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Almost completely decomposable groups and unbounded representation type

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ABSTRACT

Almost completely decomposable groups with a regulating regulator and a p-primary regulator quotient are studied. It is shown that there are indecomposable such groups of arbitrarily large rank provided that the critical typeset contains some basic configuration and the exponent of the regulator quotient is sufficiently large.

1. Introduction

A torsion-free abelian group G of finite rank is *completely decomposable* if G is isomorphic to a finite direct sum of subgroups of $\mathbb{Q}$, the additive group of rational numbers, and *almost completely decomposable* if G has a completely decomposable subgroup C with G/C a finite group. Almost completely decomposable groups are a notoriously complicated class of torsion-free abelian groups of

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finite rank, [9,1,10], the source of many pathological examples, [8], and have been generalized to infinite rank, [12].

A *type* is an isomorphism class $[X]$ of a subgroup X of $\mathbb{Q}$. The set of all types is a partially ordered set (poset), where $[X] \leqslant [Y]$ if X is isomorphic to a subgroup of Y. The *meet* of two types $[X]$ and $[Y]$ is $[X] \wedge [Y] = [X \cap Y]$ and the *join* is $[X] \vee [Y] = [X + Y]$.

Let G be an almost completely decomposable group with a completely decomposable subgroup R such that G/R is finite. The *critical typeset* of G is $\mathrm{T_{cr}}(G) = \{[X]\colon\ X \text{ rank-1 summand of } R\} = \mathrm{T_{cr}}(R)$. The *typeset* $\mathrm{Tst}(G)$ of G is $\{[X]\colon\ X \text{ pure rank-1 subgroup of } G\}$. The typeset of G is the meet closure of the critical typeset of G and is finite.

A subgroup R of an almost completely decomposable group G is a *regulating subgroup* of G if and only if R is completely decomposable (c.d.) and $|G/R|$ is the least integer in the set $\{|G/C|\colon\ C \text{ is c.d. with } G/C \text{ finite}\}$, [9].

The *regulator* $\mathrm{R}(G)$ is the intersection of all regulating subgroups of G. It is well known that the regulator is again completely decomposable, has finite index in G and is fully invariant.

Given a finite partially ordered set S of types and an integer $m \geqslant 1$, an *S-group with p^m-regulator quotient* is an almost completely decomposable group G with critical typeset $\mathrm{T_{cr}}(G) \subseteq S$ and $p^m G \subseteq \mathrm{R}(G)$, the regulator subgroup of G, e.g. see [14,15,11,13,6,10].

Let $\mathrm{w}(S)$ denote the *width of* S, the length of a maximal antichain contained in S. Define S-groups with p^m-regulator quotients to have *unbounded representation type* if there are indecomposable S-groups with p^m-regulator quotients of arbitrarily large finite rank.

The main result of this paper is:

Theorem 1. *Let p be a prime, S a finite p-locally free poset of types and $m \geqslant 1$ an integer. Then S-groups with p^m-regulator quotients have unbounded representation type if*

(1) $m = 1$, *S contains* $(1,1,1,1)$, $(2,2,2)$, $(1,3,3)$, $(1,2,5)$, *or* $(N,4)$;
(2) $m \geqslant 2$, $\mathrm{w}(S) \geqslant 3$;
(3) $m \geqslant 6$, $(1,2) \subseteq S$;
(4) $m \geqslant 4$, $(1,3) \subseteq S$;
(5) $m \geqslant 3$, $(2,2) \subseteq S$.

The class of S-groups with p^m-regulator quotients arises naturally in a more general context. If G is an almost completely decomposable group and τ is a type, then

$$G(\tau) = \sum \bigl\{X\colon\ X \text{ pure rank-1 subgroup of } G \text{ with } [X] \geqslant \tau \bigr\}$$

is a pure subgroup of G. Let T be a finite p-locally free lattice of types and S^T the poset of join irreducible elements of T. Given a prime p and positive integer m, $\mathrm{C}(T,p,m)$ denotes the isomorphism at p category of almost completely decomposable groups G with $\mathrm{Tst}(G) \subseteq T$ and $p^m G \subseteq \sum\{G(\tau)\colon\ \tau \in S^T\} \subseteq G$. The representation type of $\mathrm{C}(T,p,m)$ has been characterized in terms of m and the opposite of the poset S^T via representations of finite posets over discrete valuation rings, [4, Corollary 4.2].

Define $\mathrm{C}_{crit}(T,p,m)$ to be the full subcategory of groups G in $\mathrm{C}(T,p,m)$ with $\mathrm{T_{cr}}(G) \subseteq S^T$. In general, $\mathrm{C}_{crit}(T,p,m) \neq \mathrm{C}(T,p,m)$, each S^T-group with p^m-regulator quotient is in $\mathrm{C}_{crit}(T,p,m)$, but a group in $\mathrm{C}_{crit}(T,p,m)$ need not be an S^T-group with p^m-regulator quotient. As a result, the conditions of Theorem 1 also give S^T and m for which $\mathrm{C}_{crit}(T,p,m)$ has unbounded representation type. This extends results in [2] and answers some open questions in [1].

The converse of Theorem 1 for the case that S is an inverted forest is addressed in [3].

2. Preliminaries

A *chain* is a finite linearly ordered poset designated by n, the number of elements in the poset. The poset $\{1 < 2 > 3 < 4\}$ is denoted by N. If each S_i is a poset, then the disjoint union $S_1 \cup \cdots \cup S_m$

is a poset denoted by $(S_1, S_2, \dots, S_m)$. For example, $(1,2)$ is the disjoint union of chains of length 1 and 2 and $(N,4)$ is the disjoint union of N and a chain of length 4. A finite poset S is an *inverted forest* if for each $s \in S$, $\{t \in S\colon\ t \geqslant s\}$ is linearly ordered. For example, if S is an inverted forest with $\mathrm{w}(S) = 2$, then $S = (k,n)$ for some $1 \leqslant k \leqslant n$.

If G is an almost completely decomposable group, then $G(\tau) = G_\tau \oplus G^\sharp(\tau)$ for each type $\tau \in \mathrm{T}_{\mathrm{cr}}(G)$, where $G^\sharp(\tau)$ is the pure subgroup of G generated by $\{X\colon\ X$ pure rank-1 subgroup of G with $[X] > \tau\}$ and G_τ is a pure subgroup of G isomorphic to a finite direct sum of copies of a rank-1 group Y with $[Y] = \tau$, [5]. A regulating subgroup need not be unique. The direct sum of the subgroups G_τ for $\tau \in \mathrm{T}_{\mathrm{cr}}(G)$ is a regulating subgroup of G and conversely, if R is a regulating subgroup of G and $R = \bigoplus_{\tau \in \mathrm{T}_{\mathrm{cr}}(A)} R_\tau$ is a decomposition of R with τ-homogeneous completely decomposable summands R_τ, then $G(\tau) = R_\tau \oplus G^\sharp(\tau)$, [9].

