problem.

There is a faithful functor from **A** into Ptl(**A**). Define $F\colon \mathbf{A} \to \mathrm{Ptl}(\mathbf{A})$ as the identity on objects and to take a map $f\colon A \to B$ in **A** to $[\mathrm{id}, f]\colon A \rightharpoonup B$. Unimaginatively, we call a map in the image of F *total* and confuse **A** with the subcategory of Ptl(**A**) consisting of total maps.

It is often the case, though, that Ptl(**A**) is too big for one's purposes. This situation generally arises because the class of monics in the category **A** is too wide. The problem is particularly evident in the case of **Top**, the category of topological spaces, where monics are not, in general, even subspace inclusions and where we wish to look at the continuous functions defined on an *open subspace*; but it is clear also in the case of partial monotonic functions between posets defined on an *upward-closed subset.* We want to consider possible restrictions on the choice of the domains of definition. In order to do that, take a class $\mathscr{M}$ of subobjects and consider only the partial maps in Ptl(**A**) whose domain of definition is in $\mathscr{M}$. When these maps form a subcategory of Ptl(**A**) containing all objects of **A**, we call the class $\mathscr{M}$ *admissible*, and denote the category by $\mathscr{M}$-Ptl(**A**).

1.1. PROPOSITION. *A class $\mathscr{M}$ of subobjects in* **A** *is admissible, and hence the category $\mathscr{M}$-*Ptl(**A**) *exists, if and only if $\mathscr{M}$ satisfies the following conditions*:

(i) $[\mathrm{id}\colon A \rightarrowtail A]$ *is in $\mathscr{M}$ for every A in* **A**;

(ii) *if* $[m\colon D \rightarrowtail A]$ *and* $[n\colon E \rightarrowtail D]$ *are in $\mathscr{M}$, then* $[mn\colon E \rightarrowtail A]$ *is in $\mathscr{M}$*;

(iii) *if* $[m\colon D \rightarrowtail A]$ *is in $\mathscr{M}$ and $f\colon B \to A$ is any map in* **A**, *then* $f^{-1}[m\colon D \rightarrowtail A]$ *is in $\mathscr{M}$.*

Proof. Trivial. ∎

Notice that in order to define the category $\mathscr{M}$-Ptl(**A**), it is not necessary that **A** has all inverse images; in other words it is not necessary that Ptl(**A**) actually be a category, but only that **A** contains all inverse images of subobjects in $\mathscr{M}$. In this case, if $\mathscr{M}$ satisfies the three conditions listed in 1.1, one can define directly *$\mathscr{M}$-partial* maps $[m, \phi]\colon A \rightharpoonup B$. They are equivalence classes exactly as above, but with the further condition that $[m\colon D \rightarrowtail A]$ is in $\mathscr{M}$. It is then simple to check that the usual composition defines a category $\mathscr{M}$-Ptl(**A**).

In what follows we shall consider only the case when **A** has binary products. This will provide us sufficient definability power to carry over a study of the "algebraic" properties of such categories of partial maps but leads to no great loss of generality: any category of partial maps can be fully embedded in a category of partial maps on a category with products.

1.2. *Remark.* If **A** has binary products and $\mathcal{M}$ is an admissible class of subobjects in **A**, then $\mathcal{M}$ is closed under product. Indeed, given $m: D \rightarrowtail A$ and $n: E \rightarrowtail B$, notice that a pullback of $m: D \rightarrowtail A$ along the first projection $p: A \times B \to A$ is $m \times \mathrm{id}$: $D \times B \rightarrowtail A \times B$ as well as a pullback of $n: E \rightarrowtail B$ along the second projection $q: D \times B \to B$ is $\mathrm{id} \times n$: $D \times E \rightarrowtail D \times B$. Their composition is $m \times n$: $D \times E \rightarrowtail A \times B$. Hence the equivalence class $[m \times n: D \times E \rightarrowtail A \times B]$ is in $\mathcal{M}$.

An essential property of the product on **A** is that it extends to each category $\mathcal{M}$-Ptl(**A**).

1.3. PROPOSITION. *Suppose* **A** *has binary products and* $\mathcal{M}$ *is admissible, then the product bifunctor* $(-) \times (-)$ *on* **A** *can be extended to the whole of* $\mathcal{M}$-Ptl(**A**).

Proof. Trivial: define $[m, \phi] \times [n, \psi] = [m \times n, \phi \times \psi]$. ∎

Note, however, that $(-) \times (-)$ ceases to be a categorical product in $\mathcal{M}$-Ptl(**A**). Rather than prove this straightforward assertion (which is an entertainment left for the reader) we analyse what goes wrong in $\mathcal{M}$-Ptl(**A**) for the projections. The crucial point is that projections are not natural in *both* variables: for instance, if we let $p_{X,Y}: X \times Y \to X$ be the projection onto the first variable, then we can check that p is always natural in X, but is not natural in Y unless the category is actually a category of total maps (i.e., $\mathcal{M}$ consists solely of isomorphisms). Take $\phi: Y \rightharpoonup Z$ a genuinely partial map, then the composite $p_{X,Z}(\mathrm{id}_X \times \phi)$: $X \times Y \rightharpoonup X$ is also genuinely partial. It cannot, therefore, be equal to $\mathrm{id}_X p_{X,Y} = p_{X,Y}$, which is total. However, the diagonal $\Delta: (-) \to (- \times -)$ remains natural under the extension to partial maps, and so do the associativity and commutativity isomorphisms α and τ defined by

$$\alpha_{X,Y,Z} = ((\mathrm{id}_X \times p_{Y,Z}) \times q_{Y,Z} q_{X,Y \times Z}) \Delta_{X \times (Y \times Z)}: X \times (Y \times Z) \to (X \times Y) \times Z$$

and

$$\tau_{X,Y} = (q_{X,Y} \times p_{X,Y}) \Delta_{X \times Y}: X \times Y \to Y \times X$$

which are natural in all variables, even though the components from which they are constructed are not.

The first goal of our exposition is to give an algebraic description of the situation above. We note that, because of the declared algebraic character, a complete axiomatisation for categories of the form $\mathcal{M}$-Ptl(**A**) cannot be given, since any subcategory (closed under the structural operations) will satisfy the same identities.

DEFINITION. A *p-category* is a category **C** endowed with a bifunctor $\times : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ which is called *product*, a natural transformation $\Delta : (-) \to (- \times -)$ which is called the *diagonal* and two families of natural transformations $\{p_{-,Y} : (- \times Y) \to (-) \mid Y \in \mathrm{ob}\,\mathbf{C}\}$ and $\{q_{X,-} : (X \times -) \to (-) \mid X \in \mathrm{ob}\,\mathbf{C}\}$ which are called *projections*, satisfying the identities

$$p_{X,X}\Delta_X = \mathrm{id}_X = q_{X,X}\Delta_X \qquad (p_{X,Y} \times q_{X,Y})\Delta_{X\times Y} = \mathrm{id}_{X\times Y}$$

$$p_{X,Y}(\mathrm{id}_X \times p_{Y,Z}) = p_{X,Y\times Z} \qquad p_{X,Z}(\mathrm{id}_X \times q_{Y,Z}) = p_{X,Y\times Z}$$

$$q_{X,Y}(p_{X,Y} \times \mathrm{id}_Z) = q_{X\times Y,Z} \qquad q_{X,Z}(q_{X,Y} \times \mathrm{id}_Z) = q_{X\times Y,Z}.$$

Finally we require that the associativity and commutativity isomorphisms α and τ defined as above by

$$\alpha_{X,Y,Z} = ((\mathrm{id}_X \times p_{Y,Z}) \times q_{Y,Z}q_{X,Y\times Z})\Delta_{X\times(Y\times Z)} : X \times (Y \times Z) \to (X \times Y) \times Z$$

and

$$\tau_{X,Y} = (q_{X,Y} \times p_{X,Y})\Delta_{X\times Y} : X \times Y \to Y \times X$$

are natural in all variables—though again, their components need not be.

We again leave the reader to check that the extension of the product from **A** to $\mathcal{M}$-Ptl(**A**) induces canonically a structure of p-category on $\mathcal{M}$-Ptl(**A**). Note that in this case the identities between the products, the projections, and the diagonal are all identities between total maps; we shall see later that the projectons and the diagonal and the product of two total maps are all total in any p-category, and hence that the identities hold iff they hold in the subcategory of total maps and that they force the bifunctor $\times$ to be a product there. We also note that the requirements in Definition 1 are slightly redundant, the naturality of each projection can be made to follow in the context of the other axioms from the naturality of the other, as well as some of the identities which involve the projections. In this context we prefer symmetry to minimalism. Finally note that the p-category structure (unlike conventional product structure) is definitely an extra structure on the underlying category and not a property of it. The same category can carry more than one p-category structure. The obvious example is that of Ptl(**Set**) which not only has the p-category structure obtained from being the category of partial functions on the category **Sets**, but also the trivial p-category structure obtained from its being a category with finite products in its own right.

As we already said, p-categories are designed to be an algebraic abstraction for categories of the form $\mathcal{M}$-Ptl(**A**), and as such they include at least all subcategories of the $\mathcal{M}$-Ptl(**A**) which are closed under products of maps and under projections and diagonals. They are not though necessarily full on objects, and so we may have a p-category built as a subcategoy of $\mathcal{M}$-Ptl(**A**) which includes a partial map $[m, \phi]$, but which includes no representative of $[m]$, the subobject represented by m. Given this, we do the best that we can; we sketch an argument that shows that an arbitrary p-category is a *full* subcategory of a category of the form $\mathcal{D}$-Ptl(**D**). We shall also give a "second-order" characterisation of the categories $\mathcal{M}$-Ptl(**A**) in terms of their p-structure. The category **D** we shall use is built on domains which are defined below (a definition which stems from DiPaola and Heller, 1986). Heller's idea is to replace the subobject on which a partial map is defined by a partial endomorphism of its source, in fact, by that subfunction of the identity which is defined precisely on the domain of definition of the partial map. This bizarre-seeming idea has in fact several attractive features for the category theorist. First, it allows us to use a genuine map to replace an equivalence class, and, second, the maps involved are idempotents. After Freyd (1974), no category theorist can see an idempotent without feeling the urge to split it, and indeed the construction of **D** is a particular case.

DEFINITION. Given a map $\beta\colon X \to Y$ in the p-category **C**, the *domain* $\operatorname{dom}\beta\colon X \to X$ of β is the composite map $p_{X,Y}(\mathrm{id}_X \times \beta)\,\Delta_X\colon X \to X$.

It is easy to see that in the p-category $\mathcal{M}$-Ptl(**A**) the domain of $[m, \phi]$: $A \rightharpoonup B$ is the partial map $[m, m]$: $A \rightharpoonup A$. Obviously, these maps are in a 1–1 correspondence with the subobjects in $\mathcal{M}$, and the proof of the representation theorem in the general case reflects this relationship. We must warn the reader that from now on all indices will be deliberately omitted.

