Eigenvalues of Some Nonlinear Operators*

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1. INTRODUCTION

We apply Leray-Schauder degree to the study of eigenvalues of completely continuous operators $A$ such that $\|Ax\|$ is bounded away from zero if $\|x\|$ is bounded away from zero and such that

$$\lim_{\|x\| \to 0} \frac{\|Ax\|}{\|x\|} = 0. \tag{S}$$

Let $\text{deg}(I - \lambda A, B, 0)$ denote the Leray-Schauder degree of the mapping $I - \lambda A$ at 0 and relative to the set $B$ where $B$ is a bounded open set which contains 0. Our results are obtained essentially by showing that $\text{deg}(I - \lambda A, B, 0)$ is equal to $\text{deg}(-\lambda A_n, B \cap X_n, 0)$ where $A_n$ is a finite-dimensional approximation to $A$ on a finite-dimensional subspace $X_n$. We show (Theorem 1) that $A$ has a bounded infinite set of eigenvalues which have 0 as a limit point and that for each of these eigenvalues $1/\lambda$, the equation

$$(I - \lambda A)x = y$$

has at least two distinct solutions for each $y$ sufficiently close to zero. If $A$ is an even, finite-dimensional mapping and $\|Ax\|$ is bounded away from zero for $x$ in the range of $A$ and $\|x\|$ bounded away from zero then (Theorem 2) $A$ has a continuum of eigenvalues with zero as a limit point. Further application of the same technique yields (Theorem 3) a new proof of an eigenvalue theorem due originally to Birkhoff and Kellogg [1]. (See also Rothe [5] and Krasnosel'skii [4, pp. 184ff.]. For Theorem 3, we do not use condition (S).) Also we show (Theorem 4) that, with a somewhat stronger condition on $\|Ax\|$, the set of eigenvalues of $A$ contains the set $[-1, 0) \cup (0, 1]$. Finally, we give some examples of integral equations to which these theorems are applicable.

These results are extensions of the theorem of Birkhoff and Kellogg referred to above. Krasnosel'skii [4, pp. 240–254] has also made extensions

* The research in this paper was supported by the U.S. Army Research Office (Durham) (DA-ARO-D-31-124-G1098).

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of this theorem. However, Krasnosel'skii's extensions are in the direction of studying positive operators and the configuration of eigenfunctions whereas we impose condition (S) on the operator and obtain more definite information about the set of eigenvalues.

2. Existence of Eigenvalues

Let $A$ be a transformation defined on $D$, a set in an infinite-dimensional linear normed space $X$ such that $A : D \to X$ and assume $A$ has the following properties:

1. $A$ is completely continuous (takes bounded sets into sets which are compact in $X$);
2. there is a bounded open set $B$ such that $\bar{B} \subset D$ and $0 \in B$ and a positive constant $c$ such that if $x \in \partial B$ (the point set boundary of $B$) then $\|Ax\| \geq c$.
3. $\lim_{\|x\| \to 0} \frac{\|Ax\|}{\|x\|} = 0$.

**Theorem 1.** There exists a number $\lambda_0 > 0$ such that for each $\lambda$ with $\lambda \geq \lambda_0$ either $(1/\lambda)$ or $-(1/\lambda)$ is an eigenvalue of $A$ and for each such eigenvalue there is an eigenfunction in $B$. Also for each $\lambda$ there is an open ball $B_\lambda$ such that if $y \in B_\lambda$, the equation $(I - \lambda A)x = y$ or the equation $(I + \lambda A)x = y$ has two distinct solutions in $B$.