It can happen that an almost completely decomposable group contains exactly one regulating subgroup that then coincides with the regulator. In this case the regulator is regulating and we have a *regulating regulator*. The following lemma was first proved in [7, Satz 5.1] and reproved in [10, Proposition 4.1] and [1, Corollary 3.2.13].

Lemma 2. *Let G be an almost completely decomposable group. If $\mathrm{T}_{\mathrm{cr}}(G)$ is an inverted forest, then*

$$\mathrm{R}(G) = \sum \bigl\{G(\tau)\colon\ \tau \in \mathrm{T}_{\mathrm{cr}}(G)\bigr\} = \bigoplus \bigl\{R_\tau\colon\ \tau \in \mathrm{T}_{\mathrm{cr}}(G)\bigr\}$$

is the regulating regulator of G and

$$G(\tau) = \mathrm{R}(G)(\tau) = \bigoplus \bigl\{R_\sigma\colon\ \tau \leqslant \sigma \in \mathrm{T}_{\mathrm{cr}}(G)\bigr\}$$

for each $\tau \in \mathrm{T}_{\mathrm{cr}}(G)$.

Given a prime p, a poset $(S, \leqslant)$ of types is *p-locally free* if $pX \neq X$ for each $[X] \in S$.

Two almost completely decomposable groups G and H are *isomorphic at p* if there is an integer n prime to p and homomorphisms $f : G \to H$ and $g : H \to G$ with $fg = n1_H$ and $gf = n1_G$. The groups G and H are *nearly isomorphic* if they are isomorphic at p for every prime p. Other characterizations of isomorphism at p and near isomorphism are given in [1, Chapter 2] and [10, Chapter 9]. The regulator $\mathrm{R}(G)$ of an almost completely decomposable group G and the regulator quotient $G/\mathrm{R}(G)$ are near-isomorphism invariants. If H is nearly isomorphic to $G \oplus K$ for some almost completely decomposable group K, then H has a group summand nearly isomorphic to G, [1, Corollary 5.1.8.b].

Lemma 3. *(See [1, Lemma 5.4.1].) Assume that S is a finite p-locally free poset of types and that G and H are S-groups with p^m-regulator quotients.*

(1) *G and H are nearly isomorphic if and only if G and H are isomorphic at p.*
(2) *G is an indecomposable group if and only if G is isomorphic at p to an indecomposable group.*

3. Groups and anti-representations

Let p be a prime and $(S, \leqslant)$ a finite p-locally free inverted forest of types. Define $\mathrm{cdrep}(S, \mathbb{Z}_{p^m})$ to be the collection of objects $U = (U_0, U_s, U_*\colon\ s \in S)$ such that for each $s \in S$, there is a finitely generated free $\mathbb{Z}_{p^m}$-module V_s with $U_0 = \bigoplus_{s \in S} V_s$, $U_s = \bigoplus_{s \leqslant t \in S} V_t$, $U_t \subseteq U_s$ whenever $s \leqslant t$ (note the reversal of the order), and U_* a submodule of U_0 with $U_s \cap U_* = 0$ for each $s \in S$. Notice that U_* is finitely generated but need not be a free $\mathbb{Z}_{p^m}$-module. An object U of $\mathrm{cdrep}(S, \mathbb{Z}_{p^m})$ is called an *anti-representation* in [10].

Homomorphisms from $U = (U_0, U_s, U_*\colon\ s \in S)$ to $W = (W_0, W_s, W_*\colon\ s \in S)$ are $\mathbb{Z}_{p^m}$-homomorphisms $f : U_0 \to W_0$ with $f(U_s) \subseteq W_s$ for each $s \in S \cup \{*\}$. An object U is indecomposable if whenever $U = Y \oplus W = (Y_0 \oplus W_0, Y_s \oplus W_s, Y_* \oplus W_*\colon\ s \in S)$, then either $Y_0 = 0$ or $W_0 = 0$. It is readily verified that U is indecomposable if and only if 0 and 1 are the only idempotents of $\mathrm{End}(U)$, the endomorphism ring of U in $\mathrm{cdrep}(S, \mathbb{Z}_{p^m})$.

Suppose that $R = \bigoplus\{R_s\colon s \in \mathrm{T_{cr}}(R)\}$ is a completely decomposable group, where each R_s is a direct sum of n_s rank-1 groups of type s and $n = \sum_{s\in \mathrm{T_{cr}}(R)} n_s = \mathrm{rank}(R)$. An ordered subset $\{x_{si} \in R_s\colon s \in \mathrm{T_{cr}}(R),\ 1 \leqslant i \leqslant n_s\}$ of R is a *p-basis* of R *with coefficient groups* R_{si} if $R = \bigoplus\{R_{si}x_{si}\colon s \in \mathrm{T_{cr}}(R),\ 1 \leqslant i \leqslant n_s\}$, $R_{si} = \{q \in \mathbb{Q}\colon qx_{si} \in R\} \subseteq \mathbb{Q}$ has type s, and $1/p \notin R_{si}$ for each s and i. Notice that R has a p-basis if and only if the critical types of R are p-locally free.

Let G be an almost completely decomposable group with a completely decomposable subgroup R such that G/R is a finite p-group and $\{x_{si} \in R_s\colon s \in \mathrm{T_{cr}}(R),\ 1 \leqslant i \leqslant n_s\}$ is a p-basis of R with coefficient groups R_{si}. An integer matrix $M = (m_{si})_{r\times n}$ is a *coordinate matrix of G modulo R* if there is an ordered basis $(h_1 + R, \dots, h_r + R)$ of G/R with each $h_i = (1/p^{k_i})(\sum_{s\in S} m_{si}x_{si})$ and $p^{k_i} = \mathrm{order}(h_i + R)$. The *structure matrix belonging to* $(h_1 + R, \dots, h_r + R)$ is the diagonal matrix

$$N = \mathrm{diag}\bigl(p^{k_1}, \dots, p^{k_r}\bigr).$$

In summary, $G = R + \mathbb{Z}^r N^{-1} M\vec{x}$, where $\mathbb{Z}^r$ denotes the set of integral row vectors and $\vec{x} = (x_{si})^{tr}$ is a column vector. This is called a *standard description* of G and if $m \geqslant k_i$ for $i = 1, \dots, r$, then the matrix $p^m\mathbb{Z}^r N^{-1}M$ is integral and $M_G = p^m\mathbb{Z}^r N^{-1}M (\mathrm{mod}\ p^m)$ is called a *representing matrix of G*. By [10, Corollaries 11.2.5 and 11.3.4] $G/R \cong \mathbb{Z}_{p^{k_1}} \oplus \cdots \oplus \mathbb{Z}_{p^{k_r}}$ if and only if $\gcd(M, N) = I$.