There are two useful rules of thumb to use when dealing with domains:

$$p(\mathrm{id} \times \phi) = p(\mathrm{id} \times \operatorname{dom}\phi) \qquad (\mathrm{id} \times \operatorname{dom}\phi)\,\Delta = \Delta \operatorname{dom}\phi.$$

The first can be interpreted as showing the amount by which naturality of p in the second variable fails; the second is our first formal evidence that domains behave very much like identity maps. The proof of properties such as these is necessarily very simple (as we are working on elementary properties of an algebraic theory):

$$\begin{aligned} p(\mathrm{id} \times \phi) &= p(\mathrm{id} \times \phi q\,\Delta) = p(\mathrm{id} \times q)(\mathrm{id} \times (\mathrm{id} \times \phi)\,\Delta) \\ &= p(\mathrm{id} \times p)(\mathrm{id} \times (\mathrm{id} \times \phi)\,\Delta) = p(\mathrm{id} \times \operatorname{dom}\phi) \end{aligned}$$

$$(\mathrm{id} \times \mathrm{dom}\ \phi)\ \Delta = (p \times \mathrm{id})\ \alpha(\mathrm{id} \times (\mathrm{id} \times \phi)\ \Delta)\ \Delta = p(\mathrm{id} \times \phi)\ \alpha(\mathrm{id} \times \Delta)\ \Delta$$
$$= p(\Delta \times \phi)\ \Delta = \Delta\ \mathrm{dom}\ \phi.$$

More properties are listed in the following proposition.

1.4. PROPOSITION. *The domain operator satisfies the properties*:

(i) $\mathrm{dom}\ \mathrm{id} = \mathrm{id}$, $\mathrm{dom}\ p = \mathrm{id}$, $\mathrm{dom}\ q = \mathrm{id}$, $\mathrm{dom}\ \Delta = \mathrm{id}$

(ii) $\mathrm{dom}(\gamma\beta) = \mathrm{dom}((\mathrm{dom}\ \gamma)\beta)$

(iii) $(\mathrm{dom}\ \gamma)\beta = \beta\ \mathrm{dom}(\gamma\beta)$

(iv) $\mathrm{dom}(\beta \times \gamma) = \mathrm{dom}\ \beta \times \mathrm{dom}\ \gamma$

(v) $\mathrm{dom}\ \beta\ \mathrm{dom}\ \gamma = \mathrm{dom}\ \gamma\ \mathrm{dom}\ \beta = \mathrm{dom}(\mathrm{dom}\ \beta\ \mathrm{dom}\ \gamma)$

(vi) $\mathrm{dom}\ \beta\ \mathrm{dom}\ \beta = \mathrm{dom}\ \beta$.

Proof. The proofs here are almost all one line. A little care is needed over (iv), which comes from the naturality in all variables of the interchange isomorphism

$$(p \times p) \times (q \times q)\colon (X \times Y) \times (Z \times W) \to (X \times Z) \times (Y \times W)$$

which can be obtained as a composition of α's and τ's. ∎

One useful corollary of this lemma is that a map β is a domain ($\beta = \mathrm{dom}\ \gamma$) if and only if $\beta = \mathrm{dom}\ \beta$.

It is clear that domains enjoy properties typical of the domains of definition of partial functions. For instance, because of (v) above, composition is an operation on the set $\mathrm{Dom}(X)$ of domains on X which is associative, commutative, and idempotent with a unit. Hence it induces a partial order on $\mathrm{Dom}(X)$:

$$\mathrm{dom}\ \beta \leqslant \mathrm{dom}\ \gamma \Leftrightarrow \mathrm{dom}\ \beta = \mathrm{dom}\ \gamma\ \mathrm{dom}\ \beta.$$

The idea that $\mathrm{dom}\ \beta \circ \mathrm{dom}\ \gamma$ is the restriction (or intersection) of $\mathrm{dom}\ \gamma$ at $\mathrm{dom}\ \beta$ can be carried forward to the hom-sets $\mathbf{C}(X, Y)$ and we can define the *extension* order on maps from X to Y using

$$\beta \leqslant \gamma \Leftrightarrow \beta = \gamma\ \mathrm{dom}\ \beta.$$

We can express this more concretely in terms of the original p-category operations,

$$\beta \leqslant \gamma \Leftrightarrow p(\gamma \times \beta)\ \Delta = \beta,$$

which gives as an immediate corollary that β is a domain if and only if $\beta \leqslant \mathrm{id}$. We shall not, however, enlarge upon this at this stage. Rather, with the intuition that the domain maps "are" domains of definition we introduce the main characters in the representation theorem.

The category **D** of domains has as objects the maps of **C** of the form $\operatorname{dom}\beta\colon X \to X$. A map $\phi\colon \operatorname{dom}\beta \to \operatorname{dom}\gamma$ in **D** is defined to be a map $\phi\colon X \to Y$ in **C** such that

$$\operatorname{dom}\beta = \operatorname{dom}\phi \qquad \text{and} \qquad \phi = (\operatorname{dom}\gamma)\phi.$$

The idea is, of course, that ϕ is "defined" on $\operatorname{dom}\beta$ and "takes values" in $\operatorname{dom}\gamma$. It is a simple exercise to prove that the data given define a category (composition is composition as in **C** and the identity on $\operatorname{dom}\beta$ is $\operatorname{dom}\beta$ itself). Furthermore, **D** has binary products: the product of $\operatorname{dom}\beta$ and $\operatorname{dom}\gamma$ is $\operatorname{dom}\beta \times \operatorname{dom}\gamma$, which by 1.4(iv) is just $\operatorname{dom}(\beta \times \gamma)$.

The domains also induce an admissible collection of subobjects, but before defining this we need a further lemma.

1.5. LEMMA. *Let $\beta = \operatorname{dom}\beta$, $\gamma = \operatorname{dom}\gamma$, and $\delta = \operatorname{dom}\delta$ be domains on X. Then $\delta\colon \operatorname{dom}\beta \to \operatorname{dom}\gamma$ is a map in* **D** *if and only if $\delta = \beta$ and $\beta \leqslant \gamma$ in* $\operatorname{Dom}(X)$. *In particular, when $\beta \leqslant \gamma$, the map $\operatorname{dom}\beta\colon \operatorname{dom}\beta \to \operatorname{dom}\gamma$ is a monomorphism in* **D**.

Proof. Sufficiency is obvious. So suppose $\delta\colon \operatorname{dom}\beta \to \operatorname{dom}\gamma$. Then $\beta = \operatorname{dom}\beta = \operatorname{dom}\delta = \delta$, and $\gamma\beta = (\operatorname{dom}\gamma)\,\delta = \delta = \beta$. The last assertion is trivial. ∎

Let $\mathscr{D}$ be the class of monomorphisms in **D** of the form $\operatorname{dom}\beta$: $\operatorname{dom}\beta \rightarrowtail \operatorname{dom}\gamma$. The class is clearly closed under identities and composition. It is easy to check that the pullback of $\operatorname{dom}\beta \rightarrowtail \operatorname{dom}\gamma$ along $\phi\colon \operatorname{dom}\delta \to \operatorname{dom}\gamma$ is $\operatorname{dom}(\phi \operatorname{dom}\beta) \rightarrowtail \operatorname{dom}\gamma$. We thus conclude that $\mathscr{D}$ is admissible, and we are at last in a position to give the representation theorem:

1.6. THEOREM. *Suppose* **C** *is a p-category. Then there is a full embedding* $E\colon \mathbf{C} \to \mathscr{D}\text{-}\mathrm{Ptl}(\mathbf{D})$ *which preserves the p-structure.*

Proof. Define EX as $\mathrm{id}_X = \operatorname{dom}\mathrm{id}_X$ and $E(\beta\colon X \to Y)$ as $[\operatorname{dom}\beta, \beta]$: $\mathrm{id}_X \rightharpoonup \mathrm{id}_Y$, noting that in **D** one has $\operatorname{dom}\beta$: $\operatorname{dom}\beta \rightarrowtail id_X$ and $\beta\colon \operatorname{dom}\beta \to \mathrm{id}_Y$. ∎

In essence this theorem is in Freyd (1974). In various disguises it also appears in Schreckenberger (1981), Curien and Obtułowicz (1986), Rosolini (1986), and Carboni (1986). The maps of **C** which are taken by E into $\mathbf{D} \subset \mathscr{D}\text{-}\mathrm{Ptl}(\mathbf{D})$ are those $\phi\colon X \to Y$ such that $\operatorname{dom}\phi = \mathrm{id}$. These *total* maps form a subcateory $\mathbf{C}_t$ of **C**. Note, in particular, that

$$\mathbf{A} \simeq (\mathscr{M}\text{-}\mathrm{Ptl}(\mathbf{A}))_t.$$

We claim that the category $\mathscr{D}$-Ptl(**D**) can be constructed as Split(Dom), the free completion of **C**, where all domain idempotents split (Dom denoted the class of domains of **C**). Recall fom Freyd (1974) that, if I is a class of idempotents in the category **A**, then Split(I) is the category whose objects are the elements of $I \cup \{\mathrm{id}_A \mid A \in \mathrm{ob}\, \mathbf{A}\}$ and where a map $f: e \to d$ is a map f in **A** such that

$$dfe = f.$$

By Lemma 1.5, a map in $\mathscr{D}$-Ptl(**D**) is of the form $[\mathrm{dom}\,\phi, \phi]$: $\mathrm{dom}\,\beta \rightharpoonup \mathrm{dom}\,\gamma$. As the monomorphism $\mathrm{dom}\,\phi \rightarrowtail \mathrm{dom}\,\beta$ is uniquely determined by ϕ, the partial map $[\mathrm{dom}\,\phi, \phi]$: $\mathrm{dom}\,\beta \rightharpoonup \mathrm{dom}\,\gamma$ is defined if and only if

$$\mathrm{dom}\,\phi \leqslant \mathrm{dom}\,\beta \qquad \text{and} \qquad (\mathrm{dom}\,\gamma)\phi = \phi,$$

which is the same as requiring that $(\mathrm{dom}\,\gamma)\phi(\mathrm{dom}\,\beta) = \phi$, proving our assertion. Since a category in which a certain class of idempotents already split is equivalent (though not isomorphic) to the free splitting extension for that class given by the construction above, we have as a corollary.

1.7. THEOREM. *A p-category is of the form* $\mathscr{M}$-Ptl(**A**) *if and only if all domains split.*

Summing up, we know through Theorem 1.6 that a p-category can be regarded as a full sub-category of a category of partial maps, and we know through Theorem 1.7 that what characterises categories of partial maps is that they are complete with respect to domains splitting. Therefore, properties of categories of partial maps correspond to properties of p-categories that are independent of domains splitting. Another way to put this is that the properties we are interested in are those which hold in a p-category **C** if and only if they also hold in $\mathscr{D}$-Ptl(**D**) (since the proof of Theorem 1.7 shows that this is, in fact, the canonical completion of **C** to a category of partial maps). This is the case exactly when the property can be completely characterised in terms of **D** and $\mathscr{D}$.

As a very simple instance of what we mean, the following is a characterisation of a terminal object in $\mathbf{C}_t$ in terms of the p-structure: suppose we are given an object T in **C** and a family of maps $t_X: X \to T$ for all X in **C**. It is an easy calculation to show that the object T is terminal in $\mathbf{C}_t$ if and only if $p_{X,T}$ and $(\mathrm{id}_X \times t_X)\,\Delta$ are mutually inverse. Such a situation will be described in **C** by saying hat **C** has a *one-element* object.