**Proof.** Let $R$ be such that for each $x \in \bar{B}$, $\|x\| \leq R$. Let $\lambda_0 > 0$ be such that if $|\lambda| \geq \lambda_0$, then

$$\frac{|\lambda|c - R}{2} > R. \quad (1)$$

Then if $\lambda$ is any fixed number such that $|\lambda| \geq \lambda_0$, $\deg(I - \lambda A, \bar{B}, 0)$, where $I$ is the identity map, is defined because if $x \in \partial B$,

$$\|(I - \lambda A)x\| = \|(\lambda A - I)x\| > |\lambda|c - R > 2R > 0. \quad (2)$$

From the definition of the Leray–Schauder degree of a mapping, it follows that there exists $X_n$, a finite-dimensional subspace of $X$ which we may choose to be of odd dimension, and $A_n$, a mapping from $B$ into $X_n$, such that if $x \in \bar{B}$, then

$$\|(I - \lambda A)x - (I - \lambda A_n)x\| = \|\lambda Ax - \lambda A_nx\| < \frac{|\lambda|c - R}{2}. \quad (3)$$

Inequalities (2) and (3) show that $\deg(I - \lambda A_n, \bar{B}, 0)$ is defined and $\deg(I - \lambda A_n, \bar{B}, 0) = \deg(I - \lambda A, \bar{B}, 0)$. 

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But
\[ \text{deg}(I - \lambda A, B, 0) = \text{deg}(I - \lambda A_n/B \cap X_n, B \cap X_n, 0). \]

Thus
\[ \text{deg}(I - \lambda A, B, 0) = \text{deg}(I - \lambda A_n/B \cap X_n, B \cap X_n, 0). \] (4)

By (2) and (3), if \( x \in \partial B \) then
\[ \| (I - \lambda A_n) x \| \geq \| (I - \lambda A) x \| - \| (I - \lambda A_n) x - (I - \lambda A) x \|, \]
\[ \geq \lambda | e - R - \left[ \frac{\lambda | e - R |}{2} \right] = \frac{\lambda | e - R |}{2}. \] (5)

Also if \( x \in \partial B \), then by (1) and (5),
\[ \| (I - \lambda A_n) x + \lambda A_n x \| = \| x \| \leq R < \frac{\lambda | e - R |}{2} \leq \| (I - \lambda A) x \|. \] (6)

Inequality (6) shows that
\[ \text{deg}(\lambda A_n/B \cap X_n, B \cap X_n, 0) \]
is defined and
\[ \text{deg}(I - \lambda A_n/B \cap X_n, B \cap X_n, 0) = \text{deg}(\lambda A_n/B \cap X_n, B \cap X_n, 0). \] (7)

A similar argument shows that
\[ \text{deg}(I + \lambda A_n/B \cap X_n, B \cap X_n, 0) = \text{deg}(\lambda A_n/B \cap X_n, B \cap X_n, 0). \] (8)

Since \( X_n \) is of odd dimension, then
\[ \text{deg}(\lambda A_n/B \cap X_n, B \cap X_n, 0) \]
and
\[ \text{deg}(\lambda A_n/B \cap X_n, B \cap X_n, 0) \]
are of opposite sign. Hence at least one of them, say
\[ \text{deg}(\lambda A_n/B \cap X_n, B \cap X_n, 0), \]
is different from \(+1\). From (4) and (7), it follows that \( \text{deg}(I - \lambda A, B, 0) \) is different from \(+1\). On the other hand, by property 3) of \( A \), if \( \lambda \) is fixed, then if \( B_1 \) is a closed ball with center 0 and sufficiently small radius, then \( B_1 \subset B \) and
\[ \text{deg}(I - \lambda A, B_1, 0) = +1. \] (9)

Since \( \text{deg}(I - \lambda A, B, 0) \neq +1 \), it follows that
\[ \text{deg}(I - \lambda A, B - B_1, 0) \neq 0. \] (10)
Thus there exists $x \in B - \overline{B}_1$ such that

$$(I - \lambda A) x = 0,$$

i.e., $(1/\lambda)$ is an eigenvalue of $A$. If

$$\text{deg}(\lambda A_n/\overline{B} \cap \overline{X}_n, \overline{B} \cap \overline{X}_n, 0)$$

is different from $+1$, then it follows by the same kind of argument as immediately above that $-(1/\lambda)$ is an eigenvalue of $A$. The last conclusion of the theorem follows from (9), (10), and the properties of topological degree.

**Corollary 1.** If $A$ is even, i.e., if $A(x) = A(-x)$ for all $x \in D$, then there exists $\lambda_0 > 0$ such that for each $\lambda \geq \lambda_0$, the numbers $(1/\lambda)$ and $-(1/\lambda)$ are eigenvalues of $A$ and for each such eigenvalue there is an eigenfunction in $B$.