Lemma 4. *Assume that p is a prime and S is a finite p-locally free inverted forest of types.*

(1) *There is a bijection $[G] \mapsto [U_G]$ on isomorphism at p classes of S-groups with p^m-regulator quotients and $S = \mathrm{T_{cr}}(G)$ to isomorphism classes of objects of* $\mathrm{cdrep}(S, \mathbb{Z}_{p^m})$.
(2) *G is indecomposable if and only if U_G is indecomposable.*

Proof. (1) Let G be an S-group with p^m-regulator quotient and $S = \mathrm{T_{cr}}(G)$. By Lemma 2, G has the regulating regulator $R = \sum\{R(s)\colon s \in S\} = \bigoplus\{R_s\colon s \in S\}$ and $p^mG \subseteq R \subseteq G$, $G(s) = R(s) = \bigoplus\{R_t\colon s \leqslant t \in S\}$, and each R_s is isomorphic to a direct sum of n_s rank-1 groups of type s.

Define $U_G = (U_0, U_s, U_*\colon s \in S)$, where $U_0 = R/p^mR$, $U_s = (R(s) + p^mR)/p^mR \subseteq U_0$ for each $s \in S$, and $U_* = p^mG/p^mR \subseteq U_0$. Notice that U_0 and every $U_s \cong R(s)/p^mR(s)$ are free $\mathbb{Z}_{p^m}$-modules. Define $V_s = (R_s + p^mR)/p^mR$, a summand of U_0 for each $s \in S$. It follows that $U_0 = \bigoplus_{s\in S} V_s$ and $U_s = \bigoplus_{s\leqslant t\in S} V_t$ for each $s \in S$.

Let $G = R + \mathbb{Z}^r N^{-1}M\vec{x}$ be a standard description of G, with $\vec{x} = (x_{si})^{tr}$. Then $U_0 = \bigoplus\{\mathbb{Z}_{p^m}(x_{si} + p^mR)\colon s \in S,\ 1 \leqslant i \leqslant n_s\}$ and $U_* = (p^m\mathbb{Z}^r N^{-1}M\vec{x} + p^mR)/p^mR \subseteq U_0$. Hence, $U_* \cap U_s = 0$ for each $s \in S$, because R is the regulating regulator of G [10, Corollary 8.1.12], and so $U_G \in \mathrm{cdrep}(S, \mathbb{Z}_{p^m})$.

Assume that $f : G \to H$ is an isomorphism at p. Since S is an inverted forest, and by Lemma 2, f induces an isomorphism $\bar f : R(G)/p^mR(G) \to R(H)/p^mR(H)$ with $\bar f(G(s)/p^mG(s)) = H(s)/p^mH(s)$ for each $s \in S = \mathrm{T_{cr}}(G) = \mathrm{T_{cr}}(H)$ and $\bar f(p^mG/p^mR(G)) = p^mH/p^mR(H)$. Hence, $\bar f : U_G \to U_H$ is an isomorphism.

As for onto, let $U = (U_0, U_s, U_*\colon s \in S) \in \mathrm{cdrep}(S, \mathbb{Z}_{p^m})$ with $U_0 = \bigoplus_{s\in S} V_s$, $U_s = \bigoplus_{s\leqslant t\in S} V_t$ and $U_* \cap U_s = 0$ for each $s \in S$. Choose a completely decomposable group R_U with $\mathrm{T_{cr}}(R_U) = S$ and $R_U = \bigoplus\{R_s\colon s \in S\}$ such that $U_0 = R_U/p^mR_U$, and each $U_s = R_s/p^mR_s$. For each $s \in S$, choose a p-basis $\{x_{si} \in R_s\colon s \in S,\ 1 \leqslant i \leqslant n_s\}$ of R_U with coefficient groups R_{si} and observe that $B = \{v_{si} = x_{si} + p^mR_U\colon s \in S,\ 1 \leqslant i \leqslant n_s\}$ is a basis for U_0 and each $B_s = \{v_{si} = x_{si} + p^mR_U\colon 1 \leqslant i \leqslant n_s\}$ is a basis for U_s.

Write $U_* \cong \mathbb{Z}^{l_1}_{p^{k_1}} \oplus \cdots \oplus \mathbb{Z}^{l_r}_{p^{k_r}}$ with ordered basis $(h_1, \dots, h_l)$, where $l = l_1 + \cdots + l_r$. Then $h_i = \sum\{m_{i,sj}v_{sj}\colon s \in S,\ 1 \leqslant j \leqslant n_s\}$, $M = (m_{i,sj})$ is a $\mathbb{Z}_{p^m}$-matrix, and $p^mN_U^{-1}M$ is a $\mathbb{Z}_{p^m}$-matrix, where

$$N_U = \mathrm{diag}\bigl(p^{k_1}, \dots, p^{k_r}\bigr) \quad \text{and} \quad m = \max\{k_1, \dots, k_r\}.$$

Let M_U be an integer matrix with $M = p^m N_U^{-1} M_U (\mathrm{mod}\ p^m)$ and define $G_U = R_U + \mathbb{Z}^n N_U^{-1} M_U \vec{x}$. Then R_U is the regulating regulator of G_U since $U_* \cap U_s = 0$ for each $s \in \mathrm{T}_{\mathrm{cr}}(G_U)$, [10, Corollary 8.1.12]. It now can be readily verified that $\mathrm{T}_{\mathrm{cr}}(G_U) = S$, G_U is an S-group with p^m-regulator quotient, and $U_{G_U} = U$.

Finally, let $f : U \to W$ be an isomorphism in $\mathrm{cdrep}(S, \mathbb{Z}_{p^m})$. In view of the constructions of R_U and R_W and the fact that $f : U_0 \to W_0$ with $f(U_s) \subseteq W_s$ for each $s \in S$ there is $g : R_U \to R_W$ with g an isomorphism at p and $\bar{g} = f$. Since $f(U_*) \subseteq W_*$, g extends to a map $h : G_U \to G_W$ that is an isomorphism at p.

(2) [10, Corollary 10.7]. □

We follow up with an illustration of some notation and constructions in the proof of Lemma 4(1). It shows that, given a poset S of p-locally free types and a representing matrix M, it is easy to construct an almost completely decomposable group G whose critical typeset is S and whose anti-representation has the representing matrix M. By Lemma 4(1) the group G is unique up to isomorphism at p.

For the purposes of this paper a *rational group* is a subgroup Q of $\mathbb{Q}$ such that $1 \in Q$ but $1/p \notin Q$. If so, then $\{1\}$ is a p-basis of Q. More generally, if $R = \bigoplus_{i \in I} R_i$ is a direct sum of rational groups, then R has the natural p-basis $\{(1, 0, \ldots), \ldots, (\ldots, 0, 1)\}$.

