A taxonomy of 2-primal rings

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Abstract

Various conditions on a noncommutative ring imply that it is 2-primal (i.e., the ring’s prime radical coincides with the set of nilpotent elements of the ring). We will examine several such conditions and show that their known interdependencies are their only ones. Of particular interest will be the (PS I) condition on a ring (i.e., every factor ring modulo the right annihilator of a principal right ideal is 2-primal). We will see that even within a fairly narrow class of rings, (PS I) is a strictly stronger condition than 2-primal. We will show that the (PS I) condition is left-right asymmetric. We will also study the interplay between various types of semilocal rings and various types of 2-primal rings. The Köthe Conjecture will make a cameo appearance. In Section 6, we will examine subideals of prime ideals of commutative rings that are invariant under derivations.

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1. Introduction

The following table appeared, in the context of determining when an arbitrary direct product of 2-primal rings is 2-primal, in [42]:

\[
\begin{array}{ccc}
\text{2-primal} & \text{and} & \text{semiprime} \\
\Downarrow & & \Downarrow \\
\text{reduced} & \Rightarrow & \text{symmetric} \\
& & \Downarrow \\
& & (S I) \\
& & \Rightarrow \\
& & (PS I) \\
& & \Rightarrow \\
& & 2\text{-primal}
\end{array}
\]

\[\Leftrightarrow\]
If all rings in some set belong to one of the \((*)\) classes then their direct product must be 2-primal—indeed, their direct product belongs to the weakest \((*)\) class that contains all rings in the set—whereas if all rings in some set belong to one of the \((**)\) classes then their direct product need not be 2-primal. See [42, p. 243].

We will see that there are no additional implications between any conditions or combinations of conditions in the above table.

We follow the notation, conventions, and nomenclature of [37] and [39]. A brief review follows. All rings are associative; all rings and ring homomorphisms are unital except where explicitly indicated. By “ideal” is meant a two-sided ideal, by “noetherian” is meant right noetherian and left noetherian, etc. The ring generated over a ring \(R\) by a set \(X\) of noncommuting indeterminates (which commute with elements of \(R\)) will be denoted by \(R\langle X \rangle\). Given a set \(S\) of elements of a ring \(R\), we will let \((S)\) denote the ideal generated by \(S\) when it is clear from the context that the ring in question is \(R\). We will occasionally use the same letter to denote an indeterminate in a set \(X\) and the canonical image of this indeterminate in a factor ring of the form \(R\langle X \rangle/(S)\) or \(R[X]/(S)\) (this abuse of notation occurs in Lemma 3.10 and in Examples 3.6, 3.9, 3.11, 3.12, and 3.19).

The group of units of a ring \(R\) will be denoted by \(U(R)\). We write \(\text{Nil}_n(R)\) for the prime radical (i.e., the lower nilradical, the intersection of all prime ideals) of \(R\); recall that \(\text{Nil}_n(R)\) is the set of strongly nilpotent elements of \(R\) (see [38, Ex. 10.17]). The upper nilradical of \(R\) (that is, the sum of all nil ideals of \(R\)) will be denoted by \(\text{Nil}^*(R)\), and the Jacobson radical of \(R\) will be denoted by \(\text{rad}(R)\). We say a ring is right (resp. left) uniserial if its right (resp. left) ideals are linearly ordered by inclusion.

2. Seven classes of noncommutative rings

The classes of rings under consideration are defined as follows. A ring is called reduced if it contains no nonzero nilpotent elements. A ring is called right duo (resp. left duo) if all of its right (resp. left) ideals are two-sided. A ring \(R\) is called 2-primal if its prime radical contains every nilpotent element of \(R\). A ring \(R\) satisfies (PS I) if for every element \(a \in R\), the factor ring \(R/\text{ann}^R(aR)\) is 2-primal. A ring \(R\) satisfies (S I) if the right annihilator of each element of \(R\) is an ideal (equivalently, if for all \(a, b \in R\) we have \(ab = 0 \Rightarrow aRb = 0\)). A ring \(R\) is symmetric if for all \(a, b, c \in R\) we have \(abc = 0 \Rightarrow bac = 0\). (Note that a ring \(R\) is symmetric if and only if for any \(n \geq 3\) and for all \(a_1, a_2, \ldots, a_n \in R\), we have \(a_1a_2 \cdots a_n = 0 \Rightarrow a_{\pi(1)}a_{\pi(2)} \cdots a_{\pi(n)} = 0\) for every permutation \(\pi\) of \(\{1, 2, \ldots, n\}\), a characterization that follows from basic properties of symmetric groups.) A ring is called reversible if for all \(a, b \in R\) we have \(ab = 0 \Rightarrow ba = 0\).

An elementary theorem, first discovered by W. Krull, states that every commutative ring is 2-primal. Thus, 2-primal rings provide a sort of bridge between commutative and noncommutative ring theory. On the one hand, the 2-primal condition forces a noncommutative ring to have certain affinities with its commutative cousins (e.g., it must be Dedekind-finite, it cannot be a full matrix ring, etc.). On the other hand, the “genus” of 2-primal rings will prove to comprise many diverse “species,” as we will see in Sections 3, 4, and 5.
Essential properties of 2-primal rings are developed in [6,8,22,24,26–28,34,42–44,52], and [54]. Research on 2-primal rings was inaugurated by G. Shin in [52] (though the name “2-primal” was not coined until later). Shin proved in [52, Proposition 1.11] that a ring is 2-primal if and only if each of its minimal prime ideals is completely prime, i.e., the corresponding prime factor ring is a domain.

Shin’s main emphasis in [52] was on producing sheaf representations for a class of rings, which he called pseudo-symmetric, that are defined by the (PS I) property along with an additional condition. Shin also produced sheaf representations for a class of rings he called almost symmetric, which are defined by (S I) and another condition. See [52, Theorem 3.5] and [52, Corollary 3.7].

In near-ring theory, the study of the (S I) condition predates Shin’s work, going back at least to H.E. Bell’s paper [3] (wherein (S I) is called the insertion-of-factors-property, or I.F.P.). The (S I) condition was later studied vis-à-vis QF-3 rings in [21] by J.M. Habeb (who referred to rings satisfying (S I) as zero insertive or zi). Rings satisfying (S I) are also known as semicommutative, for example, in papers of X.N. Du, Y. Hirano, C. Huh, Y. Lee, L. Motais de Narbonne, and A. Smoktunowicz; see [15,25,29], and [48]. Elsewhere in the literature, however, “semicommutative” means other things (such as duo, for instance, in [1]).

Shin, and later S.-H. Sun (in [54]), proved that the 2-primal condition entails elegant properties for the prime spectrum of a ring (see [44, p. 2114] for a summary).

Shin’s work on sheaf representations was inspired by J. Lambek’s investigation of symmetric rings in [41], where sheaf representation theorems by A. Grothendieck and J. Dieudonné, for commutative rings, and by K. Koh, for reduced rings, are generalized to symmetric rings (see [41, p. 367]).

Shin’s study of prime ideals and stalks in sheaf representations was extended recently by G.F. Birkenmeier, J.Y. Kim, and J.K. Park in [8]. This paper contains a detailed analysis of the properties of, and interrelations among, 2-primal, (PS I), (S I), symmetric, duo, and numerous other conditions.

Reversible was defined by P.M. Cohn in [13], where a special case of the Köthe Conjecture is inferred from the relationship between the reversible condition and the 2-primal condition. It is easy to see that reversible implies (S I) and is implied by symmetric, and hence fits nicely into the table of implications given in Section 1. Prior to Cohn’s work, reversible rings were studied under the name completely reflexive by G. Mason in [46] and under the name zero commutative, or zc, by J.M. Habeb in [21]. In his monograph [56] on distributive lattices arising in ring theory, A.A. Tuganbaev investigates a property called commutative at zero, which is equivalent to the reversible condition on rings.