2. DOMINICAL CATEGORIES

These were introduced in DiPaola and Heller (1986) in order to give an abstract categorical presentation for recursion theory. From our point of view, the notion of a dominical category is a slight specialisation of the notion of p-category. Recall that a pointed category is a category $\mathbf{C}$ with a family of distinguished maps $0_{X,Y}\colon X \to Y$ stable under composiion:

$$0_{X,Y} \circ \beta = 0_{Z,Y} \qquad \gamma \circ 0_{X,Y} = 0_{X,W}.$$

Call a map $f\colon X \to Y$ in the pointed category $\mathbf{C}$ *weakly total* if

$$\forall \beta\colon Z \to X[f\beta = 0 \Rightarrow \beta = 0].$$

(DiPaola and Heller, 1986 call them total, but unfortunately we are using that name for something else). Notice that identity maps are weakly total and that a composition of weakly total maps is weakly total. Set $\mathbf{C}'$ for the subcategory of $\mathbf{C}$ consisting of weakly total maps.

DEFINITION. A pointed category $\mathbf{C}$ with a bifunctor $\times\colon \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ is *dominical* if $\times$ maps $\mathbf{C}' \times \mathbf{C}'$ into $\mathbf{C}'$, defining a cartesian product on that subcategory, in such a way that the induced commutativity and associativity isomorphisms are natural with respect to maps in $\mathbf{C}$. Moreover, for all maps β, γ in $\mathbf{C}$

$$\beta \times \gamma = 0 \Leftrightarrow \beta = 0 \vee \gamma = 0$$

$$p(\beta \times \mathrm{id}) = \beta p \qquad (\beta \times \beta)\,\Delta = \Delta\beta.$$

Partly justifying our choice of names, notice that in any pointed category that carries the structure associated with the bifunctor $\times$, and where for every map β,

$$\beta \times 0 = 0,$$

each total map is weakly total. For, if $f\colon X \to Y$ is such that $\operatorname{dom} f = \mathrm{id}$, then, given that $f\beta = 0$, we have

$$\beta = (\operatorname{dom} f)\beta = p(\beta \times f\beta)\,\Delta = 0.$$

2.1. PROPOSITION. *A dominical category* $\mathbf{C}$ *is a* p*-category. Moreover, the categories* $\mathbf{C}'$ *and* $\mathbf{C}_t$ *coincide.*

Proof. The first assertion is trivial as $\times$ is cartesian on $\mathbf{C}'$, and hence p, Δ, and q satisfy all the identities demanded of them by Definition 1. The last two identities of Definition 2 express the naturality of Δ and the naturality of p in the first coordinate. The naturality of q in the second is, as we remarked above, redundant. Now suppose that $f\colon X \to Y$ is weakly total, then $\operatorname{dom} f = p(\mathrm{id} \times f)\,\Delta = p\,\Delta = \mathrm{id}$. Hence f is total, and this yields the conclusion. ∎

It is possible to characterise dominical categories completely in terms of their category of domains. Before we embark on this, let us review what we should expect to find. There have to be empty maps: the domains of the zero maps are all isomorphic and are easily seen to be strict initial in **D**. (Recall that an initial object is said to be strict if the only maps into it are isomorphisms.) It follows that we have to express the fact that there is a strict initial object 0 in **D** and that the subobjects $[0 \rightarrowtail \operatorname{dom} \beta]$ are in $\mathscr{D}$. Second the product in **D** should have a cancellation property on objects which follows from that in the definition of dominical category: one has that

$$\beta = 0 \Leftrightarrow \operatorname{dom} \beta = 0.$$

Finally there should be a property accounting for the fact that weakly total maps are total. As the notion of weakly total map appears to have a non-positive form (more precisely, it is almost negative, allowing a map to be weakly total in a subcategory when it is not in the full category), a property that reflects this in the category **D** may be difficult to express. We begin with the easy part.

2.2. LEMMA. *In a non-empty pointed p-category* **C**, *the product* $\beta \times 0$ *is always* 0 *if and only if the category* **D** *of domains on* **C** *has a strict initial object and all minimal subobjects* $[0 \rightarrowtail \operatorname{dom} \beta]$ *are in* **D**.

Proof. ($\Rightarrow$) Let X be an object of **C** and consider $0: X \to X$. It is a domain, as $0 = p(\mathrm{id} \times 0)\Delta = \operatorname{dom} 0$. Moreover, for every object $\operatorname{dom} \beta$ in **D**, say $\operatorname{dom} \beta: Y \to Y$ in **C**, the map $0: X \to Y$ induces a map $0: 0 \to \operatorname{dom} \beta$ in **D**. Finally, any map with source or target 0 in **D** is equal to 0 because, if $\phi: 0 \to \operatorname{dom} \beta$, then $\phi = \phi 0 = 0$, and if $\psi: \operatorname{dom} \beta \to 0$, then $\psi = 0\psi = 0$. The conclusion follows. ($\Leftarrow$) is obvious. ∎

We shall refer to a p-category **C** satisfying either of the equivalent conditions in the previous lemma by saying that **C** has *zero maps*.

2.3. LEMMA. *Suppose* **C** *is a p-category with zero maps. Then* **C** *satisfies the cancellation property on maps*:

$$\beta \times \gamma = 0 \Rightarrow \beta = 0 \vee \gamma = 0$$

if and only if the category **D** *of domains satisfies the cancellation property on objects*:

$$A \times B \simeq 0 \Rightarrow A \simeq 0 \vee B \simeq 0.$$

Proof. Trivial, when one recalls that $\beta = 0$ if and only if $\operatorname{dom} \beta = 0$. ∎

Corollaries to the previous assertions are that the notion of strict initial object and the cancellation property can be axiomatised in the language of partial maps. This does not give us a complete characterisation of those p-categories which are dominical; if $\mathbf{C}$ is a pointed p-category, then $\mathbf{C}_t \subset \mathbf{C}'$. We can see, either from Proposition 2.1, or perhaps more intuitively if less precisely from the fact that for a p-category $\times$ is a categorical product on $\mathbf{C}_t$, while for a dominical category it is a product on the "rather larger" $\mathbf{C}'$, that for $\mathbf{C}$ to be dominical we must have $\mathbf{C}_t = \mathbf{C}'$, in other words, that every weakly total map is total.

The first thing to do then is to check whether all pointed p-categories are dominical. There is, however, an easy counterexample to this (in fact, a collection of counterexamples). Regarding a partially ordered set as a category in the standard way we know that a poset has binary products if and only if it has binary meets. Let $\mathbf{A}$ be a meet semi-lattice, and consider the full category of partial maps on A (a partial map from a to b is essentially an element less than or equal to their meet). This category $\mathrm{Ptl}(\mathbf{A})$ has a one-element object as defined at the end of Section 1 if and only if $\mathbf{A}$ has a top element, and it is pointed if and only if it has a bottom. By Lemma 2.3, $\mathrm{Ptl}(\mathbf{A})$ satisfies the cancellation property on maps, if and only if the bottom element is prime, i.e., if and only if $a \wedge b = 0$ implies $a = 0$ or $b = 0$. However, in the case that the bottom element is prime, a weakly total map from a to b can be characterised very simply as one that is represented by a non-zero element below $a \wedge b$. It follows that a meet semi-lattice with a prime bottom element gives rise to a category of partial maps and hence to a p-category which has zero maps and the cancellation property on maps, but in which weakly total maps are not necessarily total, and which is therefore not dominical (except in the uninteresting case of the trivial meet semi-lattice).

A different and perhaps more interesting counterexample, in that it is closer to an actual category of partial functions, was suggested by Franco Montagna (cf. Montagna, 1986): it is the syntactic p-category $\mathbf{PA}$ from the theory PA of arithmetic. The category $\mathbf{PA}$ has one object (call it $\mathbb{N}$ if you like). Maps are equivalence classes of Σ_1-formulae $F(x, y)$ of the language of PA, for which PA proves single-valuedness, with respect to provable identity. The composition of $[F(x, y)]$ and $[G(y, z)]$ is represented by the formula

$$\exists y. F(x, y) \wedge G(y, z).$$

It is easy to see that this is a p-category and that the domain of $F(x, y)$ is

$$D(x, x') \equiv \exists y[F(x, y) \wedge x = x'].$$

So that a map $[F(x, y)]$ is total if and only if

$$PA \vdash \forall x \, \exists y . F(x, y)$$

(which is what one expects). On the other hand, a weakly total map $G(x, y)$ need only be such that, for every map $[H(z, x)]$ in **PA**

$$PA \vdash \forall x, y, z . \neg [H(z, x) \wedge G(x, y)] \Rightarrow PA \vdash \forall x, z . \neg H(z, x).$$

It is possible to prove by means of a Δ_2-formula K such that both K and $\neg K$ are Π_1-conservative over PA that there are maps in **PA** which are weakly total, but not total. Hence, by 2.1, the category **PA** is not dominical.

Finally, we give a characterisation of dominical categories that depends on the presentation of the p-category (weak totality depends on which domains split).

2.4. PROPOSITION. *A non-empty pointed p-category* **C** *is dominical if and only if the associated category* **D** *of domains has the properties*:

(i) **D** *has a strict initial object* 0

(ii) **D** *satisfies the cancellation property on objects*

(iii) *all subobjects* $[0 \rightarrowtail \mathrm{dom}\, \beta]$ *are in* $\mathcal{D}$, *the class of admissible domains*

(iv) *if a subobject* $[\mathrm{dom}\, \beta \rightarrowtail id_X]$ *is not maximal, then there is a non-zero map* $\gamma : \mathrm{dom}\, \gamma \to \mathrm{id}_X$ *such that* $\gamma^{-1}(\mathrm{dom}\, \beta) = 0$.

Proof. The result follows from 2.2 and 2.3 once one notices that

$$\gamma^{-1}(\mathrm{dom}\, \beta) = \mathrm{dom}(\gamma\beta). \quad \blacksquare$$

As the counterexample before the proposition shows, there are interesting categories of partial maps where one cannot tell how big a domain is by just looking at what is *outside* of it. Nevertheless there is a large class of categories of partial maps which are dominical: the concrete ones. For this notion we shall refer to the definition of Longo and Moggi (1984) which, indeed, is the partial-map counterpart (whatever that may mean) of concrete categories with enough points and an atomic terminal object.

To explain our point of view, we must subject the reader to a half-page digression about the various degrees of *concreteness* one may encounter. In a truly set-like category **C** of partial maps, maps are indeed partial functions. They may be required to possess some additional properties, but they *are* functions. One can deal with this situation in a way similar to the way one deals with concrete categories of *total* functions, and say that **C** is *concrete* if there is a faithful *forgetful* functor $U: \mathbf{C} \to \mathrm{Ptl}(\mathbf{Set})$. The examples we gave at the beginning of Section 1 all fit into this picture by taking exactly the underlying set functor.