Example 5. Let $S = \{1, 2 < 3\}$ be a poset of p-locally free types and let $U = (U_0, U_i, U_*\colon\ i \in S)$ with the representing matrix (coefficients in $\mathbb{Z}_{p^6}$)

$$M = \begin{pmatrix} I_n & 0 & 0 & | & p^2 I_n & 0 & \vdots & I_n & 0 \\ 0 & p^2 I_n & 0 & | & p^3 I_n & p^4 I_n & \vdots & 0 & p^2 I_n \\ 0 & 0 & p^4 I_n & | & p^4 I_n & p^5 A & \vdots & 0 & 0 \end{pmatrix},$$

where A is an $n \times n$ integer matrix.

The rows of M are the generators of U_* and an arbitrary element of U_* is given by

$$\bigl(u, p^2 v, p^4 w \bigm| p^2 u + p^3 v + p^4 w, p^4 v + p^5 w A \bigm| u, p^2 v\bigr) = (u, v, w)M,$$

where $u, v, w \in \mathbb{Z}_{p^6}^n$. From this it is easily seen that $U_i \cap U_* = 0$, showing that $U \in \mathrm{cdrep}(S, \mathbb{Z}_{p^6})$. It is also obvious that the rows of M are independent, the generators of the first block have orders p^6, the generators of the second block have orders p^4, and the generators of the third block have orders p^2. Hence $U_* \cong \mathbb{Z}_{p^6}^n \oplus \mathbb{Z}_{p^4}^n \oplus \mathbb{Z}_{p^2}^n$.

Let $N = \mathrm{diag}(p^6 I_n, p^4 I_n, p^2 I_n)$. The integer matrix

$$M_0 := \begin{pmatrix} I_n & 0 & 0 & | & p^2 I_n & 0 & \vdots & I_n & 0 \\ 0 & I_n & 0 & | & p I_n & p^2 I_n & \vdots & 0 & I_n \\ 0 & 0 & I_n & | & I_n & pA & \vdots & 0 & 0 \end{pmatrix}$$

is such that $p^6 N^{-1} M_0 \equiv M \bmod p^6$. Let R_1, R_2, R_3 be rational groups such that $[R_i] = i$. Let $R = R_1^{3n} \oplus R_2^{2n} \oplus R_3^{2n}$, and let $G = R + \vec{\mathbb{Z}} N^{-1} M_0 \subseteq \mathbb{Q}^{7n}$. Then G is an almost completely decomposable group with completely decomposable subgroup R and $G/R \cong \mathbb{Z}(p^6)^n \oplus \mathbb{Z}(p^4)^n \oplus \mathbb{Z}(p^2)^n$ because obviously $\gcd(N, M_0) = I$. Also $\mathrm{T}_{\mathrm{cr}}(G) = \mathrm{T}_{\mathrm{cr}}(R) = S$. Furthermore, $U_G = (U_0^G, U_i^G, U_*^G\colon\ i \in S) \in \mathrm{cdrep}(S, \mathbb{Z}_{p^6})$ is given by

$$U_0^G = R/p^6 R, \qquad U_i^G = \bigl(R(i) + p^6 R\bigr)/p^6 R, \qquad U_*^G = \vec{\mathbb{Z}} M.$$

Clearly U and U_G are isomorphic anti-representations. In particular, $U_i \cap U_* = 0$ for every i, showing that R is the regulating regulator of G.

4. Unbounded representation type

An S-group G with p^m-regulator quotient and $\mathrm{T_{cr}}(G) = S$ will be proven indecomposable, via Lemma 4(2), by showing that $\mathrm{End}(U_G)$ has only 0 and 1 as idempotents.

Let $U = (U_0, U_*, U_s\colon\ s \in S)$ where $U_0 = \bigoplus_{s\in S} V_s$, and $U_s = \bigoplus_{s\leqslant t\in S} V_t$ for each s and let $f \in \mathrm{End}(U)$.

For convenience we assume that S is represented by $\{1, \dots, k\}$ as will be the case in all our applications.

Let $\pi_i : U_0 \to V_i$ and $\iota_i : V_i \hookrightarrow U_0$ be the projections and insertions corresponding to the decomposition $U_0 = \bigoplus_{1\leqslant i\leqslant k} V_i$. Set $f_i = \pi_i f \iota_i : V_i \to V_i$ and $h_{ij} = \pi_j f \iota_i : V_i \to V_j$. Then f can be written in matrix form as

$$f = \begin{pmatrix} f_1 & h_{12} & \cdots & h_{1k} \\ h_{21} & f_2 & \cdots & h_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ h_{k1} & h_{k2} & \cdots & f_k \end{pmatrix}.$$

The action of f on $x = (x_1, \dots, x_k) \in U_0$ is by matrix multiplication:

$$f(x) = (x_1, \dots, x_k) \begin{pmatrix} f_1 & h_{12} & \cdots & h_{1k} \\ h_{21} & f_2 & \cdots & h_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ h_{k1} & h_{k2} & \cdots & f_k \end{pmatrix} = \left(f_1(x_1) + \sum_{j>1} h_{j1}(x_j), \dots, f_k(x_k) + \sum_{j<k} h_{j1}(x_j) \right).$$

Since $f(U_i) \subseteq U_i$ the poset structure $<_S$ of $S = \{1, \dots, k\}$ requires that $h_{ij} = 0$ unless $i <_S j$.

We adopt the following notation:

- $\bar{f} = f + p\,\mathrm{End}(U_0) \in \mathrm{End}(U_0)/p\,\mathrm{End}(U_0)$ for $f \in \mathrm{End}(U_0)$,
- $\bar{f}_i = f_i + p\,\mathrm{End}(V_i) \in \mathrm{End}(V_i)/p\,\mathrm{End}(V_i)$ for $f_i \in \mathrm{End}(V_i)$,
- $E = \mathrm{End}(\mathbb{Z}_{p^m}^n)$, $\bar{E} = E/pE = \mathrm{End}(\mathbb{Z}_p^n)$, and $\bar{g} = g + pE \in \bar{E}$ for $g \in E$.

Each $\mathrm{End}(V_i)$ is a free $\mathbb{Z}_{p^m}$-module because V_i is a free $\mathbb{Z}_{p^m}$-module. Consequently, if $f_i \in \mathrm{End}(V_i)$ and $1 \leqslant j \leqslant m-1$, then $p^j f_i = 0$ if and only if $f_i \in p^{m-j}\,\mathrm{End}(V_i)$.