Duo is a natural condition, which obviously implies that every prime ideal is completely prime. Some of the work on the duo property (including connections with 2-primal rings) can be found in [1,9,10,12,14,21,23,27,30,49,50,55], and [57].

The table of implications from [42], reproduced in Section 1, can be expanded to the Venn diagram of classes of rings shown in Fig. 1. As we will see, there exist non-commutative 2-primal rings of each of the ten numbered types in Fig. 1.
3. A 2-primal ring of every type

We begin with the easy types. There is no shortage of reduced rings; however, Types I and II are of interest in view of the nice classes of rings they contain.

Recall that a ring $R$ is called von Neumann regular if for every $x \in R$ there exists some $y \in R$ such that $x = xyz$, and $R$ is called strongly regular if for every $x \in R$ there exists some $y \in R$ such that $x = x^2y$.

There are several published results on the relationship between 2-primal rings and von Neumann regular rings (e.g., [52, Proposition 1.16], [52, Theorem 1.17], [6, Proposition 2.12], [6, Corollary 2.13], [15, Theorem 1], and [30, Theorem 6]). Most of these results are encompassed in the following example.

**Example 3.1.** A Type I ring. Let $R$ be any strongly regular noncommutative ring. It is well known that a ring is von Neumann regular and reduced if and only if it is strongly regular ([18, Theorem 3.2], [18, Theorem 3.5]), and only if it is von Neumann regular and one-sided duo [38, Ex. 22.4B]. Therefore $R$ is reduced and duo.

In fact, all noncommutative 2-primal von Neumann regular rings are of Type I, since every von Neumann regular ring is semiprime.

Turning to Type II, we observe in passing that this class of rings is closed under direct products, polynomial and formal power series extensions, and Ore extensions of derivation type (though not, in general, Ore extensions of automorphism type). Further discussion of these matters can be found in [6,7,23,28,42], and [43].
Example 3.2. A Type II ring. Let $R$ be any simple domain that is not a division ring. Or, alternatively, let $R$ be any domain that is neither right nor left Ore. Then $R$ is reduced but not one-sided duo.

Our next two examples make use of skew polynomial constructions. G.F. Birkenmeier, H.E. Heatherly, and E.K. Lee proved [6, Proposition 2.6] that the 2-primal condition is inherited by ordinary polynomial extensions. This result was the impetus for the study of skew polynomial extensions of 2-primal rings in [43], where it was observed that such extensions are a “poor though not hopeless” source of 2-primal rings. Indeed, from this source will come our rings of Types III and IV.

We will use the following lemma, which extends [23, Lemma 3].

Lemma 3.3. Suppose $R$ is any ring and $\sigma$ is an automorphism of $R$. If the skew polynomial ring $S = R[x; \sigma]$ is one-sided duo, then $R$ must be commutative and $\sigma$ must be the identity automorphism.

Proof. If $\sigma$ is the identity automorphism then the result follows from [23, Lemma 3]; so assume otherwise. Choose $a \in R$ such that $\sigma(a) \neq a$. Then

$$(1 + ax + x^2)x = x + ax^2 + x^3 \notin S(1 + ax + x^2)$$

and

$$x(1 + ax + x^2) = x + \sigma(a)x^2 + x^3 \notin (1 + ax + x^2)S,$$

a contradiction. $\square$

According to [43, Proposition 3.7], if $R$ is a local, one-sided artinian ring with an automorphism $\sigma$, then $R[x; \sigma]$ satisfies (PS I). A judicious choice of $R$ in the following example will ensure that $R[x; \sigma]$ is symmetric. (Nevertheless, any temptation to try to strengthen the conclusion of [43, Proposition 3.7] will be mollified by Example 3.18.)

Example 3.4. A Type III ring. Let $p \in \mathbb{N}$ be prime, let $F$ be a field of characteristic $p$, and let $G$ be a cyclic $p$-group. Define the group ring $R = FG$, and suppose $\sigma$ is some nontrivial automorphism of $R$. (Such a $\sigma$ always exists unless $|F| = |G| = 2$.) Put $S = R[x; \sigma].$

Obviously $S$ is neither commutative nor reduced, and by Lemma 3.3, $S$ is not one-sided duo. It remains to be shown that $S$ is symmetric.

Let $m$ be the augmentation ideal of $R$, and suppose $p^n$ is the order of $G$. We note that

$$R \cong F[t]/(t^{p^n} - 1) \cong F[y]/(y)^{p^n};$$

hence, the only proper ideals of $R$ are $m^i$ for $i = 1, 2, \ldots, p^n$. The example is now complete because of the following proposition.

Proposition 3.5. Let $R$ be any right artinian, right uniserial ring, and let $\sigma$ be an automorphism of $R$. Then the skew polynomial ring $S = R[x; \sigma]$ is symmetric.
Proof. Let \( m \) denote the maximal ideal of \( R \), and choose \( n \in \mathbb{N} \) minimal such that \( m^n = (0) \). Define \( \mu : R \to \{0, 1, 2, \ldots, n\} \) to be the function such that \( a \in m^{\mu(a)} \setminus m^{\mu(a)+1} \) for all nonzero \( a \in R \), and \( \mu(0) = n \). (Here \( m^0 = R \).) It is easy to show that \( \mu(a) = \mu(\sigma(a)) \) and \( \mu(ab) = \min\{\mu(a) + \mu(b), n\} \) for all \( a, b \in R \).

We now classify the zero-divisors of \( S \).

Claim. Suppose that
\[
c_r x^r + c_{r+1} x^{r+1} + \cdots + c_s x^s
= (a_m x^m + a_{m+1} x^{m+1} + \cdots + a_n x^n)(b_p x^p + b_{p+1} x^{p+1} + \cdots + b_q x^q) \in S.
\]
If
\[
\min_{m \leq i \leq n} \{\mu(a_i)\} = K \quad \text{and} \quad \min_{p \leq i \leq q} \{\mu(b_i)\} = L,
\]
then
\[
\min_{r \leq i \leq s} \{\mu(c_i)\} = \min\{K + L, n\}.
\]

In proving the claim, we can assume that \( K < n \) and \( L < n \). Pick \( i \) and \( j \) minimal such that \( \mu(a_i) = K \) and \( \mu(b_j) = L \). Then
\[
c_{i+j} = a_i \sigma^j(b_j) + \left( \sum_{t > 0} a_{i-t} \sigma^{i-t}(b_{j+t}) \right) + \left( \sum_{t > 0} a_{i+t} \sigma^{i+t}(b_{j-t}) \right)
\begin{align*}
\in m^K m^L + m^{K+1} m^L + m^K m^{L+1}
\end{align*}
= m^{K+L},
\]
and \( a_i \sigma^j(b_j) \notin m^{K+L+1} \) unless \( K + L \geq n \). Therefore \( \mu(c_{i+j}) = \min\{K + L, n\} \). It is obvious that \( \mu(c_i) \geq \min\{K + L, n\} \) for every \( i \), so the claim is proved.

Now suppose \( a_1, a_2, \ldots, a_k \) are elements of \( S \). Using the claim, we see that the minimum value of \( \mu \) on the coefficients of the product \( a_1 a_2 \cdots a_k \) equals the sum of the minimum values of \( \mu \) on the coefficients of each of the \( a_i \)'s, or \( n \) if this sum exceeds \( n \). Thus, \( a_1 a_2 \cdots a_k = 0 \) if and only if every coefficient of \( a_i \) is contained in \( m^{\ell_i} \) and \( \ell_1 + \ell_2 + \cdots + \ell_k \geq n \). This condition is invariant under all permutations of the indices of the \( a_i \)'s; therefore, the ring \( S \) is symmetric. \( \square \)

Remark 1. If \( R \) is a local ring with maximal ideal \( m \neq (0) \) satisfying \( m^2 = (0) \), and \( \sigma \) is a nontrivial automorphism of \( R \), then the skew polynomial ring \( R[x; \sigma] \) is of Type III, by a straightforward modification of the argument given above.