**Proof.** If $(1/\lambda)$ is an eigenvalue of $A$, i.e., if there exists $f \neq 0$ such that

$$(I - \lambda A) x = 0,$$

then

$$-(\lambda) A(-x) = -[x - \lambda A(x)] = 0.$$ 

That is, $-(1/\lambda)$ is an eigenvalue and $-x$ is a corresponding eigenfunction.

Next suppose $A$ is a transformation defined on $D$, a set in a normed linear space $X$ such that $A : D \to X$ and assume $A$ has the following properties:

1. $A$ takes bounded sets into bounded sets;
2. $A(D)$ is contained in a finite dimensional subspace $N$ of $X$;
3. there is a bounded open set $B$ such that $B \subset D$ and $0 \in B$ and a constant $c > 0$ such that if $x \in N \cap \partial B$ then $\|Ax\| \geq c$;
4. $\lim_{\|x\| \to 0} \frac{\|Ax\|}{\|x\|} = 0$;
5. $A$ is even.

**Theorem 2.** There exists a number $\lambda_0 > 0$ such that the set $[-(1/\lambda_0), 0) \cup (0, (1/\lambda]]$ is contained in the set of eigenvalues of $A$ and for each eigenvalue in $[-(1/\lambda_0), 0) \cup (0, (1/\lambda]]$ there is an eigenfunction in $B$. Also for each $\lambda$ such that $|\lambda| \geq \lambda_0$ there is an open ball $B_\lambda$ such that if $y \in B_\lambda$, the equation

$$(I - \lambda A) x = y$$

and the equation

$$(I + \lambda A) x = y$$

each have two distinct solutions.
Proof. By the same kind of argument as in the proof of Theorem 1 we obtain: if $|\lambda|$ is sufficiently large,
\[
\deg(I - \lambda A/B \cap \overline{N}, B \cap \overline{N}, 0) = \deg(-\lambda A/B \cap \overline{N}, B \cap \overline{N}, 0).
\]
From the definition of Leray–Schauder degree, it follows that $\deg(I - \lambda A, B, 0)$ is defined and equals $\deg(I - \lambda A/B \cap \overline{N}, B \cap \overline{N}, 0)$. Since $A$ is even, $-\lambda A/B$ is even and hence by Krasnosel'skii [4, p. 223, fn.]
\[
\deg(-\lambda A/B \cap \overline{N}, B \cap \overline{N}, 0) \neq +1.
\]
The remainder of the proof is similar to that of Theorem 1.

Theorem 3. Let $A$ be a completely continuous transformation defined on $\partial B$, where $B$ is a bounded open set in a normed linear space $X$ such that $0 \in B$, such that $A : \partial B \to X$. Suppose there exists a positive constant $c$ such that if $x \in \partial B$, then
\[
\|Ax\| \geq c.
\]
Then $A$ has an eigenfunction on $\partial B$.

Proof. First by Dugundji's Extension Theorem [3] there exists an extension $\tilde{A}$ of $A$ to the set $\overline{B}$ such that $\tilde{A}$ is completely continuous on $\overline{B}$. We denote the extension $\tilde{A}$ by $A$. Since $\tilde{A}$ is completely continuous and hence bounded, then there exists $\lambda_1 > 0$ such that if $|\lambda| \leq \lambda_1$, then
\[
\deg(I - \lambda A, B, 0) = +1. \tag{11}
\]
By the same argument as used in the proof of Theorem 1, there exists $\lambda_2 > 0$ such that if $|\lambda| \geq \lambda_2$, then
\[
\deg(I - \lambda A, B, 0) = \deg(-\lambda A_n/B \cap \overline{X_n}, B \cap \overline{X_n}, 0), \tag{12}
\]
where $X_n$, $A_n$ have the same meaning as in the proof of Theorem 1. Since $X_n$ is of odd dimension, then either
\[
\deg(\lambda A_n/B \cap \overline{X_n}, B \cap \overline{X_n}, 0)
\]
or
\[
\deg(-\lambda A_n/B \cap \overline{X_n}, B \cap \overline{X_n}, 0)
\]
is different from $+1$. Suppose
\[
\deg(-\lambda A_n/B \cap \overline{X_n}, B \cap \overline{X_n}, 0)
\]
is different from $+1$. Then it follows from (11) and (12) that
\[
I - \lambda_1 A - t(\lambda_2 - \lambda_1) A/\partial B \quad 0 \leq t \leq 1
\]
is not a homotopy in $X - \{0\}$. That is, there exists $t_0 \in (0, 1)$ and $x_0 \in \partial B$ such that