Given a prime p, a positive integer n, and an $n \times n$ $\mathbb{Z}$-matrix $A = (a_{ij})$, define

$$A\big(\mathrm{mod}\ p^m\big) = \big(a_{ij}\big(\mathrm{mod}\ p^m\big)\big),$$

an $n \times n$ $\mathbb{Z}_{p^m}$-matrix. Choose A such that the minimal polynomial $m_{A(\mathrm{mod}\ p)}(x)$ of $A(\mathrm{mod}\ p)$ has degree n and is a power of an irreducible polynomial in $\mathbb{Z}_p[x]$.

For example, let $\lambda \in \mathbb{Z}$ and

$$A = J_n(\lambda) = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{pmatrix}$$

be an $n \times n$ Jordan block matrix. Then $m_{A(\mathrm{mod}\ p)}(x) = (x - \lambda(\mathrm{mod}\ p))^n \in \mathbb{Z}_p[x]$, and

$$\mathbb{Z}_p[x]/\langle m_{A(\text{mod } p)}(x)\rangle = \mathbb{Z}_p^n$$

is an indecomposable $\mathbb{Z}_p[x]$-module denoted by K^A.

Proposition 6. *(See [2], [1, Example 5.2.3].) Let p be a prime, $(1,1,1) \subseteq S$ a p-locally free poset of types, and $m \geqslant 2$. There are indecomposable S-groups with p^m-regulator quotients of arbitrarily large finite rank.*

In the proof of Proposition 6, $S = \{1 \mid 2 \mid 3\}$, $m = 2$ and

$$M_G = \begin{pmatrix} I_n & | & I_n & | & A \\ 0 & | & pI_n & | & pI_n \end{pmatrix},$$

where A is an $n \times n$ $\mathbb{Z}_{p^2}$-matrix with K^A indecomposable.

Proposition 7. *If p is a prime, $(1,2) \subseteq S$ is a finite poset of p-locally free types, and $m \geqslant 6$, then there are indecomposable S-groups with p^m-regulator quotients of arbitrarily large finite rank.*

Proof. Consider Example 5 where A is such that K^A is an indecomposable module.

It is left to show that G is indecomposable. We have the decomposition $U_0 = V_1 \oplus V_2 \oplus V_3$, where $V_1 \cong R_1^{3n}/p^6 R_1^{3n} \cong \mathbb{Z}_{p^6}^{3n}$, $V_2 \cong R_2^{2n}/p^6 R_2^{2n} \cong \mathbb{Z}_{p^6}^{2n}$, and $V_3 \cong R_3^{2n}/p^6 R_3^{2n} \cong \mathbb{Z}_{p^6}^{2n}$.

Let $f^2 = f \in \operatorname{End}(U)$. Then

$$f = \begin{pmatrix} f_1 & 0 & 0 \\ 0 & f_2 & h_{23} \\ 0 & 0 & f_3 \end{pmatrix},$$

where $f_i : V_i \to V_i$ and $h_{23} : V_2 \to V_3$.

As $f^2 = f$

$$\begin{pmatrix} f_1^2 & 0 & 0 \\ 0 & f_2^2 & f_2 h_{23} + h_{23} f_3 \\ 0 & 0 & f_3^2 \end{pmatrix} = \begin{pmatrix} f_1 & 0 & 0 \\ 0 & f_2 & h_{23} \\ 0 & 0 & f_3 \end{pmatrix}^2 = \begin{pmatrix} f_1 & 0 & 0 \\ 0 & f_2 & h_{23} \\ 0 & 0 & f_3 \end{pmatrix}$$

so that

$$f_i^2 = f_i, \qquad f_2 h_{23} + h_{23} f_3 = h_{23}.$$

Write $V_1 = V_{11} \oplus V_{12} \oplus V_{13}$, $V_2 = V_{21} \oplus V_{22}$, and $V_3 = V_{31} \oplus V_{32}$ with each $V_{ij} = \mathbb{Z}_{p^m}^n$ and let $\pi_{ij} : U_0 \to V_{ij}$ and $\iota_{ij} : V_{ij} \to U_0$ be the corresponding projections and insertions. Define $f_{ijk} = \pi_{ik} f_j \iota_{ij} : V_{ij} \to V_{ik}$ so that, in matrix form,

$$f_1 = \begin{pmatrix} f_{111} & f_{112} & f_{113} \\ f_{121} & f_{122} & f_{123} \\ f_{131} & f_{132} & f_{133} \end{pmatrix}, \qquad f_i = \begin{pmatrix} f_{i11} & f_{i12} \\ f_{i21} & f_{i22} \end{pmatrix}, \quad \text{where } i = 2, 3.$$

The homomorphism $f_{ijk} : V_{ij} = \mathbb{Z}_{p^m}^n \to V_{ik} = \mathbb{Z}_{p^m}^n$ is regarded as an element of $E := \operatorname{End}(\mathbb{Z}_{p^m}^n)$.

Let $x \in \mathbb{Z}_{p^m}^n$, and recall that $f(U_*) \subseteq U_*$.

Then $(x, 0, 0 \mid p^2x, 0 \mid x, 0)$ is in the span of the first block of generators of U_* and

$$f\bigl(x, 0, 0 \bigm| p^2x, 0 \bigm| x, 0\bigr) = \bigl(f_1(x, 0, 0) \bigm| f_2\bigl(p^2x, 0\bigr) \bigm| f_3(x, 0) + h_{23}\bigl(p^2x, 0\bigr)\bigr)$$
$$= \bigl(u_1, p^2v_1, p^4w_1 \bigm| p^2u_1 + p^3v_1 + p^4w_1, p^4v_1 + p^5w_1A \bigm| u_1, p^2v_1\bigr)$$

for some $u_1 = u_1(x)$, $v_1 = v(x)$, $w_1 = w(x) \in \operatorname{End}\mathbb{Z}^n_{p^6}$. Hence

$$f_1(x, 0, 0) = \bigl(u_1, p^2v_1, p^4w_1\bigr), \tag{1}$$

$$f_2\bigl(p^2x, 0\bigr) = \bigl(p^2u_1 + p^3v_1 + p^4w_1, p^4v_1 + p^5w_1A\bigr), \tag{2}$$

$$f_3(x, 0) + h_{23}\bigl(p^2x, 0\bigr) = \bigl(u_1, p^2v_1\bigr). \tag{3}$$

Then by (1), (2), and (3) we have

$$f_1\bigl(p^5x, 0, 0\bigr) = \bigl(p^5u_1(x), 0, 0\bigr), \qquad f_2\bigl(p^5x, 0\bigr) = \bigl(p^5u_1(x), 0\bigr), \qquad f_3\bigl(p^5x, 0\bigr) = \bigl(p^5u_1(x), 0\bigr).$$