Remark 2. In Example 3.4, it is insufficient to take \( G \) a finite \( p \)-group (cf. Example 3.18).

The Type IV ring in the next example is inspired by the constructions in \([38, \text{Ex. 19.12}]\) and \([38, \text{Ex. 22.4A}]\).
Example 3.6. A Type IV ring. Let $A$ be a left duo domain with a nonidentity injective ring homomorphism $\sigma : A \to A$ for which $\text{im} \sigma \subseteq \{0\} \cup U(A)$. (For instance, one could take $A$ to be any field whose automorphism group is nontrivial.) Let $S = A[x; \sigma]$ and $I = Sx^2 = Sx^2S$. Our example will be the factor ring $R = S/I$, which obviously is neither commutative nor reduced.

If every zero-divisor of a ring is contained in a completely prime ideal $p$ for which $p^2 = (0)$, then the ring must be symmetric. Hence $R$ is symmetric (take $p = RxR$ in this case).

For any $a, b, c, d \in A$, one can find $t_0, t_1 \in A$ such that

$$(a + bx)(c + dx) = ac + [ad + b\sigma(c)]x = t_0a + [t_0b + t_1\sigma(a)]x$$

because of the hypotheses on $A$ and $\sigma$. So $R$ is left duo.

Remark. In the last example, if $\sigma$ is not surjective then $R$ will not be right duo (e.g., $xR \not\subseteq R$ will not be an ideal).

Rings of Types V and VI were constructed in [45] in order to rectify the occasional assertion in the literature that there do not exist rings that are reversible but not symmetric. We will content ourselves with a brief sketch of two counterexamples to this assertion; details can be found in [45]. The following ring occurs as [45, Example 5]:

Example 3.7. A Type V ring. Let $k$ be a field, define the free algebra $F = k\langle x, y, z \rangle$, and let

$$I = (FxF)^2 + (FyF)^2 + (FzF)^2 + FxyzF + FyFx + Fzx\subseteq F.$$

Put $R = F/I$. Clearly, $R$ is neither symmetric nor one-sided duo; however, a direct calculation shows that $R$ is reversible.

The ring in the next example occurs as [45, Example 7]:

Example 3.8. A Type VI ring. Let $Q_8$ denote the quaternion group of order 8, and define the group algebra $R = \mathbb{F}_2Q_8$. As shown in [45, Example 7], the ring $R$ is duo and reversible but not symmetric.

To obtain a Type VII ring, we make use of a construction from [57, Example 2]:

Example 3.9. A Type VII ring. Let $F$ be a field, and let

$$R = F(x, y)/(x^3, y^3, yx, xy - x^2, xy - y^2).$$

Note that $R = \{a_0 + a_1x + a_2y + a_3xy : a_0, a_1, a_2, a_3 \in F\}$ is clearly not reversible.
Let us show that an arbitrary principal right ideal \((a_0 + a_1x + a_2y + a_3xy)R\) is an ideal. We can assume that \(a_0 = 0\), since otherwise \(a_0 + a_1x + a_2y + a_3xy\) is a unit. Then

\[
(b_0 + b_1x + b_2y + b_3xy)(a_1x + a_2y + a_3xy) = (a_1x + a_2y + a_3xy)r
\]

provided

\[
r = \begin{cases} 
  b_0 + (b_1 + b_2a_1^{-1} + b_2a_2a_1^{-1})x & \text{if } a_1 \neq 0 \\
  b_0 + (b_1 + b_2)y & \text{if } a_1 = 0, a_2 \neq 0 \\
  b_0 & \text{if } a_1 = a_2 = 0.
\end{cases}
\]

Thus, \(R\) is right duo.

**Remark 1.** The ring \(R\) in Example 3.9 is also left duo. This can be seen directly, or deduced from results of R.C. Courter [14, Corollary 2.3] or Y. Hirano, C.-H. Hong, J.-Y. Kim, and J.K. Park [23, Theorem 3], which say that right duo implies left duo for, respectively, a finite-dimensional algebra over a field, or (more generally) a right artinian ring that has finite length as a module over its center.

**Remark 2.** In Example 3.9, the commutativity of \(F\) is crucial (cf. Example 3.12).

To broaden our forthcoming construction of a class of Type VIII rings, we will use the following lemma on lifting subdirect product representations.

**Lemma 3.10.** Suppose that

\[
\varepsilon : S \to \prod_{\alpha} S_{\alpha}
\]

is a subdirect product representation of \(S\). Denote each induced homomorphism \(S \to S_{\alpha}\) by \(\varepsilon_{\alpha}\). Let \(\{x_i\}_{i \in I}\) be any set of noncommuting indeterminates, and let \(\{m_j\}_{j \in J}\) be any set of monic monomials in the \(x_i\)’s. Then the map

\[
S[\{x_i\}_{i \in I}] / (\{m_j\}_{j \in J}) \to \prod_{\alpha} S_{\alpha}[\{x_i\}_{i \in I}] / (\{m_j\}_{j \in J})
\]

given by

\[
x_i \mapsto (x_i)_\alpha, \quad s \mapsto (\varepsilon_{\alpha}(s))_\alpha \quad (\text{for every } s \in S)
\]

is a subdirect product representation of \(S[\{x_i\}_{i \in I}] / (\{m_j\}_{j \in J})\).
Proof. Since $S_{\alpha} \subseteq S_{\alpha}\langle\{x_i\}_{i \in I}\rangle/\langle\{m_j\}_{j \in J}\rangle$, we can construct a well-defined homomorphism

$$\varphi : S\langle\{x_i\}_{i \in I}\rangle \to \prod_{\alpha} S_{\alpha}\langle\{x_i\}_{i \in I}\rangle/\langle\{m_j\}_{j \in J}\rangle$$

by defining the restriction of $\varphi$ to $S$ as the composition

$$S \xrightarrow{\varepsilon} \prod_{\alpha} S_{\alpha} \subseteq \prod_{\alpha} S_{\alpha}\langle\{x_i\}_{i \in I}\rangle/\langle\{m_j\}_{j \in J}\rangle$$

and setting $\varphi(x_i) = (x_i)_{\alpha}$ for every $i$.

Obviously $\langle\{m_j\}_{j \in J}\rangle \subseteq \ker \varphi$. Conversely, consider any element in $\ker \varphi$, written in the form

$$\sum_{\ell=1}^{n} s_{\ell}x_{i(\ell,1)}x_{i(\ell,2)} \cdots x_{i(\ell,k(\ell))} (i(\ell, j) \in I, k(\ell) \in \mathbb{N})$$

with $n$ minimal. Then for every $\alpha$,

$$\sum_{\ell=1}^{n} \varepsilon_{\alpha}(s_{\ell})x_{i(\ell,1)}x_{i(\ell,2)} \cdots x_{i(\ell,k(\ell))} \in \langle\{m_j\}_{j \in J}\rangle \subseteq S_{\alpha}\langle\{x_i\}_{i \in I}\rangle.$$

Now, the ideal $\langle\{m_j\}_{j \in J}\rangle \subseteq S_{\alpha}\langle\{x_i\}_{i \in I}\rangle$ equals the set of elements

$$\sum_{\ell=1}^{n'} t_{\ell}x_{i(\ell,1)}x_{i(\ell,2)} \cdots x_{i(\ell,k(\ell))} (t_{\ell} \in S_{\alpha})$$

for which for each $\ell$ the monomial $x_{i(\ell,1)}x_{i(\ell,2)} \cdots x_{i(\ell,k(\ell))}$ contains some $m_j$ as an internal factor. Because $\varepsilon$ is injective, every term of the sum

$$\sum_{\ell=1}^{n} s_{\ell}x_{i(\ell,1)}x_{i(\ell,2)} \cdots x_{i(\ell,k(\ell))}$$

contains some $m_j$ as an internal factor, which shows that the sum lies in $\langle\{m_j\}_{j \in J}\rangle$. Therefore $\langle\{m_j\}_{j \in J}\rangle = \ker \varphi$.