Let us make life a bit easier and suppose $\mathbf{C}$ is a p-category with a one-element object T and zero maps. Say that $\mathbf{C}$ has *enough points* if the functor

$$\mathbf{C}(T, -): \mathbf{C} \to 1/\mathbf{Set}$$

is faithful, where we are giving the hom-sets $\mathbf{C}(T, X)$ the canonical point given by $0_{T,X}$. Since

$$\hom(1, -): \mathrm{Ptl}(\mathbf{Set}) \xrightarrow{\simeq} 1/\mathbf{Set}$$

is an equivalence between the p-category of sets and partial functions and the category of pointed sets with p-category structure given by the smash product, it follows that a p-category with enough points is concrete.

We recall that we gave the presentation of concrete p-categories in order to discuss dominicality! But before stating that result we still need to give a result from folklore.

We say that the one-element object T of a p-category with zero maps is *atomic* if it has precisely two domains—the identity and the zero map. This agrees with the usage of DiPaola and Heller (1986), since T is atomic iff the identity on T is an atom in their sense.

2.5. LEMMA. *Suppose* $\mathbf{C}$ *is a* p*-category with an atomic one-element object and zero maps. Then* $\mathbf{C}(T, -): \mathbf{C} \to 1/\mathbf{Set}$ *preserves the* p*-structure. In particular, it preserves the one-element object and zero maps.*

2.6. THEOREM. *Suppose* $\mathbf{C}$ *is a* p*-category with an atomic one-element object and zero maps. If* $\mathbf{C}$ *has enough points, then* $\mathbf{C}$ *is dominical.*

Proof. First observe that the p-category of sets and partial functions, and hence the category $1/\mathbf{Set}$, is dominical. We must show that $\mathbf{C}$ has the cancellation property on maps, and that all weakly total maps are total. Now, since $\mathbf{C}$ has enough points, $\beta \times \gamma = 0$ iff $\mathbf{C}(T, \beta) \times \mathbf{C}(T, \gamma) = 0$ in $1/\mathbf{Set}$. But this now holds iff one of two is 0, say $\mathbf{C}(T, \beta) = 0$. As the functor $\mathbf{C}(T, -)$ is faithful and preserves zero maps, we have that $\beta = 0$. To show that weakly total maps are total, we assume that we are given a map β which is not total, so we have $\operatorname{dom} \beta: X \to X$ and $\operatorname{dom} \beta \neq \mathrm{id}$. It follows that $\mathbf{C}(T, \operatorname{dom} \beta) \neq \mathrm{id}$. Pick $\gamma: T \to X$ such that $\mathbf{C}(T, \operatorname{dom} \beta)(\gamma) \neq \gamma$. Now since $\mathbf{C}(T, -)$ preserves domains by 2.5, it follows that $\mathbf{C}(T, \operatorname{dom} \beta)$ is a sub-function of the identity, and hence that $\mathbf{C}(T, \operatorname{dom} \beta)(\gamma)$ must be the zero map. Thus $(\operatorname{dom} \beta)\gamma = 0$ revealing the fact that β is not weakly total either. ∎

We can now see the importance of the atomicity hypothesis: if $\mathscr{E}$ is a non-Boolean localic topos—for example, $\mathscr{S}^{\rightarrow}$, the category of sheaves on Sierpinski space—then Ptl($\mathscr{E}$), the full category of partial maps on $\mathscr{E}$ is a p-category with enough points which is not dominical. Again, we can if we wish find other recursion-theoretic counterexamples (cf. Montagna, 1986).

The careful reader will have noticed that whenever we considered aspects of concreteness we always considered pointed categories. We shall try to argue that this is the most useful case by a comparison with another notion of concreteness, the concept of *concrete categories of domains* of Moggi (1985). These are the categories of the form $\mathbf{C} = \mathscr{M}\text{-Ptl}(\mathbf{A})$ with a one-element object T such that not only does T generate $\mathbf{A}$ in the sense that $\mathbf{A}(T, -)\colon \mathbf{A} \to \mathbf{Set}$ is faithful, but it can also detect subdomains: $\mathbf{A}(T, -)\colon \mathscr{M}(X) \to \mathrm{Sub}(\mathbf{A}(T, X))$ is 1–1, and hence for all pairs of subobjects $[m\colon D \rightarrowtail X]$ and $[n\colon E \rightarrowtail X]$ in $\mathscr{M}$,

$$\forall x\colon T \to X \text{ in } \mathbf{A}[x \leqslant m \Leftrightarrow x \leqslant n] \Rightarrow [m] = [n].$$

2.7. Proposition. *Suppose $\mathbf{C} = \mathscr{M}\text{-Ptl}(\mathbf{A})$ is a concrete category of domains. Then either $\mathscr{M}$ consists only of maximal subobjects (i.e., those generated by the identities) and $\mathbf{C}(T, -)\colon \mathbf{C} \to \mathbf{Set}$ is faithful, or $\mathbf{C}$ is pointed, $\mathbf{C}(T, -)\colon \mathbf{C} \to 1/\mathbf{Set}$ is faithful and the one element object is atomic.*

Proof. Notice first that since T is terminal in $\mathbf{A} = \mathbf{C}_t$, there is only one map in $\mathbf{A}$ from T to T, and hence that there can be at most two subobjects of T in $\mathscr{M}$. There are thus two cases to consider: either $[\mathrm{id}\colon T \rightarrowtail T]$ is the only subobject of T in $\mathscr{M}$, or there is also another one $[z\colon Z \rightarrowtail T]$. In the first case it is easily seen that $\mathscr{M}$ contains only maximal subobjects, and so

$$\mathbf{C} \simeq \mathscr{M}\text{-Ptl}(\mathbf{A}) \simeq \mathbf{A}$$

and $\mathbf{C}(T, -)\colon \mathbf{C} \to \mathbf{Set}$ is faithful by definition. In order to treat the second case, notice first of all that there are no maps from T into Z. Then consider the following two full subcategories of $\mathbf{A}$: the category $\mathbf{A}_0$ consisting of all objects W of $\mathbf{A}$ with no map $T \to W$ from T, and the category $\mathbf{A}_1$ consisting of all objects X of $\mathbf{A}$ with a map $Z \to X$ from Z. We have the following lemma.

2.8. Lemma. *With the notation above*

(i) *The category $\mathbf{A}_0$ is a pre-order with Z as terminal element. Furthermore, $\mathscr{M}$ restricted to $\mathbf{A}_0$ contains only maximal subobjects.*

(ii) *Z is a strict initial object in $\mathbf{A}_1$, and hence $\mathbf{A}_0$ and $\mathbf{A}_1$ intersect only in objects isomorphic to Z.*

(iii) *For every A in $\mathbf{A}_1$ the subobject $[Z \rightarrowtail A]$ is in $\mathscr{M}$.*

Proof. (i) Since no element of $\mathbf{A}_0$ is the codomain of any total map from T, $\mathcal{M}$-subobjects are unique, and the second part of the statement is clear. It follows that if W is an element of $\mathbf{A}_0$, then the unique map from W into T must factor through Z (or by pulling back we could obtain another subobject). However, $\mathbf{A}(T, -)$ is faithful, and careful empty set calculations show that maps between elements of $\mathbf{A}_0$ must be unique.

(ii) According to the definition, there is a map from Z into any element of $\mathbf{A}_1$, but by faithfulness of $\mathbf{A}(T, -)$ these maps must be unique. Hence Z is initial in $\mathbf{A}_1$. If we now suppose that there is a map from some $Y \in \mathbf{A}_1$ into Z, then there can be no map from T into Y, and faithfulness implies that Y and Z are isomorphic. Z is thus strictly initial.

(iii) Since Z is strictly initial, for any $W \in \mathbf{A}_1$, $W \times_T Z$ is isomorphic to Z. But $[W \times_T Z \rightarrowtail W]$ is the pullback of $[Z \rightarrowtail T]$, and is thus in $\mathcal{M}$. ∎

From the lemma it follows that $\mathbf{A}$ is the glueing together of a partial order and a category with a strict initial object. It now follows from the results about $\mathcal{M}$-subobjects that the p-category $\mathbf{C}$ is pointed by the partial maps $[Z \rightarrowtail A, Z \rightarrowtail B]: A \rightharpoonup B$. It is also straightforward to check that the functor $\mathbf{C}(T, -): \mathbf{C} \to 1/\mathbf{Set}$ is faithful. ∎

2.9. *Remark.* This is perhaps the most appropriate place to talk about the notion of *partial cartesian category* appearing in Asperti and Longo (1986): this is a pointed category $\mathbf{C}$ such that the category $\mathbf{C}'$ of weakly total maps has a cartesian product and the bifunctor $\times : \mathbf{C}' \times \mathbf{C}' \to \mathbf{C}'$ extends to $\mathbf{C}$ in such a way that

$$\phi h \text{ weakly total} \wedge \psi k \text{ weakly total} \Rightarrow (\phi \times \psi)\langle h, k\rangle = \langle \phi h, \psi k\rangle. \quad (*)$$

We can follow Asperti and Longo in defining a pre-order on maps

$$\phi \leqslant \psi \Leftrightarrow \forall h[\phi h \text{ weakly total} \Rightarrow \phi h = \psi h].$$

This order is not necessarily a partial order, but it is easy to check that it is preserved by composition, and so we can consider the quotient category $\mathbf{C}/\sim$ obtained by quotienting the hom-sets by the relations

$$\phi \sim \psi \Leftrightarrow \phi \leqslant \psi \wedge \psi \leqslant \phi.$$

Then $(\mathbf{C}/\sim)' \simeq \mathbf{C}'$, but $\mathbf{C}/\sim$ is still not necessarily dominical because, although

$$\Delta\phi \leqslant (\phi \times \phi)\,\Delta,$$

one cannot prove that they are equivalent. Indeed, consider the category $\mathbf{B}$

of partial functions on powers of the set $\mathbb{N}$ of natural numbers (*all* partial functions). This category is pointed, and the weakly total maps are the total functions. Now consider the following bifunctor $\times : \mathbf{B} \times \mathbf{B} \to \mathbf{B}$ which extends the product on $\mathbf{B}'$:

$$(\phi \times \psi)(\mathbf{n}, \mathbf{m}) = \begin{cases} (\phi(\mathbf{n}), \psi(\mathbf{m})) & \text{if } \phi(\mathbf{n})\downarrow \text{ and } \psi(\mathbf{m})\downarrow \\ \mathbf{0} & \text{otherwise,} \end{cases}$$

where $\mathbf{0}$ is the appropriate k-tuple of zeroes. Clearly, $(*)$ holds in $\mathbf{B}$; so $\mathbf{B}$ is a partial cartesian category, but its product is not the one inherited from the partial map structure. In particular, $\mathbf{B}/\sim$ is not dominical. (Note that the construction is much more general than it looks.)

In order to have $\mathbf{C}/\sim$ dominical one needs an extra condition such as

$$(\phi \times \psi)\langle h, k\rangle \text{ weakly total} \Rightarrow \phi h \text{ weakly total} \wedge \psi k \text{ weakly total}; \quad (*_c)$$

after all one expects that in $(*)$ one side is defined just when the other is.