Therefore,

$$p^5f_{111} = p^5u_1, \qquad p^5f_{112} = 0, \qquad p^5f_{113} = 0, \tag{4}$$

$$p^5f_{211} = p^5u_1, \qquad p^5f_{212} = 0, \tag{5}$$

$$p^5f_{311} = p^5u_1, \qquad p^5f_{312} = 0, \tag{6}$$

The element $(0, p^2x, 0 \mid p^3x, p^4x \mid 0, p^2x)$ is in the span of the second block of generators of U_* and

$$f\bigl(0, p^2x, 0 \bigm| p^3x, p^4x \bigm| 0, p^2x\bigr)$$
$$= \bigl(f_1\bigl(0, p^2x, 0\bigr) \bigm| f_2\bigl(p^3x, p^4x\bigr) \bigm| f_3\bigl(0, p^2x\bigr) + h_{23}\bigl(p^3x, p^4x\bigr)\bigr)$$
$$= \bigl(u_2, p^2v_2, p^4w_2 \bigm| p^2u_2 + p^3v_2 + p^4w_2, p^4v_2 + p^5w_2A \bigm| u_2, p^2v_2\bigr),$$

for some $u_2 = u_2(x)$, $v_2 = v_2(x)$, $w_2 = w_2(x) \in \operatorname{End}\mathbb{Z}^n_{p^m}$. Hence

$$f_1\bigl(0, p^2x, 0\bigr) = \bigl(u_2, p^2v_2, p^4w_3\bigr), \tag{7}$$

$$f_2\bigl(p^3x, p^4x\bigr) = \bigl(p^2u_2 + p^3v_2 + p^4w_2, p^4v_2 + p^5w_2A\bigr), \tag{8}$$

$$f_3\bigl(0, p^2x\bigr) + h_{23}\bigl(p^3x, p^4x\bigr) = \bigl(u_2, p^2v_2\bigr). \tag{9}$$

By (7), (8), and (9) we have $u_2 = p^2u'$ and

$$f_1\bigl(0, p^5x, 0\bigr) = \bigl(p^5u', p^5v_2, 0\bigr) \qquad f_2\bigl(p^5x, 0\bigr) = \bigl(p^5v_2, 0\bigr) \qquad f_3\bigl(0, p^5x\bigr) = \bigl(p^5u', p^5v_2\bigr).$$

Also

$$f_2\bigl(0, p^5x\bigr) = pf_2\bigl(p^3x, p^4x\bigr) - f_2\bigl(p^4x, 0\bigr) \overset{(8),(2)}{=} \bigl(p^5u' + p^4v_2 + p^5w_2, p^5v_2\bigr) - \bigl(p^4u_1 + p^5v_1, 0\bigr)$$
$$= \bigl(p^5u' + p^4v_2 + p^5w_2 - p^4u_1 - p^5v_1, p^5v_2\bigr). \tag{10}$$

Multiplying (10) by p we get

$$0=\left(p^5v_2-p^5u_1,0\right),\quad \text{so}\quad p^5v_2=p^5u_1. \tag{11}$$

Therefore, using (11),

$$p^5f_{121}=p^5u',\qquad p^5f_{122}=p^5u_1,\qquad p^5f_{123}=0, \tag{12}$$

$$p^5f_{211}=p^5u',\qquad p^5f_{212}=0, \tag{13}$$

$$p^5f_{222}=p^5u_1, \tag{14}$$

$$p^5f_{321}=p^5u',\qquad p^5f_{322}=p^5u_1. \tag{15}$$

Finally, $(0,0,p^4x\mid p^4x,p^5xA\mid 0,0)$ is in the span of the third block of generators of U_* and

$$\begin{aligned} &f\left(0,0,p^4x\,\middle|\,p^4x,p^5xA\,\middle|\,0,0\right)\\ &\quad=\left(f_1\left(0,0,p^4x\right)\,\middle|\,f_2\left(p^4x,p^5xA\right)\,\middle|\,f_3(0,0)+h_{23}\left(p^4x,p^5xA\right)\right)\\ &\quad=\left(u_3,p^2v_3,p^4w_3\,\middle|\,p^2u_3+p^3v_3+p^4w_3,p^4v_3+p^5w_3A\,\middle|\,u_3,p^2v_3\right)\end{aligned}$$

for some $u_3=u_3(x)$, $v_3=v_3(x)$, $w_3=w_3(x)\in\operatorname{End}\mathbb{Z}_{p^m}^n$. Hence

$$f_1\left(0,0,p^4x\right)=\left(u_3,p^2v_3,p^4w_3\right), \tag{16}$$

$$f_2\left(p^4x,p^5xA\right)=\left(p^2u_3+p^3v_3+p^4w_3,p^4v_3+p^5w_3A\right), \tag{17}$$

$$h_{23}\left(p^4x,p^5xA\right)=\left(u_3,p^2v_3\right). \tag{18}$$

By (16) and (17), we have that $u_3=p^4u''$, $v_3=p^2v'$, and

$$f_1\left(0,0,p^5x\right)=\left(p^5u'',p^5v',p^5w_3\right),\qquad f_2\left(p^5x,0\right)=\left(p^5w_3,0\right).$$

Therefore

$$p^5f_{131}=p^5u'',\qquad p^5f_{132}=p^5v',\qquad p^5f_{133}=p^5w_3, \tag{19}$$

$$p^5f_{211}=p^5w_3,\qquad p^5f_{212}=0. \tag{20}$$

From (20) and (5) we conclude that

$$p^5w_3=p^5u_1. \tag{21}$$

We use that for any $g_1,g_2\in E:=\operatorname{End}\mathbb{Z}_{p^6}^n$ it follows from $p^5g_1=p^5g_2$ that $\bar g_1=\bar g_2\in\bar E=E/pE$. Setting $a:=\bar u_1\in\bar E$, it follows from (4), (5), (6), (12), (13), (14), and (15) that:

$$\bar f_1=\begin{pmatrix} a & 0 & 0\\ \bar u' & a & 0\\ \bar u'' & \bar v' & a\end{pmatrix},\qquad \bar f_2=\begin{pmatrix} a & 0\\ \bar f_{221} & a\end{pmatrix},\qquad \bar f_3=\begin{pmatrix} a & 0\\ \bar u' & a\end{pmatrix}.$$

Finally,

$$f_2\big(0, p^5xA\big) = f_2\big(p^4x, p^5xA\big) - f_2\big(p^4x, 0\big) \overset{(17)}{=} \big(*, p^4v_3 + p^5w_3A\big) - f_2\big(p^4x, 0\big)$$
$$\overset{(2)}{=} \big(*, p^4v_3 + p^5w_3A\big) - (*, 0) = \big(*, p^5w_3A\big),$$

because $p^4v_3 = p^6v' = 0$. Thus,

$$p^5u_1(xA) \overset{(14)}{=} f_{222}\big(p^5xA\big) = p^5w_3(x)A \overset{(21)}{=} p^5u_1(x)A,$$

so $a(xA) = a(x)A$ or $Aa = aA$.