We obtain an injective homomorphism

$$S\langle\{x_i\}_{i \in I}\rangle/\langle\{m_j\}_{j \in J}\rangle \to \prod_{\alpha} S_{\alpha}\langle\{x_i\}_{i \in I}\rangle/\langle\{m_j\}_{j \in J}\rangle.$$

Let us denote each induced homomorphism

$$S\langle\{x_i\}_{i \in I}\rangle/\langle\{m_j\}_{j \in J}\rangle \to S_{\alpha}\langle\{x_i\}_{i \in I}\rangle/\langle\{m_j\}_{j \in J}\rangle$$
by \( \varphi_\alpha \). It remains only to be seen that each map \( \varphi_\alpha \) is surjective. But this follows immediately from the fact that each \( \varepsilon_\alpha \) is surjective. 

\[ \blacksquare \]

**Remark.** It is vital for the monomials \( m_j \) in Lemma 3.10 to be monic. For instance, if \( \Omega \) denotes the set of odd prime integers, then we have a subdirect product representation

\[ \mathbb{Z} \to \prod_{p \in \Omega} \mathbb{F}_p; \]

however, the ring \( \mathbb{Z}(x)/(2x) \) does not embed at all in the ring

\[ \prod_{p \in \Omega} \mathbb{F}_p(x)/(2x) \cong \prod_{p \in \Omega} \mathbb{F}_p. \]

The ring in the next example comes from [52, Example 5.2] and generalizes [28, Example 1.5].

**Example 3.11.** A Type VIII ring. Let \( S \) be any reduced ring, and fix any \( n \in \mathbb{N} \). Let

\[ R = S\langle x, y \rangle / (x^{n+1}, y^{n+1}, yx). \]

Note that every element of \( R \) can be written in the form

\[ \sum_{i=0}^{n} \sum_{j=0}^{n} a_{i,j} x^i y^j \quad (a_{i,j} \in S). \]

Clearly, \( R \) is neither reversible nor one-sided duo.

It remains only to be shown that \( R \) satisfies (S I). Since the (S I) condition is inherited by subdirect products, we can assume without loss of generality that \( S \) is a domain, by dint of Lemma 3.10.

Observe that

\[ \left( \sum_{i=0}^{n} \sum_{j=0}^{n} a_{i,j} x^i y^j \right) \left( \sum_{i=0}^{n} \sum_{j=0}^{n} b_{i,j} x^i y^j \right) \]

\[ = \sum_{i=0}^{n} \sum_{j=0}^{n} \left( \sum_{k=0}^{i} a_{i,k} b_{i-k,j} \right) \left( \sum_{k=0}^{j} a_{i,k} b_{i-k,j} \right) x^i y^j. \]  (1)

We will use this equation to classify the zero-divisors of \( R \).

**Case 1.** Suppose \( a_{i,0} = 0 \) for all \( i = 1, 2, \ldots, n \). Let \( m > 0 \) be the smallest integer for which \( a_{i,m} \neq 0 \) for some \( i \). Then we claim that

\[ \left( \sum_{i=0}^{n} \sum_{j=0}^{n} a_{i,j} x^i y^j \right) \left( \sum_{i=0}^{n} \sum_{j=0}^{n} b_{i,j} x^i y^j \right) = 0 \]
if and only if

\[ b_{0,j} = 0 \quad \text{for each } j = 0, 1, 2, \ldots, n - m. \]

The “if” statement is clear from Eq. (1). To see the “only if” assertion, note that if \( b_{0,t} \neq 0 \) for some \( t < n - m \) with \( t \) chosen minimal, then in Eq. (1) the coefficient of \( x^i y^{m+t} \) is

\[
\left[ \sum_{k=0}^{i} a_{k,0} b_{(i-k),m+t} \right] + \left[ \sum_{k=1}^{m+t} a_{i,k} b_{0,(m+t-k)} \right] = \sum_{k=1}^{m+t} a_{i,k} b_{0,(m+t-k)} = a_{i,m} b_{0,t} \neq 0,
\]

a contradiction.

**Case 2.** Suppose \( a_{i,0} \neq 0 \) for some \( i \). Let \( d \) be the smallest integer such that \( a_{d,0} \neq 0 \). Then we claim that

\[
\left( \sum_{i=0}^{n} \sum_{j=0}^{n} a_{i,j} x^i y^j \right) \left( \sum_{i=0}^{n} \sum_{j=0}^{n} b_{i,j} x^i y^j \right) = 0
\]

if and only if

\[ b_{i,j} = 0 \quad \text{whenever } 0 \leq i \leq n - d \text{ and } 0 \leq j \leq n. \]

Again, the “if” statement is clear from Eq. (1); to see the “only if” assertion, fix \( j \) as the smallest integer such that \( b_{i,j} \neq 0 \) for some \( i \), and then let \( t < n - d \) be the smallest integer for which \( b_{t,j} \neq 0 \). Then in Eq. (1) the coefficient of \( x^t y^j \) is

\[
\left[ \sum_{k=0}^{t+d} a_{k,0} b_{(t+d-k),j} \right] + \left[ \sum_{k=1}^{j} a_{i+d,k} b_{0,(j-k)} \right] = \sum_{k=0}^{t+d} a_{k,0} b_{(t+d-k),j} = a_{d,0} b_{1,j} \neq 0,
\]

a contradiction.

Now, to show that \( R \) satisfies (S I), suppose the elements

\[
\alpha = \sum_{i=0}^{n} \sum_{j=0}^{n} a_{i,j} x^i y^j \in R \quad \text{and} \quad \beta = \sum_{i=0}^{n} \sum_{j=0}^{n} b_{i,j} x^i y^j \in R
\]

satisfy \( \alpha \beta = 0 \).

Fix any \( s \in S \). If \( \alpha \) falls into Case 1, then \( sb_{0,j} = 0 \) for each \( j = 0, 1, 2, \ldots, n - m \), which shows that \( \alpha (s \beta) = 0 \). If \( \alpha \) falls into Case 2, then \( sb_{i,j} = 0 \) whenever \( 0 \leq i \leq n - d \) and \( 0 \leq j \leq n \), which shows that \( \alpha (s \beta) = 0 \). Consequently \( \alpha S \beta = 0 \).

Now we insert the factor \( x \). If \( \alpha \) falls into Case 1, then automatically \( \alpha (x \beta) = 0 \). If \( \alpha \) falls into Case 2, then \( b_{i-1,j} = 0 \) whenever \( 1 \leq i \leq n - d \) and \( 0 \leq j \leq n \). Therefore \( \alpha (x \beta) = 0 \).

Finally, we insert the factor \( y \). If \( \alpha \) falls into Case 1, then \( \alpha y \) also falls into Case 1 but with the value of \( m \) increased by 1, whereupon \( (\alpha y) \beta = 0 \). If \( \alpha \) falls into Case 2, then \( b_{0,j} = 0 \) whenever \( 0 \leq j \leq n \); thus, since \( \alpha y \) falls into Case 1, we have \( (\alpha y) \beta = 0 \).
Since $S$, $x$, and $y$ generate $R$, we conclude that if $\alpha \beta = 0$ then $\alpha R \beta = 0$. Hence, $R$ satisfies (S I).