3. Other Approaches

As we have seen, one feature of categories of partial maps (and of p-categories) is that the hom-sets come equipped with a canonical partial order. There are several approaches to the notion of an "abstract category of partial maps" which take this partial order as part of the basic structure (and not, as in the examples above, as merely derivable from it). The concept of a category with structured hom-sets has a long and honourable history in the literature, and the hom-sets can carry many different types of structure. Particularly important are categories whose hom-sets carry the structure of abelian groups (for example, categories of modules) and those where the hom-sets are themselves categories (we recall that functors can be made the objects of a category by taking the arrows to be natural transformations). Study of these examples has led to a well-developed theory of enriched categories.

In enriched category theory one abstracts away from the idea of a hom-set being a set—even a set with structure—a hom-set is simply an object of some category. A $\mathbf{V}$-enriched categoy $\mathbf{C}$ is given by a collection of objects C_0 and a map which associates to each pair of objects (a, b) of C_0 an object $h(a, b)$ of the category $\mathbf{V}$, which plays the role of their "hom-set." It is, of course, also necessary to give interpretations of composition and the identity maps. The obvious solution is to give composition as a map

$$h(a, b) \times h(b, c) \to h(a, c),$$

where $(-)\times(-)$ is the categorical product, and to interpret the identity maps as categorical elements, i.e. via maps from the terminal object. This, however, does not suffice for all the examples we wish to consider; it does not work, for example, in the case of modules over a ring. The accepted solution is to relax the definition somewhat and take $\mathbf{V}$ to be a *monoidal* category.

A *monoidal category* consists of a structure $(\mathbf{V}, \otimes, I, \alpha, \rho, \lambda)$, where $\mathbf{V}$ is a category, $\otimes: \mathbf{V}\times\mathbf{V}\to\mathbf{V}$ is a bifunctor (usually called *tensor*), I is an object in $\mathbf{V}$, and

$$\alpha_{X,Y,Z}: X\otimes(Y\otimes Z)\to(X\otimes Y)\otimes Z$$

$$\rho_X: X\otimes I\to X$$

$$\lambda_X: I\otimes X\to X$$

are natural isomorphisms satisfying all the identities which hold in case $\otimes$ is a cartesian product on $\mathbf{V}$, I is terminal and α, ρ, and λ are the canonical isomorphisms (cf. Eilenberg and Kelly, 1966). In most cases, and all the cases we shall ever be interested in, $\mathbf{V}$ is *symmetric monoidal*, in which case one has a natural symmetry isomorphism

$$\tau_{X,Y}: X\otimes Y\to Y\otimes X$$

which must also satisfy the coherence axioms given by requiring that $\otimes$ satisfy all the identities which would hold if it were a cartesian product. It is a remarkable theorem of MacLane (cf. MacLane, 1971) that in both these cases the coherence conditions are essentially finite in number; in the case of symmetric monoidal categories it is possible to reduce the list of equations to five.

We have already seen an example of a symmetric monoidal category in which $\otimes$ is not the categorical product. Any p-category with a one-element object automatically inherits a symmetric monoidal structure from its category of total maps.

As a side remark we stress the fact that $\otimes$ is not reqired to be associative "on the nose," but only up to specified isomorphism. This is not an example of category theorists habitually working in ludicrous generality—an important application of this theory is to study definitions made up to coherent isomorphism. Furthermore, even products are very rarely associative up to actual equality—readers may wish to amuse themselves by verifying that there is no pairing operation on **Set** which is strictly associative in this sense.

Composition in a $\mathbf{V}$-category is given by maps $h(a,b)\otimes h(b,c)\to h(a,c)$, and the identities by maps $I\to h(a,b)$. We require that the obvious diagrams expressing the associativity of composition and the unit property of the identities commute (see Kelly, 1982). Note that these diagrams are more complicated than the ones usually given in category theory textbooks, owing to the need to make explicit use of α, λ, and ρ, but that no use is made of τ. It follows that a **Set**-enriched category is just an ordinary (locally small) category, and that a category enriched over **Po**, the category of partial orders and monotone maps, is a category whose hom-sets are partially ordered and in which composition is a monotone operation.

We recall from Section 1 that any p-category (any category of partial maps) has a natural order (the extension ordering) induced on its hom-sets by stipulating that $\phi \leqslant \psi$ if trivial matter to check that composition is monotone for this ordering, and hence that any p-category may be regarded naturally as a **Po**-enriched category. There is one significant advantage to this—the order-enriched structure determines the p-category structure on the underlying category uniquely. Suppose **C** is a p-category which is also an order-enriched category via the extension ordering. We have $\phi \leqslant \psi$ if and only if $\psi(\operatorname{dom} \phi) = \phi$, and hence in particular $\phi \leqslant \mathrm{id}$ iff $\phi = \operatorname{dom} \phi$. In other words, we can identify the domain maps of the p-category from the order-enriched structure as those maps less than or equal to the identities. Moreover, given any map $\phi : X \to Y$, the domain of ϕ can be characterised as the least domain map $\psi : X \to X$ such that $\phi \circ \psi = \phi$. It follows that were **C** to carry another p-structure inducing the same partial order structure, then the domain operators for both p-structures would have to be the same. This, however, is enough to show that the p-structures themselves are isomorhic (we get the same category of domains and the same full embedding of the p-category in both cases and recall that the p-structure is inherited from conventional product structure on the category of domains). This is in marked contrast with the unenriched situation, where the same category can carry more than one p-structure. (A discussion on this point appears also in Carboni, 1986.)

To return to a more abstract level, note that order-enriched categories provide an example of a more general and sophisticated theory. Since any partial order can be regarded as a category in which the hom-sets contain at most one element, any **Po**-enriched category is automatically a **Cat**-enriched category or *2-category*.

TERMINOLOGY. If **A** is a 2-category, and a and b are objects of **A**, then we call the objects of $h(a, b)$ *1-cells* and arrows *2-cells*. So in the case of the 2-category **Cat**, 1-cells are functors and 2-cells are natural transformations.

One of the major reasons for studying 2-categories is that categorical structure (for example, products) is defined only up to natural isomorphism. Suppose, however, that we have two categories with designated products, then the natural notion of a product-preserving functor between them is not one that preserves the structure; in other words, it is not one that sends designated products to designated products, but one that sends a designated product to something that will serve as a product. It is useful to put this in terms of the relevant 2-diagram, for it is not the case that a product-preserving functor is naturally equivalent to one that preserves products on the nose, but we can express the property required by saying that there is a 2-iso making

$$\begin{tikzcd} \mathbf{C}\times\mathbf{C} \arrow[r, "\times"] \arrow[d, "F\times F"'] \arrow[dr, phantom, "\Rightarrow"] & \mathbf{C} \arrow[d, "F"] \\ \mathbf{D}\times\mathbf{D} \arrow[r, "\times"] & \mathbf{D} \end{tikzcd}$$

commute; in other words, although the two paths are not necessarily equal we are given a natural isomorhism between them.

The idea that what is important in 2-categories is diagrams commuting up to specified 2-isos gives rise to the notion of a pseudo-limit. In this paper we shall, however, be concerned only with **Po**-enriched categories (usually called *locally ordered categories*), and for these the theory of 2-categories simplifies in several important respects—mainly due to the fact that 2-cells, where they exist, are unique. This means that the notions of limit and pseudo-limit coincide, and also that we can forget about troublesome coherence conditions on 2-cells. There is one further important simplification; there is in the literature a notion of bicategory (cf. Bénabou, 1973). A bicategory is similar to a 2-category, except that the composition of 1-cells is associative only up to coherent 2-isomorhism (and identities act as identities only up to 2-isomorphism); hence a bicategory is not necessarily a category, whereas a 2-category is. Fortunately for us, any partially ordered bicategory is necessarily a **Po**-enriched category, and so we can forget about this little unpleasantness. To return to our theme, there is, however, one 2-categorical notion which does not trivialise. Instead of requiring diagrams to commute up to 2-iso, we can allow them to commute only up to some more general 2-cell. This gives us the notion of laxity, lax adjoints, lax limits, etc. For instance, $\mu: F \to G$ is a (*right*) *lax natural transformation* between 2-functors $F, G: \mathbf{C} \to \mathbf{D}$, with $\mathbf{C}$ and $\mathbf{D}$ locally ordered categories, if

$$\mu_B \circ Ff \leqslant Gf \circ \mu_A$$

for all $f: A \to B$ in $\mathbf{C}$: diagrammatically

$$\begin{tikzcd} FA \arrow[r, "\mu_A"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\leqslant"] & GA \arrow[d, "Gf"] \\ FB \arrow[r, "\mu_B"] & GB \end{tikzcd}$$

The notion of lax natural transformation is used implicitly by Curien and Obtułowicz in their axiomatisation of the notion corresponding to terminal object in a category of partial maps regarded as a **Po**-enriched category.

DEFINITION. A *partial cartesian category* (Curien and Obtułowicz, 1986) consists of a locally ordered category $\mathbf{C}$ together with an operation $A \times B$ on objects and a particular object T, a lax natural transformation $!: (-) \to T$ into the constant functor T, and, for each pair of objects A and B in $\mathbf{C}$, monotone natural transformations

$$\mathrm{Pair}: \mathbf{C}(-, A) \times \mathbf{C}(-, B) \to \mathbf{C}(-, A \times B)$$
$$\mathrm{Unpair}: \mathbf{C}(-, A \times B) \to \mathbf{C}(-, A) \times \mathbf{C}(-, B)$$

(let $\mathrm{Fst}_{A,B}$ and $\mathrm{Snd}_{A,B}$ be the two maps representing the second transformation) such that

(a) $\mathrm{Pair} \circ \mathrm{Unpair} = \mathrm{Id}$ and $\mathrm{Unpair} \circ \mathrm{Pair} \leqslant \mathrm{Id}$ (p3, p4)

(b) for every A and B in $\mathbf{C}$, $!_A \, \mathrm{Fst}_{A,B} = !_{A \times B} = !_B \, \mathrm{Snd}_{A,B}$ (p1)

(c) for every A in $\mathbf{C}$, the map $!_A$ is the greatest element in $\mathbf{C}(A, T)$ (t1)

(d) for $\phi, \psi, \theta: A \to B$, if both $\phi, \psi \leqslant \theta$, and $!_B \phi \leqslant !_B \psi$, then $\phi \leqslant \psi$ (t3)

(e) for $\phi \leqslant \psi: A \to B$ and $\theta: B \to T$, one has $\theta\phi = \theta\psi \wedge !_B \phi$ (t4)

(f) for $\phi: A \to B$ and $\psi: A \to C$, one has $!_B \phi \wedge !_C \psi = !_{B \times C} \, \mathrm{Pair}(\phi, \psi)$ (p6).

Strictly speaking, the definition we have given above is not the one given in their paper: we have rephrased it in an equivalent, but slightly more compact form. Curien and Obtułowicz make little use of any kind of natural transformation; they are concerned mainly with operations on objects and maps, and with equations between them. We refer the reader to the original paper for the axioms in their original form, but provide a guide to the differences:

- The operation $\langle -, + \rangle$ has been renamed Pair, and axioms (p2) and (p5) state it is a monotone natural transformation

- (t2) simply defines the notion of total map as those f such that $!f = !$
- the rest of the axiomatisation is as indicated to the right of (a)–(f) above.