$$[I - \lambda_1 A - t_0(\lambda_2 - \lambda_1) A] (x_0) = 0.$$ 

Thus $\{1/[\lambda_1 + t_0(\lambda_2 - \lambda_1)]\}$ is an eigenvalue and $x_0$ is the corresponding eigenfunction on $\partial B$. If

$$\deg(\lambda A_n/\bar{B} \cap X_n, \bar{B} \cap X_n, 0)$$

is different from $+1$, then a similar argument shows that

$$\{1/[\lambda_1 - t_0(\lambda_2 - \lambda_1)]\}$$

is an eigenvalue with corresponding eigenfunction on $\partial B$.

Notice that we obtain bounds on the eigenvalue, i.e., the eigenvalue is in the interval $(\lambda_1, \lambda_2)$ or $(-\lambda_2, -\lambda_1)$.

Now we sharpen the hypotheses on $A$ and obtain correspondingly sharper results concerning the set of eigenvalues. We assume that $A$ is a completely continuous transformation from a normed linear space $X$ into itself and that $A$ satisfies the following conditions:

1. there exists a positive constant $M > 0$ such that if $\|x\| > M$, then
   $$\|Ax\| > c \|x\|^{1+\delta},$$
   where $\delta$ and $c$ are positive constants.

2. $\lim_{\|x\| \to 0} \frac{\|Ax\|}{\|x\|} = 0$.

**Theorem 4.** If $\mu \in [0, 1]$, then either $\mu$ or $-\mu$ is an eigenvalue of $A$ and there is an open ball $B_\mu$ with center 0 such that for each $y \in B_\mu$, the equation

$$[I - (1/\mu) A] x = y$$

or the equation

$$[I + (1/\mu) A] x = y$$

has at least two distinct solutions.

**Proof.** Let $\epsilon_0$ be a fixed positive number. If $\bar{B}$ is a closed ball with center 0 and sufficiently large radius and if $x \in \partial B$ and $|\lambda| \geq 1$, then

$$\|x\| < \epsilon \|x\|^{1+\delta} - \epsilon < \|\lambda Ax\| - \epsilon$$

for all $\epsilon$ such that $0 < \epsilon \leq \epsilon_0$. Take $B$ fixed and take $\lambda$ fixed such that $|\lambda| \geq 1$. Let

$$r = \max_{x \in \partial B} \|(I - \lambda A)x\|.$$
If \( r = 0 \), then since \( A \) is completely continuous, \((1/\lambda)\) is an eigenvalue of \( A \) and the corresponding eigenfunction is in \( \partial B \). Suppose \( r > 0 \). Let \( \epsilon = \min(r, \epsilon_0) \). From the definition of the Leray-Schauder degree, there exists \( X_n \), a finite-dimensional space of odd dimension, and a continuous mapping \( A_n \) from \( X \) into \( X \) such that if \( x \in \overline{B} \), then
\[
\text{deg}(I - \lambda A_n, B \cap X_n, 0) = \text{deg}(I - \lambda A, \overline{B}, 0)
\]
and
\[
\| \lambda Ax - \lambda A_n x \| < \epsilon.
\]
Then
\[
\| \lambda Ax \| - \| \lambda A_n x \| < \epsilon \quad \text{and} \quad \| \lambda Ax \| - \epsilon < \| \lambda A_n x \|.
\] (14)

If \( x \in X_n \cap \partial B \), then by (13) and (14),
\[
\| (I - \lambda A_n)x \| - \| \lambda A_n x \| < \epsilon \quad \text{and} \quad \| \lambda Ax \| - \epsilon < \| \lambda A_n x \|.
\]

Therefore
\[
\text{deg}(I - \lambda A_n, B \cap X_n, 0) = \text{deg}(-\lambda A_n, B \cap \overline{X_n}, 0).
\]

The remainder of the proof is the same as the proof of Theorem 1.