We will now produce rings that satisfy (PS I) but not (S I). Note that a ring $R$ satisfies (PS I) if every factor ring $R/I$ is 2-primal, i.e., if every prime ideal $p \subset R$ is completely prime. (For instance, this is the case when $R$ is local artinian.)

The next example provides the promised justification for Remark 2 following Example 3.9.

**Example 3.12.** A Type IX ring. Let $F = \mathbb{R} + \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$ be the division ring of real quaternions, and let

$$R = F(x, y) / \langle x^3, y^3, xy - x^2, xy^2 - y^2 \rangle.$$ 

As in Example 3.9, $R$ is a local artinian ring and hence satisfies (PS I). But, in contrast to Example 3.9, the present ring is not duo; in fact, it does not even satisfy (S I):

$$ix - \frac{1}{2}(i + j)y \in \text{ann}_R^R(ix + jy) \quad \text{but} \quad i \left( ix - \frac{1}{2}(i + j)y \right) \notin \text{ann}_R^R(ix + jy).$$

Taking $R$ as in Example 3.12, if we form $R[[t]]$, we obtain another 2-primal ring of the same type, as Example 3.13 will show. Not all formal power series rings over 2-primal base rings are 2-primal: see the discussion following Example 3.20 below.

**Example 3.13.** A Type IX ring. Let $R$ be any local artinian ring that does not satisfy (S I). Let $t$ be a central indeterminate, and let $S = R[[t]]$ be the ring of formal power series over $R$. One easily sees that $S$, which does not satisfy (S I), is a (local noetherian) ring in which every prime ideal is completely prime, hence $S$ satisfies (PS I).

The next example occurs in [5, Example 9] and [4, Example 9]. If $T$ is a non-unital ring, its **Dorroh extension** is the unital ring $T' = \mathbb{Z} \oplus T$, with componentwise addition, and multiplication given by $(n_1, t_1)(n_2, t_2) = (n_1n_2, t_1t_2 + n_1t_2 + n_2t_1)$.

**Example 3.14.** A Type IX ring. Let $S = \{a, b\}$ be the semigroup with multiplication $a^2 = ab = a$, $b^2 = ba = b$. Put $T = \mathbb{F}_2 S$, which is a four-element semigroup ring without identity. As shown in [45, Example 2], the Dorroh extension $T'$ satisfies (PS I) but not (S I).

Before constructing further examples, we will obtain some sufficient conditions for a ring to satisfy (PS I). Following a jocular suggestion by T.Y. Lam, let us call a ring $\frac{1}{4}$-**perfect** if it is semilocal and its Jacobson radical is nil. Then

$$\text{one-sided perfect } \Rightarrow \text{ } \frac{1}{4}-\text{perfect } \Rightarrow \text{ semiperfect}$$

(hence Professor Lam’s suggestion), but both implications’ converses are false, even for commutative local rings (e.g., consider \( k[x_1, x_2, x_3, \ldots] / (x_1^2, x_2^2, x_3^2, \ldots) \) and \( k[[x]] \), with \( k \) a field).

Note that a \( \frac{3}{4} \)-perfect ring that is one-sided Goldie, or satisfies the ascending chain condition on both left and right annihilators, must be semiprimary [16, Corollaries 1.7, 1.8]. Thus, a ring is left (resp. right) noetherian and \( \frac{3}{4} \)-perfect if and only if it is left (resp. right) artinian. The \( \frac{3}{4} \)-perfect condition is inherited by finite direct products, and a factor ring of a \( \frac{3}{4} \)-perfect ring is easily seen to be \( \frac{3}{4} \)-perfect (cf. [37, Corollary (24.17)] and [37, Corollary (24.19)]).

Since right perfect, left perfect, and semiperfect are Morita-invariant properties, one is inclined to ask whether \( \frac{3}{4} \)-perfect is as well. If a ring \( R \) is \( \frac{3}{4} \)-perfect, and \( e \in R \) is any idempotent (full or not), then the corner ring \( eRe \) is clearly \( \frac{3}{4} \)-perfect. It remains only to ascertain whether \( R \) being \( \frac{3}{4} \)-perfect implies that the matrix ring \( M_n(R) \) be \( \frac{3}{4} \)-perfect for every \( n \in \mathbb{N} \) (cf. [39, Corollary (18.35)]). Of course, \( M_n(R) \) will be semilocal, so the only outstanding issue is whether its Jacobson radical, \( \text{rad}(R) \), must in this case be nil. If it is true that \( \text{Nil}^*(M_n(R)) = \text{Nil}^*(M_n(S)) \) for every ring \( R \) and every \( n \in \mathbb{N} \), then the answer is “yes,” and the \( \frac{3}{4} \)-perfect condition is Morita-invariant. On the other hand, suppose that for some field \( F \) there exists an \( F \)-algebra \( S \) with the property that \( M_n(F) \text{Nil}^*(S) \) is not nil for some \( n \in \mathbb{N} \). Given such an \( F \) and an \( S \), we could define the ring \( R = F \oplus \text{Nil}^*(S) \) with multiplication given by \( (a_1, s_1)(a_2, s_2) = (a_1a_2, a_1s_2 + s_1a_2 + s_1s_2) \), and the answer would be a \( \frac{3}{4} \)-perfect ring such that \( M_n(R) \) was not \( \frac{3}{4} \)-perfect.

Therefore, by means of [35, Theorem 2] and Reduction 1 of [51, §III], we infer that Morita-invariance of the \( \frac{3}{4} \)-perfect condition is equivalent to the truth of the Köthe Conjecture. We leave this matter unresolved.

**Proposition 3.15.** Suppose \( R \) is a 2-primal, \( \frac{3}{4} \)-perfect ring. Then \( R \) satisfies (PS I).

**Proof.** The hypotheses imply \( \text{rad}(R) = \text{Nil}_n(R) \). The factor ring \( R / \text{rad}(R) = R / \text{Nil}_n(R) \) is semisimple and reduced, i.e., isomorphic to a finite direct product of division rings. Thus, every prime ideal of \( R \) is completely prime; so \( R \) satisfies (PS I). \( \square \)

One cannot weaken \( \frac{3}{4} \)-perfect to semiperfect in Proposition 3.15. In Example 5.4, we will see a 2-primal noetherian local ring that does not satisfy (PS I).

In order to apply Proposition 3.15, let us record the following analogue of [6, Proposition 2.5]:

**Lemma 3.16.** Suppose \( S \) and \( T \) are rings, and \( M \) is an \( (S, T) \)-bimodule. Let

\[
R = \begin{bmatrix} S & M \\ 0 & T \end{bmatrix}.
\]

Then \( R \) is \( \frac{3}{4} \)-perfect if and only if \( S \) and \( T \) are \( \frac{3}{4} \)-perfect.

Consequently, given any \( n \in \mathbb{N} \), the ring \( S \) is \( \frac{3}{4} \)-perfect if and only if the ring of upper triangular \( n \) by \( n \) matrices over \( S \) is \( \frac{3}{4} \)-perfect.
We will omit the (trivial) proof of this lemma. We are now ready to construct another Type IX ring.

**Example 3.17.** A Type IX ring. Let $n > 1$ be an integer, and let $R$ be the ring of upper triangular $n$ by $n$ matrices over any 2-primal, $\frac{3}{4}$-perfect ring. By [6, Proposition 2.5(i)] and Lemma 3.16, we can apply Proposition 3.15 to conclude that $R$ satisfies (PS I). But $R$ does not satisfy (S I): for example, if $E_{ij}$ denote the matrix units, then $\text{ann}_R(E_{11} + E_{12})$ contains $E_{1n} - E_{2n}$ but not $E_{11}(E_{1n} - E_{2n})$, and so is not an ideal.