One may notice that given the lax naturality of !, condition (c) is ensured by the one equality $!_T = \mathrm{id}$. Also, it takes one only a second's thought to see that (b) is implied by the other axioms, as Pair(Fst, Snd) = id. Finally, condition (f) can be replaced by a weaker form of its with $B = C = T$. Indeed, under the weaker condition, (b) is equivalent to (f). The reader may now reformulate the definition to please his own taste.

The intuitive idea behind the way in which this definition is formed is that once we have ensured that the object T is terminal in the category of total maps, then a partial map $A \to T$ is determined by its domain of definition. Axiom (d) ensures that the mapping

$$\delta \mapsto !\delta : \{\delta \in \mathbf{C}(A, A) \mid \delta \leqslant \mathrm{id}\} \to \mathbf{C}(A, T)$$

is monotone, but it is also onto. Given any map $\phi : A \to T$, let $\mathrm{dom}(\phi) = \mathrm{Snd}\ \mathrm{Pair}(\phi, \mathrm{id}_A)$. Axiom (a) implies immediately that $\mathrm{dom}(\phi) \leqslant \mathrm{id}$, but we have also that

$$\mathrm{dom}(!\phi) = !\ \mathrm{Snd}\ \mathrm{Pair}(\phi, \mathrm{id}) = !\ \mathrm{Pair}(\phi, \mathrm{id}) = !\phi \wedge ! = \phi,$$

which completes the proof of the correspondence.

In fact, a little extra effort produces something stronger: given a map $\phi : A \to B$, one has that

$$\phi = \phi\ \mathrm{dom}(!\phi),$$

because by (e)

$$!\phi\ \mathrm{dom}(!\phi) = !\phi \wedge !\ \mathrm{dom}(!\phi) = !\phi \wedge !\phi = !\phi,$$

the equality now following from (d). A corollary of this is that the elements of the set $D_A = \{\delta \in \mathbf{C}(A, A) \mid \delta \leqslant \mathrm{id}\}$ are idempotent. It is now easy to see that D_A is a meet-semilattice with composition as meet, and that $! \circ -$ and $\mathrm{dom}(-)$ are inverse meet-semilattice isomorphisms between D_A and $\mathbf{C}(A, T)$. Thus, for instance, (f) can be rewritten as

$$\mathrm{dom}(!\ \mathrm{Pair}(\phi, \psi)) = \mathrm{dom}(!\phi)\ \mathrm{dom}(!\psi).$$

Still commenting on the definition, one can easily check that every category $\mathscr{M}$-Ptl($\mathbf{A}$) of partial maps on a category $\mathbf{A}$ with *all* finite products is a partial cartesian category. In fact, 3.1. PROPOSITION. *Suppose* $\mathbf{A}$ *is a category with finite products and* $\mathcal{M}$ *is an admissible class of subobjects. Then every full subcategory of* $\mathcal{M}$-Ptl($\mathbf{A}$) *which is closed under formation of finite products of objects is a partial cartesian category.*

Proof. Trivial. ▮

There is a converse to 3.1: a sketch of an elementary proof that every partial cartesian category satisfies the axioms for p-categories with a one-element object is contained in Curien and Obtułowicz (1986). For the sake of completeness, we produce one here. Let $\mathbf{C}$ be a partial cartesian category. The first step in the proof is to show that $\mathbf{C}$ has a subcategory on which the operation $A \times B$ defines a cartesian product. Let $\mathbf{C}_T$ be the subcategory of $\mathbf{C}$ which consists of the total maps as defined above: the objects of $\mathbf{C}_T$ are those of $\mathbf{C}$, a map $f\colon A \to B$ is in $\mathbf{C}_T$ if $!f = !$.

3.2. LEMMA. (i) *Given* $\phi\colon X \to A$ *and* $\psi\colon X \to B$, *suppose* $!\phi \leqslant !\psi$. *Then*

$$\phi = \text{Fst Pair}(\phi, \psi) = \text{Snd Pair}(\psi, \phi).$$

(ii) *The operation* $A \times B$ *defines a cartesian product on* $\mathbf{C}_T$ *and* T *is terminal in* $\mathbf{C}_T$.

Proof. (i) As $!\,\text{Fst} = !$, one has

$$!\,\text{Fst Pair}(\phi, \psi) = !\phi \wedge !\psi = !\phi.$$

Hence the first equality follows because Fst Pair$(\phi, \psi) \leqslant \phi$. The second is proved similarly.

(ii) From (i) it follows that Pair and Unpair are natural isomorphisms between

$$\mathbf{C}_T(-, A) \times \mathbf{C}_T(-, B) \cong \mathbf{C}_T(-, A \times B).$$

Thus $A \times B$ is a product. Obviously, ! is the only element in $\mathbf{C}_T(A, T)$. ▮

As Pair is defined on all of $\mathbf{C}(X, A) \times \mathbf{C}(X, B)$, we can extend the product functor from $\mathbf{C}_T$ to all of $\mathbf{C}$ by letting

$$\phi \times \psi = \text{Pair}(\phi\,\text{Fst}, \psi\,\text{Snd}).$$

Now the definition of the possible p-structure on the partial cartesian category $\mathbf{C}$ is easy: take $p = \text{Fst}$ and $q = \text{Snd}$, and let $\Delta = \text{Pair(id, id)}$. By 3.2(ii) above, the seven abominable equalities in the definition of p-category hold for $\mathbf{C}$, as they must hold in $\mathbf{C}_T$. We are now left with proving that some five families of maps are natural on the whole of $\mathbf{C}$ (we make extensive use of 3.2(i)):

- diagonal: $(\phi \times \phi)\,\Delta = \mathrm{Pair}(\phi p, \phi q)\,\Delta = \mathrm{Pair}(\phi p\,\Delta, \phi q\,\Delta)$
$= \mathrm{Pair}(\phi, \phi) = \Delta\phi$
- projections: $p(\phi \times \mathrm{id}) = p\,\mathrm{Pair}(\phi p, q) = \phi p$
- commutativity: let $\delta = \mathrm{dom}(!(\phi \times \psi))$, so that

$$\delta = \mathrm{dom}(!\phi p)\,\mathrm{dom}(!\psi q) = \mathrm{dom}(!\phi p) \wedge \mathrm{dom}(!\psi q) = \mathrm{dom}(!\psi q)\,\mathrm{dom}(!\phi p).$$

Then

$$\tau(\phi \times \psi) = \tau(\phi \times \psi)\,\delta = \mathrm{Pair}(q\,\mathrm{Pair}(\phi p\,\delta, \psi q\,\delta), p\,\mathrm{Pair}(\phi p\,\delta, \psi q\,\delta))$$
$$= \mathrm{Pair}(\psi q, \phi p)\,\delta = \mathrm{Pair}(\psi q, \phi p) = (\psi \times \phi)\tau$$

- associativity: similar to the proof of commutativity.

Finally, $p: X \times T \to X$ and $(\mathrm{id} \times !)\,\Delta = \mathrm{Pair}(\mathrm{id}, !)$: $X \to X \times T$ are inverse because T is terminal in $\mathbf{C}_T$. This concludes the proof that a partial cartesian is a p-category with a one-element object. Recalling 3.1 and 1.6 we can then state

3.3. THEOREM. *Partial cartesian categories are exactly the p-categories with a one-element object.*

The notion of partial cartesian category uses the order inherited by hom-sets in categories of partial maps as an essential feature in the definition of the structure. The rest of the definition has however a distinctly algebraic rather than categorical flavour; we are, for example, given the product solely as an operation on objects and have to derive its effect on arrows from Pair, Fst, and Snd. Indeed Curien and Obtułowicz use the definition of a partial cartesian category to go on to prove an elegant algebraic characterisation of categories of partial maps on a category with finite products. They retain the operations Pair, Fst, Snd, and !, but forget the partial order structure on the hom-sets; instead they introduce a new binary operation on arrows: $f \upharpoonright g$ means "f restricted to the domain of g." Using this new operation they give a list of twelve equalities which axiomatise the category structure of partial cartesian categories. Curien and Obtułowicz call these structures *precartesian categories*, and they are again equivalent to p-categories with a one-element object (this time in the strong sense that their structure is equationally interdefinable). Again the stress is on operations and equations, rather than on conventional categorical structure and, for this reason, we omit the specific axiomatisation given in the paper; it is, however, quite compact and would be particularly attractive to any reader interested in automated category theory.

Perhaps one drawback of the use of locally ordered categories to describe categories of partial maps is that it complicates the structure when there is an alternative order on the hom-sets. Consider the category Ptl(**Po**) of posets and partial monotone functions. On the set of partial monotone functions Ptl(**Po**)(P, Q), there are two definable orders (which coincide only when the order of Q is discrete): the extension order, and the pointwise order

$$\phi \sqsubseteq \psi \Leftrightarrow \forall p \in P[\phi p \downarrow \Rightarrow \psi p \downarrow \wedge \phi p \sqsubseteq \psi p].$$

The work of Plotkin (1986) is based on the latter. In *loc. cit.* the main lines of the abstract theory for these *richer* structures are suggested, and it is no surprise to find that category theorists have developed part of this theory from a different approach: they appear under the name of *bicategories of partial maps* in Carboni (1986).

Carboni gives a subtle and very elegant description of the monoidal categories one is concerned with if dealing with partial maps. To understand Carboni's definition note that an object in a category of partial maps (say a p-category with one element object) carries a unique structure of cocommutative comonoid: use the counit identities to show first that the multiplication is total, and then the commutativity to show that it is the diagonal, and now the counits again to show that the counit is the total map into the one element object.

DEFINITION. A *bicategory of partial maps* is a symmetric monoidal structure on a locally ordered category **B** such that

(a) the tensor product is a 2-functor (= monotone on arrows)

(b) there is a unique cocommutative comonoid structure $\Delta_X: X \to X \otimes X$, $t_X: X \to I$ on each object X

(c) every map in **B** is a strict comultiplication homomorphism and a lax counit homomorphism:

$$\Delta_Y f = (f \otimes f)\Delta_X, \qquad t_Y f \sqsubseteq t_X.$$

Condition (c) can be put more compactly: it is simply that t is a lax natural and Δ is a natural transformation, while condition (b) in the definition requires that the maps Δ_X and t_X satisfy the three identities:

coassociativity:

$$\begin{tikzcd} X \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & X \otimes X \arrow[r, "\mathrm{id} \otimes \Delta"] & X \otimes (X \otimes X) \arrow[d, "\alpha"] \\ X \otimes X \arrow[rr, "\Delta \otimes \mathrm{id}"] & & (X \otimes X) \otimes X \end{tikzcd}$$

cocommutativity:

$$\begin{tikzcd} X \arrow[r, "\Delta"] \arrow[dr, "\Delta"'] & X \otimes X \arrow[d, "\tau"] \\ & X \otimes X \end{tikzcd}$$

counit:

$$\begin{tikzcd} X \arrow[r, "\Delta"] \arrow[d, "\mathrm{id}"'] & X \otimes X \arrow[d, "\mathrm{id} \otimes t"] \\ X \arrow[r, "\rho"] & X \otimes I \end{tikzcd}$$

(we apologise for the boring display of these explicit isomorphisms, but they play a crucial role in the treatment). Note that we require that any other pair of maps satisfying the three identities above must coincide with Δ and t.