Moreover, a is idempotent as $\bar{f}_1$ is idempotent. As $a \in \bar{E}$ with $aA = Aa$, a is an idempotent endomorphism of the indecomposable $\mathbb{Z}_p[x]$-module K^A. Hence, $a = 0$ or 1.

If $a = 0$, then $\bar{f}$ is idempotent and nilpotent and so $\bar{f} = 0$. Write $f = pg$ for some $g \in \mathrm{End}(U_0)$. Since f is idempotent, $f = f^6 = p^6g^6 = 0$.

If $a = 1$, then $\bar{1} - \bar{f}$ is idempotent and nilpotent, hence $\bar{f} = 1$. Write $f = 1 + pg$ for some $g \in \mathrm{End}(U_0)$. Then f is a unit of $\mathrm{End}(U_0)$, because pg is nilpotent, and $f = 1$ because $f^2 = f$.

As 0 and 1 are the only idempotents of $\mathrm{End}(U)$, $U = U_G$ is indecomposable. By Lemma 4(2), G is an indecomposable $(1,2)$-group with p^6-regulator quotient group and rank $7n$. If $m \geqslant 6$ and $(1,2) \subseteq S$, then it is easy to see that G is an indecomposable S-group with p^m-regulator quotient group and rank $7n$. $\square$

Proposition 8. *Let p be a prime, $(1,3) \subseteq S$ a p-locally free poset of types, and $m \geqslant 4$. There are indecomposable S-groups with p^m-regulator quotients of arbitrarily large finite rank.*

Proof. Let $S = (1,3) = \{1 \mid 2 < 3 < 4\}$, $m = 4$, and n a positive integer. Let $U = (U_0, U_i, U_*\colon\ i \in S) \in \mathrm{cdrep}(S, \mathbb{Z}_{p^4})$ with representing matrix

$$M_G = \begin{pmatrix} I_n & 0 & | & p^2I_n & \vdots & pI_n & \vdots & I_n \\ 0 & p^2I_n & | & p^3A & \vdots & p^2I_n & \vdots & 0 \end{pmatrix}.$$

Observe that an arbitrary element of U_* is

$$\big(u, p^2v \mid p^2u + p^3vA \mathbin{\vdots} pu + p^2v \mathbin{\vdots} u\big) = (u, v)M_G$$

for some $u, v \in \mathbb{Z}_{p^4}^n$. It follows easily that $U_* \cap U_i = 0$ for each $1 \leqslant i \leqslant 4$ showing that indeed $U \in \mathrm{cdrep}(S, \mathbb{Z}_{p^4})$.

By Lemma 4(1) there is an S-group G such that $U_G \cong U$. We only need to show that U, and so G, are indecomposable.

Let $f^2 = f \in \mathrm{End}(U_G)$. Then

$$f = \begin{pmatrix} f_1 & 0 & 0 & 0 \\ 0 & f_2 & h_{23} & h_{24} \\ 0 & 0 & f_3 & h_{34} \\ 0 & 0 & 0 & f_4 \end{pmatrix}.$$

As $f^2 = f$, it follows that $f_i^2 = f_i$ for $i = 1, 2, 3, 4$. We will use the matrix form of f_1: $f_1 = \left(\begin{smallmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{smallmatrix}\right)$ corresponding to the decomposition $R_1^{2n} = R_1^n \oplus R_1^n$.

We now use the fact that $f(U_*) \subseteq U_*$. Let $x \in \mathbb{Z}_{p^4}^n$.

Then $(x, 0 \mid p^2x \vdots px \vdots x)$ is in the span of the first block of generators of U_* and

$$f\big(x, 0 \mid p^2x \vdots px \vdots x\big) = \big(f_1(x, 0) \mid f_2\big(p^2x\big) \vdots f_3(px) + h_{23}\big(p^2x\big) \vdots f_4(x) + h_{24}\big(p^2x\big) + h_{34}(px)\big)$$
$$= \big(u_1, p^2v_1 \mid p^2u_1 + p^3v_1A \vdots pu_1 + p^2v_1 \vdots u_1\big) \in U_*$$

for some $u_1 = u_1(x)$, $v_1 = v_1(x)$, $w_1 = w_1(x) \in \operatorname{End} \mathbb{Z}^n_{p^4}$. Then

$$\begin{aligned} f_1(x, 0) &= \big(u_1, p^2v_1\big), \\ f_2\big(p^2x\big) &= p^2u_1 + p^3v_1A, \\ f_3(px) + h_{23}\big(p^2x\big) &= pu_1 + p^2v_1, \\ f_4(x) + h_{24}\big(p^2x\big) + h_{34}(px) &= u_1. \end{aligned}$$

This yields

$$\bar{f}_{11} = \bar{u}_1, \qquad \bar{f}_{12} = 0, \qquad \bar{f}_2 = \bar{u}_1, \qquad \bar{f}_3 = \bar{u}_1, \qquad \bar{f}_4 = \bar{u}_1. \tag{22}$$

The element $(0, p^2x \mid p^3xA \vdots p^2x \vdots 0)$ is in the span of the second block of generators of U_* and

$$\begin{aligned} &f\big(0, p^2x \mid p^3xA \vdots p^2x \vdots 0\big) \\ &\quad = \big(f_1\big(0, p^2x\big) \mid f_2\big(p^3xA\big) \vdots f_3\big(p^2x\big) + h_{23}\big(p^3xA\big) \vdots f_4(0) + h_{24}\big(p^3xA\big) + h_{34}\big(p^2x\big)\big) \\ &\quad = \big(u_2, p^2v_2 \mid p^2u_2 + p^3v_2A \vdots pu_2 + p^2v_2 \vdots u_2\big) \in U_* \end{aligned}$$

for some $u_2 = u_2(x)$, $v_2 = v_2(x)$, $w_2 = w_2(x) \in \operatorname{End} \mathbb{Z}^n_{p^4}$. It follows that

$$\begin{aligned} f_1\big(0, p^2x\big) &= \big(u_2, p^2v_2\big), \\ f_2\big(p^3xA\big) &= p^2u_2 + p^3v_2A, \\ f_3\big(p^2x\big) + h_{23}\big(p^3xA\big) &= pu_2 + p^2v_2, \\ h_{24}\big(p^2xA\big) + h_{34}\big(p^2x\big) &= u_2. \end{aligned}$$

Hence $u_2 = p^2u'$, $p^2u_2 = 0$, and

$$\bar{f}_{21} = \bar{u}', \qquad \bar{f}_{22} = \bar{v}_2, \qquad \bar{f}_2(xA) = \bar{v}_2(x)A, \qquad \bar{f}_3 = \bar{v}_2. \tag{23}$$