If $R$ is a local, one-sided artinian ring with an automorphism $\sigma$, then [43, Proposition 3.7] says that the skew polynomial ring $R[x; \sigma]$ satisfies (PS I). It is possible to weaken the hypotheses of [43, Proposition 3.7] (see [44, Theorem 3.4(i)]); however, the following example shows that the conclusion of [43, Proposition 3.7] is the strongest possible with respect to the classes of 2-primal rings we are considering.

**Example 3.18.** A Type IX ring. Let $p \in \mathbb{N}$ be prime, let $F$ be a field of characteristic $p$, and let $G$ be the noncyclic group of order $p^2$, with generators $g, h \in G$. Define the group algebra $R = FG$, and let $\sigma : R \to R$ be the $F$-automorphism that interchanges $g$ and $h$.

By [43, Proposition 3.7], the skew polynomial ring $S = R[x; \sigma]$ satisfies (PS I). Since $\text{ann}_S(x(\sum_{i=0}^{p-1} g^i) but not the element $x(\sum_{i=0}^{p-1} g^i)$, the ring $S$ does not satisfy (S I).

As we come to our final type of 2-primal rings, we recall that whenever a direct product of 2-primal rings is not 2-primal, infinitely many of the 2-primal constituent rings must be of Type IX or Type X, by [42, p. 243]. (For this reason, the ring $R$ in [42, Example 1] is of Type IX.) So the examples constructed in [28] to show that the 2-primal condition does not go up to direct products might seem good candidates to land in Type X and complete our taxonomy. But we will show that none of these rings is of Type X (see Examples 3.19 and 3.20 below). Indeed, the literature is quite sparsely populated with Type X rings (cf. [44, p. 2116]).

The following example is the first of the “good candidates” for Type X that falls short. The ring in this example occurs in [28, Example 1.6].

**Example 3.19.** A Type IX ring. Let $k$ be a division ring, and for each integer $n \geq 2$, put $S_n = k[x]/(x^n)$. Let $m_n$ be the maximal ideal of $S_n$, and define

$$R_n = \begin{bmatrix} S_n & m_n \\ m_n & S_n \end{bmatrix}.$$ 

Since $R_n/\text{Nil}_*(R_n) \cong k \times k$, every prime ideal of $R_n$ is completely prime.

Note that $R_n$ does not satisfy (S I), since

$$\begin{bmatrix} 1 & x \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & x \\ 0 & -1 \end{bmatrix} = 0 \neq \begin{bmatrix} 1 & x \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & x \\ 0 & -1 \end{bmatrix}.$$
Let \( T = \bigoplus_{n=2}^{\infty} R_n \), and define \( R = k \oplus T \) with multiplication given by
\[
(\alpha_1, t_1)(\alpha_2, t_2) = (\alpha_1\alpha_2, \alpha_1t_2 + t_1\alpha_2 + t_1t_2).
\]
To see that \( R \) is of Type IX, first note that since it contains the \( R_n \)'s as non-unital subrings, it does not satisfy (S I). Furthermore, [6, Propositions 3.9(i), 3.10] imply that every prime ideal of \( R \) is completely prime. Thus, \( R \) satisfies (PS I).

**Remark.** It is shown in [28, Example 1.7] that the direct product of power series rings \( \prod_{n=2}^{\infty} R_n[t] \) is not 2-primal. Indeed, its subring \( \prod_{n=2}^{\infty} R_n \) is not 2-primal, providing another example of a non-2-primal direct product of finite 2-primal rings (cf. [7, Example 1.6]), which satisfy (PS I) (cf. [42, Example 1]). The nilpotent element
\[
\left(\begin{array}{cc}
0 & x \\
0 & 0
\end{array}\right)_{n=2}^{\infty} \in \prod_{n=2}^{\infty} R_n
\]
is not strongly nilpotent, since if we put
\[
r_1 = \left(\begin{array}{cc}
0 & x \\
0 & 0
\end{array}\right)_{n=2}^{\infty} \in \prod_{n=2}^{\infty} R_n,
\]
\[
r_{i+1} = r_i \cdot \left(\begin{array}{cc}
0 & 0 \\
x & 0
\end{array}\right)_{n=2}^{\infty} \cdot r_i \quad \text{(for all } i \in \mathbb{N}),
\]
then
\[
r_i = \left(\begin{array}{cc}
0 & x^{i-1} \\
0 & 0
\end{array}\right)_{n=2}^{\infty} \in \prod_{n=2}^{\infty} R_n \setminus \{0\}
\]
for every \( i \in \mathbb{N} \).

The next example generalizes [28, Example 1.7] (cf. Example (5) on p. 476 of H. Bass’s paper [2]).

**Example 3.20.** A Type IX ring. Let \( S \) be any ring all of whose prime ideals are completely prime (cf. [6, §3]). For each integer \( n > 1 \), let \( A_n \) be the ring of upper triangular \( n \) by \( n \) matrices over \( S \), and let \( I_n \subset A_n \) be the ideal consisting of matrices with 0’s down the main diagonal. When \( m < n \), define an \((S, S)\)-bimodule homomorphism \( A_m \to A_n \) by sending each matrix unit \( E_{ij} \in A_m \) to the corresponding matrix unit \( E_{ij} \in A_n \); with respect to these injective non-unital ring homomorphisms, define the non-unital ring and \((S, S)\)-bimodule
\[
A = \lim_{\to} A_n.
\]
Let \( I = \lim_{\to} I_n \subset A \).

Our Type IX ring is \( R = S \oplus A \) with multiplication given by
\[
(s_1, a_1)(s_2, a_2) = (s_1s_2, s_1a_2 + a_1s_2 + a_1a_2).
\]
Since (as in Example 3.17) the \( A_n \)'s do not satisfy (S I), and they embed (as non-unital subrings) in \( R \), we see that \( R \) does not satisfy (S I).

Now, \( 0 \oplus I \subset \text{Nil}_e(R) \), since every element of the ideal \( 0 \oplus I \) generates a nilpotent subideal. Since \( R/(0 \oplus I) \) is isomorphic to the ring of all eventually-constant sequences in \( \prod_{i=1}^{\infty} S \), every prime factor ring of which is isomorphic to a prime factor ring of \( S \), it follows that every prime ideal of \( R \) is completely prime; therefore, \( R \) satisfies (PS I).
Taking the ring $R$ in either Example 3.19 or Example 3.20 above, the formal power series ring $R[[t]]$ is not 2-primal, by C. Huh, H.K. Kim, and Y. Lee’s arguments at the end of Examples 1.1 and 1.6 of [28]. There are three examples in the literature that I know of that show that the 2-primal condition is not inherited by power series extensions (namely, [42, Example 2], [28, Examples 1.1, 1.6]), and it is curious that all three of these examples are Type IX rings. Of course, it is easy enough to turn any of these examples into a Type X example by taking the direct product with some Type X ring. This leads us to the following question:

**Question.** In which of the ten types of 2-primal rings in Fig. 1 does there exist a ring $R$ whose formal power series ring $R[[t]]$ is not 2-primal?

Obviously Types I and II are out, and Types IX and X are in. By [28, Proposition 1.2], $R[[t]]$ is 2-primal whenever $R$ is 2-primal with a nilpotent prime radical (and thus, for example, whenever $R$ is 2-primal and noetherian); however, this criterion seems to shed little light on the outstanding six parts of our question.

It is known, for instance, that a power series ring over a duo ring need not be duo (on either side), as shown by Example 4 on pp. 2211–2212 of [23]. The ring $R$ in that example, however, is of Type I; so it has no bearing on our question.

Returning to the taxonomy, we will construct rings of Type X by means of the following proposition. Proposition 3.21 shows that upon triangular ring constructions the (PS I) condition is not so well behaved as the weaker condition 2-primal or the stronger condition that all primes be completely prime: cf. [6, Proposition 2.5(ii)] and [6, Proposition 3.11(ii)].