The category Ptl(**Po**) is a bicategory of partial maps when we provide it with either of the orders defined above. Indeed, any category of the form $\mathscr{M}$-Ptl(**A**), where **A** has finite products, is a bicategory of partial maps when it is given the extension order

$$[m, \phi] \leqslant [n, \psi] \Leftrightarrow [m] \leqslant [n] \wedge [m, \phi] = [m, \psi].$$

The following is trivial.

3.4. Proposition. *Let* **C** *be a p-category with a one-element object* T. *If on every hom-set* $\mathbf{C}(X, Y)$ *the extension order is defined*

$$\phi \leqslant \psi \Leftrightarrow \phi = \psi \operatorname{dom} \phi,$$

then this defines a structure of bicategory of partial maps on **C**.

There cannot be a complete converse to 3.4; the different bicategory structures on Ptl(**Po**) show that we can have bicategories of partial maps in which the ordering on hom-sets is not the extension ordering. The example shows that bicategories of partial maps really represent the theory of the richer structure where we have well-behaved partial orders on the hom-sets. In other words, bicategories of partial maps should be algebraic 2-theory for categories of the form $\mathscr{M}$-Ptl(**A**), where **A** is locally ordered and has finite 2-products (= the product functor is monotone). A proof of this is given below. We proceed through the following steps:

- forgetting the partial-order structure of a bicategory of partial maps still leaves enough information in order to recognise it as a category of partial maps. Indeed, we shall describe the categories so obtainable in terms of pre-*dht*-symmetric categories

- pre-*dht*-symmetric categories correspond to *p*-categories
- the partial-order structure on the bicategory can be completely described in terms of a locally ordered structure on its category of domains.

It is important to notice that the partial order structure actually plays only a very small role in the definition. But, first, let us define pre-*dht*-symmetric and *dht*-symmetric categories.

DEFINITION. A *pre-dht-symmetric category* is a structure $(\mathbf{C}, \otimes, I, a, r, l, c, d, t)$ such that the part $(\mathbf{C}, \otimes, I, a, r, l, c)$ is symmetric monoidal, $d: (-) \to (- \otimes -)$ is a natural transformation, and $\{t_X: X \to I \mid X \in \mathrm{ob}\,\mathbf{C}\}$ is a family of maps which satisfy

(a) $r_X(\mathrm{id}_X \otimes t_X)\, d_X = \mathrm{id}_X = l_X(t_X \otimes \mathrm{id}_X)\, d_X$

(b) $(r_X(\mathrm{id}_X \otimes t_Y) \otimes l_Y(t_X \otimes \mathrm{id}_Y))\, d_{X \otimes Y} = \mathrm{id}_{X \otimes Y}$

(c) $r_I(t_X \otimes t_Y) = t_{X \otimes Y}$.

Note that $l = rc$, and that either of the identities in (a) can be proved from the other; we retain both for reasons of symmetry.

A *dht-symmetric category* is a pre-*dht*-symmetric category $\mathbf{C}$ together with an object O and a map $o: I \to O$ such that the composite $o_X = o \circ t_{X \otimes O}$: $X \otimes O \to O$ is iso, and that and $o \circ t_X$: $X \to O$ and $r_X(\mathrm{id}_X \otimes t_O)$: $O = X \otimes O \to X$ are the only maps with source or target O.

We would like to thank an anonymous referee for pointing out that Hoehnke has now changed his terminology from that of Hoehnke (1977). We have used new terminology above, in contrast to the first draft of this paper, where we used his older nomenclature, and called pre-*dht*-symmetric categories *dt*-symmetric. We believe that he now uses the name *dt*-symmetric to refer to a slightly different concept. The definition of *dht*-symmetric category is drawn from Hoehnke (1977a) although the form in which is given here is essentially that of Vogel (1979). We have split its definition into two as we think that fits better with our general philosophy, and we have added condition (c) to the latter. It is clear in the claim of Hoehnke (1977a) that *dht*-symmetric categories want to capture as many algebraic properties of the category Ptl(**Set**) as possible that (c) must hold in *dht*-symmetric categories, but we must declare our incapability of proving that it follows from the others. We note that the useful fact that $t_I = \mathrm{id}_I$ can be made to follow from (a)–(c) using the equality $l_I = r_I$ true for all monoidal categories. The notion of pre-*dht*-symmetric category will be useful for the next representation theorem.

3.5. PROPOSITON. *Suppose* **B** *is a bicategory of partial maps. Then the structure* $(\mathbf{B}, \otimes, I, \alpha, \rho, \lambda, \Delta, t)$ *is a pre-dht-symmetric category.*

Proof. By definition, $(\mathbf{B}, \otimes, I, \alpha, \rho, \lambda, \tau)$ is symmetric monoidal. Condition (a) is the statement that $t_X: X \to I$ is a counit for Δ_X. As for conditions (b) and (c), let $\sigma: (X \otimes Y) \otimes (Z \otimes W) \to (X \otimes Z) \otimes (Y \otimes W)$ be the obvious natural isomorphism, and notice that the maps $\sigma(\Delta_X \otimes \Delta_Y)$ and $\rho_I(t_X \otimes t_Y)$ define a cocommutative comonoid on $X \otimes Y$. Hence

$$\Delta_{X \otimes Y} = \sigma(\Delta_X \otimes \Delta_Y) \qquad t_{X \otimes Y} = \rho_I(t_X \otimes t_Y).$$

Thus

$$\begin{aligned}
&(\rho_X \otimes \lambda_Y)((\mathrm{id}_X \otimes t_Y) \otimes (t_X \otimes \mathrm{id}_Y))\, \Delta_{X \otimes Y} \\
&\quad = (\rho_X \otimes \lambda_Y)\, \sigma((\mathrm{id}_X \otimes t_X)\, \Delta_X \otimes (\mathrm{id}_Y \otimes t_Y)\, \Delta_Y) \\
&\quad = (\rho_X \otimes \rho_Y)((\mathrm{id}_X \otimes t_X)\, \Delta_X \otimes (\mathrm{id}_Y \otimes t_Y)\, \Delta_Y) \\
&\quad = \mathrm{id}_X \otimes \mathrm{id}_Y = \mathrm{id}_{X \otimes Y}. \quad \blacksquare
\end{aligned}$$

3.6. PROPOSITION. *Every pre-dht-symmetric category is a p-category with one-element object.*

Proof. Suppose $(\mathbf{C}, \otimes, a, r, l, c, d, t)$ is pre-*dht*-symmetric. Let $p_{X,Y} = r_X(\mathrm{id}_X \otimes t_Y)$: $X \otimes Y \to X$, $q_{X,Y} = l_Y(t_X \otimes \mathrm{id}_Y)$: $X \otimes Y \to Y$ and $\Delta_X = d_X$: $X \to X \otimes X$. Then $p_{-,Y}$, $q_{X,-}$, and $\Delta_{(-)}$ are natural transformations. As for the seven equalities, (a) and (b) are exactly the first three. To obtain the fourth we have

$$p(\mathrm{id} \otimes p) = r(\mathrm{id} \otimes t)(\mathrm{id} \otimes r(\mathrm{id} \otimes t)) = r(\mathrm{id} \otimes r(t \otimes t)) = r(\mathrm{id} \otimes t) = p.$$

The other three are proved similarly. Finally, in order to prove that the commutativity isomorphism $\tau = (q \otimes p)\,\Delta$ is natural in two variables, we show that it coincides with c:

$$\tau = (l(t \otimes \mathrm{id}) \otimes r(\mathrm{id} \otimes t))\,\Delta = c(r(\mathrm{id} \otimes t) \otimes l(t \otimes \mathrm{id}))\,\Delta = c.$$

Similarly for the associativity isomorphism α. ∎

3.7. COROLLARY. *The dht-symmetric categories are exactly the p-categories with a one-element object, zero object, and zero maps.*

Suppose then **B** is a bicategory of partial maps. By putting together 3.5 and 3.6, one gets that $(\mathbf{B}, \otimes, p, q, \Delta)$ is a p-category, where $p = \rho(\mathrm{id} \otimes t)$ and $q = \lambda(t \otimes \mathrm{id})$, and I is a one-element object.

3.8. LEMMA. *Suppose* **B** *is a bicategory of partial maps and suppose a structure of p-category is defined on* **B** *as above. Let* $\leqslant$ *be the extension order induced by the p-structure. Let* $\phi, \psi : X \to Y$. *Then*

(i) $\operatorname{dom} \phi \sqsubseteq \mathrm{id}$

(ii) $\operatorname{dom} \phi \leqslant \operatorname{dom} \psi \Leftrightarrow \operatorname{dom} \phi \sqsubseteq \operatorname{dom} \psi$

(iii) $\phi \sqsubseteq \psi \Leftrightarrow \operatorname{dom} \phi \leqslant \operatorname{dom} \psi \wedge \phi \sqsubseteq -\operatorname{dom} \phi$

(iv) $\phi \leqslant \psi \Leftrightarrow \operatorname{dom} \phi \sqsubseteq \operatorname{dom} \psi \wedge \phi = \psi \operatorname{dom} \phi$.

Proof. (i) $\operatorname{dom} \phi = p(\mathrm{id} \otimes \phi)\, \Delta = \rho(\mathrm{id} \otimes t\phi)\, \Delta \sqsubseteq \rho(\mathrm{id} \otimes t)\, \Delta = \mathrm{id}$.

(ii) If $\operatorname{dom} \phi \leqslant \operatorname{dom} \psi$, then

$$\operatorname{dom} \phi = \operatorname{dom} \phi \operatorname{dom} \psi \sqsubseteq \mathrm{id} \operatorname{dom} \psi = \operatorname{dom} \psi.$$

Suppose, conversely, that $\operatorname{dom} \phi \sqsubseteq \operatorname{dom} \psi$. Then

$$\operatorname{dom} \phi = \operatorname{dom} \phi \operatorname{dom} \phi \sqsubseteq \operatorname{dom} \phi \operatorname{dom} \psi \sqsubseteq \operatorname{dom} \phi\, \mathrm{id} = \operatorname{dom} \phi.$$

Therefore, $\operatorname{dom} \phi = \operatorname{dom} \phi \operatorname{dom} \psi$, or, equivalently, $\operatorname{dom} \phi \leqslant \operatorname{dom} \psi$.

(iii) Suppose $\phi \sqsubseteq \psi$. Then $\operatorname{dom} \phi \sqsubseteq \operatorname{dom} \psi$ because $\otimes$ and composition are monotone. Hence by (ii), one has $\operatorname{dom} \phi \leqslant \operatorname{dom} \psi$. Also

$$\phi = \phi \operatorname{dom} \phi \sqsubseteq \psi \operatorname{dom} \phi.$$

The converse follows from (ii).