By (22) and (23) $\bar{u}_1 = \bar{f}_2 = \bar{f}_3 = \bar{v}_2 \in \bar{E}$, and setting $a = \bar{u}_1$ we have $a(xA) = a(x)A$ or $aA = Aa$ and

$$\bar{f}_1 = \begin{pmatrix} a & 0 \\ \bar{u}' & a \end{pmatrix}.$$

As in the proof of Proposition 7, a is an idempotent endomorphism of the indecomposable module K^A, $a = 0$ or $a = 1$, $f = 0$ or 1, so $U = U_G$ and G are indecomposable. $\square$

Proposition 9. *Let p be a prime, $(2,2) \subseteq S$ a p-locally free poset of types, and $m \geqslant 3$. There are indecomposable S-groups with p^m-regulator quotients of arbitrarily large finite rank.*

Proof. Let $S = (2,2) = \{1 < 2, 3 < 4\}$, $m = 3$, and let n be a positive integer. Further let $U = (U_0, U_i, U_*\colon\ i \in S) \in \operatorname{cdrep}(S, \mathbb{Z}_{p^3})$ with representing matrix

$$M = \begin{pmatrix} pI_n & \vdots & I_n & | & pA & \vdots & I_n \\ p^2 I_n & \vdots & 0 & | & p^2 I_n & \vdots & 0 \end{pmatrix},$$

where A is as always. An arbitrary element of U_* is

$$\bigl(pu + p^2 v \mathrel{\vdots} u \bigm| puA + p^2 v \mathrel{\vdots} u\bigr) = (u, v)M$$

for some $u, v \in \mathbb{Z}^n_{p^3}$. It is easily seen that $U_* \cap U_i = 0$ for each $1 \leqslant i \leqslant 4$, hence $U \in \operatorname{cdrep}(S, \mathbb{Z}_{p^3})$.

The proof of indecomposability of G follows that of Proposition 7. Let $f \in \operatorname{End}(U_G)$ be an idempotent. Then

$$f = \begin{pmatrix} f_1 & h_{12} & 0 & 0 \\ 0 & f_2 & 0 & 0 \\ 0 & 0 & f_3 & h_{34} \\ 0 & 0 & 0 & f_4 \end{pmatrix},$$

and $f_i^2 = f_i$ for $i = 1, 2, 3, 4$.

Let $x \in \mathbb{Z}^n_{p^3}$. Then $(px \mathrel{\vdots} x \mid pxA \mathrel{\vdots} x) \in U_*$ is in the span of the first block of generators of U_* and

$$\begin{aligned} f(px \mathrel{\vdots} x \mid pxA \mathrel{\vdots} x) &= \bigl(f_1(px) \mathrel{\vdots} f_2(x) + h_{12}(px) \bigm| f_3(pxA) \mathrel{\vdots} f_4(x) + h_{34}(pxA)\bigr) \\ &= \bigl(pu_1 + p^2 v_1 \mathrel{\vdots} u_1 \bigm| pu_1 A + p^2 v_1 \mathrel{\vdots} u_1\bigr) \in U_* \end{aligned}$$

for some $u_1 = u_1(x)$ and $v_1 = v_1(x)$. Hence

$$\begin{aligned} f_1(px) &= pu_1 + p^2 v_1, \\ f_2(x) + h_{12}(px) &= u_1, \\ f_3(pxA) &= pu_1 A + p^2 v_1, \\ f_4(x) + h_{34}(pxA) &= u_1. \end{aligned}$$

Consequently,

$$a := \bar{f}_1 = \bar{f}_2 = \bar{f}_4 = \bar{u}_1 \in \bar{E} \quad \text{and} \quad \bar{f}_3(xA) = a(x)A. \tag{24}$$

Further, $(p^2 x \mathrel{\vdots} 0 \mid p^2 x \mathrel{\vdots} 0)$ is in the span of the second block of generators of U_* and

$$\begin{aligned} f\bigl(p^2 x \mathrel{\vdots} 0 \mid p^2 x \mathrel{\vdots} 0\bigr) &= \bigl(f_1(p^2 x) \mathrel{\vdots} h_{12}(p^2 x) \mid f_3(p^2 x) \mathrel{\vdots} h_{34}(p^2 x)\bigr) \\ &= \bigl(pu_2 + p^2 v_2 \mathrel{\vdots} u_2 \bigm| pu_2 A + p^2 v_2 \mathrel{\vdots} u_2\bigr) \in U_* \end{aligned}$$

for some $u_2 = u_2(x) \in \operatorname{End} \mathbb{Z}_{p^3}^n$ and $v_2 = v_2(x) \in \operatorname{End} \mathbb{Z}_{p^3}^n$. Then

$$\begin{aligned} f_1(p^2x) &= pu_2 + p^2v_2, \\ h_{12}(p^2x) &= u_2, \\ f_3(p^2x) &= pu_2A + p^2v_2, \\ h_{34}(p^2x) &= u_2. \end{aligned}$$

Consequently,

$$u_2 = p^2u', \qquad pu_2 = 0, \qquad \bar{f}_3 = \bar{v}_2 = \bar{f}_1 = a. \tag{25}$$

By (24) and (25) $aA = \bar{f}_3A = \bar{v}_2(x)A = Aa$.

As in the proof of Proposition 7, a is an idempotent endomorphism of the indecomposable module K^A, $a = 0$ or $a = 1$, $f = 0$ or $f = 1$, U_G is indecomposable, and G is an indecomposable $(2,2)$-group with p^3-regulator quotient group of rank $4n$.

Hence there exist indecomposable S-groups of rank $4n$ with p^m-regulator quotient whenever $m \geqslant 3$ and $(2,2) \subseteq S$. $\square$

5. Proof of Theorem 1

Proof. (1) If S is a p-locally free inverted forest of types, then, by Lemma 4(1), there is a bijection from isomorphism at p classes of indecomposable S-groups with p-regulator quotients to isomorphism classes of indecomposable anti-representations in $\operatorname{cdrep}(S, \mathbb{Z}_p)$. The category $\operatorname{cdrep}(S, \mathbb{Z}_p)$ is equivalent to the category $\operatorname{rep}(S, \mathbb{Z}_p)$ of $\mathbb{Z}_p$-representations of S, [1, Theorem 5.2.8(b)]. By Kleiner's theorem for representations of finite posets over a field, [1, Theorem 1.3.6], if S contains $(1,1,1,1)$, $(2,2,2)$, $(1,3,3)$, $(1,2,5)$, or $(N,4)$, then $\operatorname{rep}(S, \mathbb{Z}_p)$ has unbounded representation type.

(2)–(5) are Propositions 6, 7, 8 and 9, respectively. $\square$

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