**Proposition 3.21.** Let $T$ be a 2-primal ring that contains a prime ideal $p$ that is not completely prime. Let $S$ be any nonzero subring of $T$. Regard $T/p$ as an $(S, T)$-bimodule. Then the ring

$$ R = \begin{bmatrix} S & T/p \\ 0 & T \end{bmatrix} $$

is 2-primal but does not satisfy (PS I).

**Proof.** By [6, Proposition 2.2] and [6, Proposition 2.5(ii)], $R$ is 2-primal. Let

$$ a = \begin{bmatrix} 0 & 1 + p \\ 0 & 0 \end{bmatrix} \in R. $$

Then

$$ \text{ann}^R_a (aR) = \begin{bmatrix} S & T/p \\ 0 & p \end{bmatrix}, $$

whence $R/\text{ann}^R_a (aR) \cong T/p$, which is a prime ring with zero-divisors. So $R$ does not satisfy (PS I). $\square$
Proposition 3.21 reveals another way in which the (PS I) condition is not so well behaved as the weaker condition 2-primal or the stronger condition that all primes be completely prime. Given rings $S \subseteq T$ that satisfy (PS I), and given a prime but not completely prime ideal $p \subset T$, let

$$R = \begin{bmatrix} S & T/p \\ 0 & T \end{bmatrix}, \quad I = \begin{bmatrix} 0 & T/p \\ 0 & 0 \end{bmatrix}.$$  

As a non-unital ring, the ideal $I$ of $R$ satisfies (PS I) (even under Shin’s definition in [52] of (PS I) for rings without unity, which requires that the ring be 2-primal in addition to the factor rings by right annihilators of principal right ideals being 2-primal). Also, the factor ring $R/I$ satisfies (PS I), since $R/I \cong S \times T$, and $S$ and $T$ both satisfy (PS I). Yet $R$ does not satisfy (PS I), even though $I$ and $R/I$ do (cf. [6, Proposition 2.4(i)] and [6, Proposition 3.9(i)]).

**Example 3.22.** A Type X ring. Let $\mathbb{H} = \mathbb{R} + \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$ be the ring of real quaternions. Let

$$R = \begin{bmatrix} \mathbb{H}[x] & \mathbb{H}[x]/(x^2 + 1)\mathbb{H}[x] \\ 0 & \mathbb{H}[x] \end{bmatrix}.$$  

By [47, Lemma 9.6.3(ii)], the ideal $(x^2 + 1)\mathbb{H}[x] \subset \mathbb{H}[x]$ is prime (even maximal), though not completely prime (neither $x + i$ nor $x - i$ belongs to $(x^2 + 1)\mathbb{H}[x]$). By Proposition 3.21, $R$ is 2-primal but does not satisfy (PS I).

The next example demonstrates that the condition $\frac{1}{2}$-perfect cannot be weakened to semilocal in Proposition 3.15. A further demonstration will occur in Example 5.4.

**Example 3.23.** A Type X ring. Let $T = \mathbb{Z} + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}k$ be the ring of quaternions with integer coefficients. Fix a prime integer $p \geq 3$, let $S = \mathbb{Z} \setminus p\mathbb{Z}$, and put $A = S^{-1}T$. By [20, Exercise 2A], the only nonzero prime ideal of $T$ that does not meet $S$ is $pT$. Inasmuch as $T$ is noetherian, the only nonzero prime ideal of $A$ is $pA$ (see [39, Proposition (10.33)(3)]). One easily sees that $A/pA \cong M_2(F_p)$ (e.g., by [36, p. 61]), since $A/pA$ is isomorphic to the quaternion algebra $(\frac{-1}{-1})$. Thus, $A$ is a semilocal domain whose unique nonzero prime ideal is not completely prime.

Define

$$R = \begin{bmatrix} A & A/pA \\ 0 & A \end{bmatrix},$$  

a 2-primal, noetherian, semilocal ring. By Proposition 3.21, $R$ does not satisfy (PS I).

Example 3.23 shows that a countable 2-primal ring need not satisfy (PS I). By contrast, every finite 2-primal ring must satisfy (PS I), though not (S I) necessarily, by our foregoing results.
4. (PS I) versus (I SP)

For six of the seven classes of rings defined in Section 2, closure of the class under passage to opposite rings is either completely trivial or else false (recall Example 3.6 and the subsequent remark). The one hitherto unsettled case is that of the (PS I) condition, for which left-right symmetry has never been ascertained in the literature on the subject. As we will now see, the (PS I) condition is not left-right symmetric.

**Proposition 4.1.** Let \( A \) be a domain. Suppose that \( A \) contains a prime ideal \( P \) that is not completely prime, and suppose that \( A \) contains a nonzero subring \( B \) with the property that every prime ideal of \( B \) is completely prime. Then the ring

\[
R = \begin{bmatrix} B & A/P \\ 0 & A \end{bmatrix}
\]

does not satisfy (PS I), but its opposite ring \( R^{\text{op}} \) does.

**Proof.** By Proposition 3.21, the ring \( R \) does not satisfy (PS I).

For convenience, write \( \mathfrak{A} = A^{\text{op}} \) and \( \mathfrak{B} = B^{\text{op}} \), and note that every prime ideal of \( \mathfrak{B} \) is completely prime. The ideal of \( \mathfrak{A} \) that corresponds to \( P \subset A \) is prime but not completely prime; let us denote this ideal by \( p \subset \mathfrak{A} \). Put

\[
\mathcal{R} = \begin{bmatrix} \mathfrak{A} & \mathfrak{A}/p \\ 0 & \mathfrak{B} \end{bmatrix};
\]

then \( R^{\text{op}} \cong \mathcal{R} \).

To show that \( \mathcal{R} \) satisfies (PS I), choose any nonzero element

\[
a = \begin{bmatrix} a_1 & a_2 + p \\ 0 & b \end{bmatrix} \in \mathcal{R}.
\]

Let \( M \) be the cyclic right \( \mathfrak{B} \)-submodule of \( \mathfrak{A}/p \) generated by \( a_2 + p \), and let

\[
I = \text{ann}_{\mathfrak{B}^{\text{B}}} (M) \cap \text{ann}_{\mathfrak{B}^{\text{B}}} (b \mathfrak{B}) \subset \mathfrak{B}.
\]

There are four different cases.

**Case 1.** Suppose that \( a_1 = 0 \). Then

\[
\text{ann}_{\mathcal{R}} (a\mathcal{R}) = \begin{bmatrix} \mathfrak{A} & \mathfrak{A}/p \\ 0 & I \end{bmatrix} \implies \mathcal{R}/\text{ann}_{\mathcal{R}} (a\mathcal{R}) \cong \mathfrak{B}/I.
\]

**Case 2.** Suppose that \( a_1 \not\in p \). Then, setting \( J = p \cap \text{ann}_{\mathfrak{B}^{\text{B}}} (b \mathfrak{B}) \subset \mathfrak{B} \), we have

\[
\text{ann}_{\mathcal{R}} (a\mathcal{R}) = \begin{bmatrix} 0 & 0 \\ 0 & J \end{bmatrix} \implies \mathcal{R}/\text{ann}_{\mathcal{R}} (a\mathcal{R}) \cong \begin{bmatrix} \mathfrak{A} & \mathfrak{A}/p \\ 0 & \mathfrak{B}/J \end{bmatrix}.
\]
Case 3. Suppose that $a_1 \in p \setminus (0)$ and $a_2 \in p$. Then
\[ \text{ann}_R^n(aR) = \begin{bmatrix} 0 & \mathfrak{A}/p \\ 0 & \text{ann}_B^n(bB) \end{bmatrix} \implies R/\text{ann}_R^n(aR) \cong \mathfrak{A} \times (\mathfrak{B}/\text{ann}_B^n(bB)). \]

Case 4. Suppose that $a_1 \in p \setminus (0)$ and $a_2 \notin p$. Then
\[ \text{ann}_R^n(aR) = \begin{bmatrix} 0 & \mathfrak{A}/p \\ 0 & I \end{bmatrix} \implies R/\text{ann}_R^n(aR) \cong \mathfrak{A} \times (\mathfrak{B}/I). \]

Because all prime ideals of $\mathfrak{B}$ are completely prime, all factor rings of $\mathfrak{B}$ are 2-primal. Thus (using [6, Proposition 2.5(ii)] and [6, Proposition 2.2]) the factor ring $R/\text{ann}_R^n(aR)$ is 2-primal in all four cases, and these cases are exhaustive. So $R^{\text{op}}$ satisfies (PS I). \(\square\)

In the setting of Proposition 4.1, any nonzero subring of the center of $A$ is obviously a viable choice for $B$. Concrete examples illustrating the left-right asymmetry of the (PS I) condition can now be obtained as subrings of the rings in Examples 3.22 and 3.23.