**Corollary 4.** If \( A \) is even, the set of eigenvalues of \( A \) contains the set \([-1, 0) \cup (0, 1] \). If \( \mu \in [-1, 0) \cup (0, 1] \), there is a ball \( B_\mu \) with center 0 such that for each \( y \in B_\mu \), the equation
\[
[I - (1/\mu) A] x = y
\]
has two distinct solutions.

### 3. Application to Integral Equations

The applications to be described are in the simplest possible space of continuous functions. They could be given in considerably more general spaces.

Let \( C \) be the space of continuous functions on \([0, 1]\) with the \( L_p \) norm where \( p \) is any positive number such that \( p > 1 \). Let \( K(s, t) \) be a continuous function on \([0, 1] \times [0, 1]\) and suppose there exist positive constants \( m, M \) such that for all \((s, t) \in [0, 1] \times [0, 1]\),
\[
m < K(s, t) < M.
\]
The equation
\[ \mu \phi(s) - \int_0^1 K(s, t) |\phi(t)|^\mu \, dt = 0 \]

has a nonzero solution \( \phi(t) \in \mathcal{C} \) for each \( \mu \in [-1, 0) \cup (0, 1] \).

Proof. We must show that the hypotheses of Theorem 4 are satisfied. First the mapping

\[ A : \phi(t) \rightarrow \int_0^1 K(s, t) |\phi(t)|^\mu \, dt \]

is clearly a mapping from \( \mathcal{C} \) into \( \mathcal{C} \). Also the mapping is a completely continuous mapping from \( \mathcal{C} \) into \( \mathcal{C} \) because if \( \{\phi_n(t)\} \) is a sequence such that

\[ \| \phi_n \|^p = \int_0^1 |\phi_n(t)|^p \, dt < M_0, \]

where \( M_0 \) is a positive constant, then

\[ \max_{s \in [0, 1]} |\int_0^1 K(s, t) |\phi_n(t)|^\mu \, dt| \leq MM_0. \]

Also since \( K(s, t) \) is continuous, the functions

\[ \int_0^1 K(s, t) |\phi_n(t)|^\mu \, dt \]

are equicontinuous on \([0, 1]\). Hence by Arzela's theorem there exists a subsequence \( \phi_{n_j}(t) \) and a continuous function \( g(t) \in \mathcal{C} \) such that the sequence

\[ f_j(s) = \int_0^1 K(s, t) |\phi_{n_j}(t)|^\mu \, dt \]

converges uniformly to \( g(t) \). But then the sequence \( \{f_j(t)\} \) converges in the \( L_p \) norm to \( g(t) \).

Next, if \( \phi(t) \in \mathcal{C} \),

\[ \frac{\|A\phi\|^p}{\|\phi\|^p} = \frac{\left\| \int_0^1 K(s, t) |\phi(t)|^\mu \, dt \right\|^p}{\|\phi\|^p} \leq \frac{\int_0^1 \int_0^1 K(s, t) |\phi(t)|^\mu \, dt \, ds}{\|\phi\|^p} \]

\[ \leq \frac{M^p \int_0^1 \int_0^1 |\phi(t)|^\mu \, dt \, ds}{\|\phi\|^p} \leq M_p \frac{\|\phi\|^p}{\|\phi\|^p}. \]
Hence
\[ \lim_{\|\phi\| \to 0} \frac{\|A\phi\|}{\|\phi\|} = 0. \]

Finally if \( \phi(t) \in \mathcal{C} \), then
\[
\|A\phi\|^p = \int_0^1 \left( \int_0^1 K(s, t) |\phi(t)|^p \, dt \right)^p \, ds \\
= m^p \|\phi(t)\|^p.
\]

Therefore
\[ \|A\phi\| \geq m \|\phi\|^p. \]

REFERENCES