(iv) Trivial. ∎

3.9. PROPOSITION. *Suppose* **B** *is a bicategory of partial maps and let* **D** *be the category of domains associated to the p-structure on* **B**. *Then the category* **D** *is locally ordered with finite 2-products, and the embedding* $\mathbf{B} \to \mathscr{D}\text{-}\mathrm{Ptl}(\mathbf{D})$ *is a monoidal 2-functor.*

Proof. A trivial application of 3.8. ∎

3.10. COROLLARY. *Let* **B** *be a bicategory of partial maps. Then* **B** *is obtained from a p-category by adding the extension order if and only if for any* $\phi, \psi : X \to Y$

$$\phi \sqsubseteq \psi \wedge \operatorname{dom} \phi = \operatorname{dom} \psi \Rightarrow \phi = \psi.$$

4. Representation of Partial Maps

Representing $\mathscr{M}$-partial maps from A to B in a category $\mathbf{A}$ means having an object $[A \rightharpoonup B]$ such that

$$\mathscr{M}\text{-Ptl}(\mathbf{A})(A \times X, B) \simeq \mathbf{A}(X, [A \rightharpoonup B])$$

naturally as X varies in $\mathbf{A}$. Clearly such an object is determined up to unique isomorphism in either the category of total or the category of partial maps.

The problem of representation arose in topos theory with the notion of a partial map classifier (cf. Johnstone, 1977), but was never singled out clearly till its appearance in Longo and Moggi (1984). We think that is the first time the notion appeared in the literature in the form given above, and it has been a proliferant one ever since. Indeed, though the name is never the same in any two of the references listed, the notion we are going to define below has been standardly accepted. We think that in the best tradition of cartesian closed categories, the object $[A \rightharpoonup B]$ should be christened *partial-function space*. The reason for having $\mathbf{A}(X, [A \rightharpoonup B])$ on the right-hand side in the isomorphism and not partial maps, should be transparently clear from the example of Ptl(**Set**).

In the case of a p-category $\mathbf{C}$, the partial-function space $[A \rightharpoonup B]$ is an object such that there is a natural isomorphism

$$\lambda_p \colon \mathscr{D}\text{-Ptl}(\mathbf{D})(A \times X, B) \xrightarrow{\sim} \mathbf{D}(X, [A \rightharpoonup B])$$

as X varies in the category $\mathbf{D}$ of domains (which looks like a ghastly condition). But there are ways of expressing the condition above in terms of the p-structure on $\mathbf{C}$.

4.1. Proposition. *Let $\mathbf{C}$ be a p-category. Suppose $\mathbf{D}$ is the category of domains and $\leqslant$ is the extension order defined on the hom-sets of $\mathbf{C}$. Suppose A, B, and $[A \rightharpoonup B]$ are objects in $\mathbf{C}$. Then the following are equivalent:*

(i) *there is a natural isomorphism $\lambda_p \colon \mathscr{D}\text{-Ptl}(\mathbf{D})(A \times X, B) \xrightarrow{\sim} \mathbf{D}(X, [A \rightharpoonup B])$ as X varies in $\mathbf{D}$*

(ii) *there is a lax natural injection $\Lambda \colon \mathbf{C}(A \times C, B) \rightarrowtail \mathbf{C}(C, [A \rightharpoonup B])$ as C varies in $\mathbf{C}$, whose images are the sets $\mathbf{C}_t(C, [A \rightharpoonup B])$ of total maps*

(iii) *there is a natural isomorphism $\lambda \colon \mathbf{C}(A \times C, B) \xrightarrow{\sim} \mathbf{C}_t(C, [A \rightharpoonup B])$ as C varies in $\mathbf{C}_t$ and, for $\psi \colon C \to [A \rightharpoonup B]$ in $\mathbf{C}$,*

$$\psi = \lambda(\varepsilon(\mathrm{id} \times \psi)) \circ \mathrm{dom}\, \psi,$$

where $\varepsilon = \lambda^{-1}(\mathrm{id}_{[A \rightharpoonup B]})$: $A \times [A \rightharpoonup B] \to B$ *is the evaluation map and is partial, in general.*

Proof. (i) $\Rightarrow$ (ii) Define Λ as

$$\Lambda : \mathbf{C}(A \times C, B) \to \mathbf{C}(C, [A \rightharpoonup B])$$
$$\phi \mapsto \lambda_p([\mathrm{dom}\,\phi, \phi]).$$

We only need to check that it is lax-natural. So let $\psi : D \to C$ be a map in $\mathbf{C}$. We must show that $\Lambda(\phi) \circ \psi \leqslant \Lambda(\phi(\mathrm{id} \times \psi))$. Compose the map $\Lambda(\phi(\mathrm{id} \times \psi))$: $D \to [A \rightharpoonup B]$ with $\mathrm{dom}\,\psi$: $\mathrm{dom}\,\psi \to D$ which is an honest map in $\mathbf{D}$: by naturality of λ_p,

$$\begin{aligned}\Lambda(\phi(\mathrm{id} \times \psi))\,\mathrm{dom}\,\psi &= \lambda_p([\mathrm{dom}\,\phi, \phi](\mathrm{id} \times [\mathrm{dom}\,\psi, \psi])(\mathrm{id} \times \mathrm{dom}\,\psi)) \\ &= \lambda_p([\mathrm{dom}\,\phi, \phi](\mathrm{id} \times [\mathrm{dom}\,\psi, \psi])) \\ &= \lambda_p([\mathrm{dom}\,\phi, \phi])\psi = \Lambda(\phi)\psi.\end{aligned}$$

Hence lax-naturality follows.

(ii) $\Rightarrow$ (iii) Let λ be the corestriction of Λ. Suppose $\psi : D \to C$. As $\psi = \lambda(\varepsilon)\psi \leqslant \lambda(\varepsilon(\mathrm{id} \times \psi))$ and $\lambda(\varepsilon(\mathrm{id} \times \psi))$ is total, the final equality follows.

(iii) $\Rightarrow$ (i) Suppose β is a domain, and define

$$\lambda_p : \mathscr{D}\text{-}\mathrm{Ptl}(\mathbf{D})(A \times \beta, B) \simeq \mathbf{D}(\beta, [A \rightharpoonup B])$$
$$\phi \mapsto \lambda(\phi)\beta.$$

It is straightforward to check that this definition is natural in $\mathbf{D}$. Its inverse is the function which takes $\psi : \beta \to [A \rightharpoonup B]$ to $[\mathrm{dom}(\varepsilon(\mathrm{id} \times \psi)), \varepsilon(\mathrm{id} \times \psi)]$: $A \times \beta \to A \times [A \rightharpoonup B] \to B$. ∎

We shall only mention that the various definitions that appear for partial-function spaces in concrete or dominical categories are always a rephrasing of one of the equivalent conditions above.

Because of 4.1(ii), if the existence of the partial-function spaces is assured for every object B in $\mathbf{C}$, then there is an adjunction

$$(A \times -) \dashv [A \rightharpoonup -] : \mathbf{C} \to \mathbf{C}_t$$

and, by (i), this can be extended to all of $\mathscr{D}$-Ptl($\mathbf{D}$).

Another corollary of 4.1 is that the partial-function space $[A \rightharpoonup B]$ represents a certain set-valued functor on $\mathbf{C}$: consider the following, very simple, idempotent natural transformation on the functor $\mathbf{C}(A \times -, B) \times \mathrm{Dom}(-) : \mathbf{C} \to \mathbf{Set}$,

$$\mathbf{C}(A \times C, B) \times \mathrm{Dom}(C) \to \mathbf{C}(A \times C, B) \times \mathrm{Dom}(C)$$
$$\langle \phi, \delta \rangle \mapsto \langle \phi(\mathrm{id} \times \delta), \delta \rangle$$

and let $F_B^A(C)$ be the set of fixed points. This obviously gives a functor $F_B^A(-) : \mathbf{C} \to \mathbf{Set}$ which is a natural retract of $\mathbf{C}(A \times -, B) \times \mathrm{Dom}(-)$. Moreover,

$$\mathbf{C}(C, [A \rightharpoonup B]) \simeq F_B^A(C)$$
$$\psi \mapsto \langle \varepsilon(\mathrm{id} \times \psi), \mathrm{dom}\, \psi \rangle$$

is natural as C varies in $\mathbf{C}$. We do not have space here to survey the extensive collection of results already shown which have made use of this notion (cf. Longo and Moggi, 1984; Curien and Obtułowicz, 1986; Rosolini, 1986; Asperti and Longo, 1986). The reader should, however, realise that in most cases the proofs of results in these papers are presented in an algebraic style. The representation theory discussed in the present paper allows us to give alternative (and we believe more intuitive, though not necessarily shorter) proofs based on the representation isomorphism above and the Yoneda lemma.

A Note on Partial Algebra

The conventional categorical approach to algebra requires a slight (and slightly pedantic) modificaton if it is to work in categories of partial maps. We recall that, for example, a monoid structure on an object A in a category $\mathbf{C}$ is given by two maps $\mu : A \times A \to A$ and $1 : 1 \to A$. The convention is that $(-) \times (-)$ is the categorical product and that 1 is the categorical terminal object. One of the recurring themes of this paper has been that in dealing with categories of partial maps the product structure on categories of total maps gets relaxed to a symmetric monoidal structure (plus diagonal). This is, of course, the modification necessary here. The effect can be seen particularly clearly if one follows the approach of Lawvere: an algebraic theory "is" a category $\mathbf{T}$ whose objects are the natural numbers, such that n is the product of n-copies of one, and where the maps from n to 1 correspond to equivalence classes of terms under provable equality. A $\mathbf{T}$-algebra in a category $\mathbf{C}$ is then given by a product-preserving map $\mathbf{T} \to \mathbf{C}$. The partial viewpoint uses the same categories to represent algebraic theories (we are not interested in partiality at the level of syntax!), but takes $\mathbf{T}$-algebras to be given by monoidal functors.

Matters become more interesting when we start to deal with homomorphisms. As we would expect from 2-category theory, the notion of homomorphism splits into inequivalent lax and pseudo versions. Suppose we have algebra objects A and B in a category of partial maps $\mathbf{C}$. Then a lax homomorphism $\theta: A \to B$ is given by a partial map θ such that for each n-ary operation f of the theory the square

$$\begin{tikzcd} A \otimes \cdots \otimes A \arrow[r, "\theta \otimes \cdots \otimes \theta"] \arrow[d, "f"'] \arrow[dr, phantom, "\leqslant"] & B \otimes \cdots \otimes B \arrow[d, "f"] \\ A \arrow[r, "\theta"] & B \end{tikzcd}$$

commutes, where $\leqslant$ is the extension ordering. In the literature these homomorphisms are often called *weak*. A *strong homomorphism* is defined as a map θ making the diagram commute exactly, and thus corresponds to the 2-categorical pseudo as well as the exact notions. We warn readers that this terminology, although widespread, is not completely standard, and that the maps giving either sort of homomorphism may, in addition, be required to be total. Furthermore, both Hoehnke (1977a) and Vogel (1979) discuss partial algebras, but use terminology which actually conflicts with conventional 2-categorical usage: their pseudo is our lax.

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