Example 4.2. Let $H = \mathbb{R} + \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$ be the ring of real quaternions. The noetherian ring
\[ R = \begin{bmatrix} \mathbb{R} & \mathbb{H}[x]/(x^2 + 1)\mathbb{H}[x] \\ 0 & \mathbb{H}[x] \end{bmatrix} \]
does not satisfy (PS I) although its opposite ring $R^{\text{op}}$ does.

Example 4.3. Let $T = \mathbb{Z} + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}k$ be the ring of quaternions with integer coefficients. Fix a prime integer $p \geq 3$, let $S = \mathbb{Z} \setminus p\mathbb{Z}$, and form the central localization $A = S^{-1}T$. The semilocal noetherian ring
\[ R = \begin{bmatrix} \mathbb{Z}(p) & A/pA \\ 0 & A \end{bmatrix} \]
does not satisfy (PS I) although its opposite ring $R^{\text{op}}$ does.

5. A local noetherian example

In this section, we will construct a local noetherian 2-primal ring that does not satisfy (PS I) (cf. Proposition 3.15). Since the construction will at a certain stage make use of a differential operator ring $A[x; \delta]$, let us now establish some properties of such a ring.

Recall that if $\delta$ is a derivation on a ring $A$, then an ideal $I \subseteq A$ satisfying $\delta(I) \subseteq I$ is called a $\delta$-ideal. An ideal $I \subseteq A$ is called $\delta$-prime if it is a proper $\delta$-ideal and the only time a product of two $\delta$-ideals is contained in $I$ is when at least one of the two is contained in $I$. Recall from [32, Lemma 1.3] that if $Q \subset A[x; \delta]$ is a prime ideal then $Q \cap A \subset A$ is
a \delta\text{-prime ideal, and if } P \subset A \text{ is a } \delta\text{-prime ideal then } P[x; \delta] \subset A[x; \delta] \text{ is a prime ideal. Given an ideal } I \subset A, \text{ we follow the notation of [19] and define}

\[(I : \delta) = \{a \in I \mid \delta^n(a) \in I \text{ for all } n \in \mathbb{N}\},\]

which is the largest \delta\text{-ideal of } A \text{ contained in } I, \text{ and which is } \delta\text{-prime if } I \text{ is prime (see [40, Proposition 1.1]).}

The following lemma, closely related to [11, Proposition 5.3], is probably known.

**Lemma 5.1.** Suppose \(A\) is a commutative ring with a derivation \(\delta\), and suppose \(t \in A\) is an element with the property that the ideal \(tA \subset A\) is prime but not a \(\delta\text{-ideal}. Then}

\[(tA : \delta) = \begin{cases} \sum_{n=1}^{\infty} t^n A & \text{if } \text{char}(A/tA) = p > 0, \\ \text{A } & \text{if } \text{char}(A/tA) = 0. \end{cases}\]

**Proof.** Since \(tA \subset A\) is not a \(\delta\text{-ideal, } \delta(t) \notin tA\).

First, assume \(\text{char}(A/tA) = p > 0\). Since \(\delta(t^p) = 0\), we have only to show that \((tA : \delta) \subset t^p A\). Suppose instead that there exists some \(\alpha \in (tA : \delta) \setminus t^p A\). Then we can write \(\alpha \in t^m A \setminus t^{m+1} A\), where \(1 \leq m < p\). Thus, \(\alpha = t^m a\), where \(a \in A \setminus tA\).

Now, for each \(k = 1, 2, \ldots, m\), we have

\[\delta^k(t^m a) = \binom{m}{k} t^{m-k} a_k + t^{m-k+1} b_k\]  \hspace{1cm} (2)

for some \(a_k \in A \setminus tA\) and \(b_k \in A\). In particular, \(\delta^m(t^m a) = m!a_m + tb_m\), which is not in the prime ideal \(tA\) since \(a_m\) is not and \(\text{char}(A/tA) = p\) does not divide \(m!\). This contradicts \(\alpha = t^m a \in (tA : \delta)\), establishing \((tA : \delta) = t^p A\).

Finally, assume \(\text{char}(A/tA) = 0\). Suppose \(\alpha \in \bigcap_{n=1}^{\infty} t^m A\), and let \(k \in \mathbb{N}\) be arbitrary. To show that \(\delta^k(\alpha) \in tA\) (whence \(\alpha \in (tA : \delta)\)), choose an integer \(m > k\), and since \(\alpha \in t^m A\), Eq. (2) yields \(\delta^k(\alpha) \in tA\). (Equation (2) remains true, except \(a_k \notin tA\) in this case.) So \((tA : \delta) \supseteq \bigcap_{m=1}^{\infty} t^m A\). On the other hand, suppose \(\alpha \notin \bigcap_{m=1}^{\infty} t^m A\). If \(\alpha \notin tA\) then \(\alpha \notin (tA : \delta)\); thus, assume instead that \(\alpha \in t^m A \setminus t^{m+1} A\) for some \(m \in \mathbb{N}\). Exactly as in the characteristic \(p\) case, Eq. (2) yields \(\delta^m(\alpha) = m!a_m + tb_m \notin tA\), and again \(\alpha \notin (tA : \delta)\). So \((tA : \delta) = \bigcap_{m=1}^{\infty} t^m A\). \(\Box\)

**Remark 1.** In Lemma 5.1, in the case where \(\text{char}(A/tA) = 0\) the \(\delta\text{-prime ideal } (tA : \delta)\) is actually prime, by [19, Proposition 1.1]. So when \(A\) is artinian, this case cannot occur.

**Remark 2.** If \(A\) is a commutative noetherian domain in Lemma 5.1, then in the case where \(\text{char}(A/tA) = 0\) we see that \(tA \subset A\) is a height 1 prime that contains no nonzero \(\delta\text{-ideals (see, e.g., [58, p. 216, Corollary 1]).}\)

One might wonder how much of Lemma 5.1 remains if the principal prime ideal \(tA \subset A\) is replaced by a prime ideal \(p \subset A\) that is not principal. We will explore this matter in Section 6 below.
We now generalize a construction by D.A. Jordan in [33, §4]. (One should be aware that in [33], what we are calling a local ring, i.e., a ring with a unique maximal left ideal, is termed “scalar local” by Jordan; he uses the term “local” to refer to what we are calling semilocal rings, i.e., rings \( R \) for which \( R/\text{rad}(R) \) is artinian.) Unfortunately, the argument behind Jordan’s ingenious example in [33, §4] is slightly misleading. Jordan localizes an \( \mathbb{F}_p \)-algebra at a certain prime ideal (see [33, p. 80], the paragraph preceding Proposition 1.1) on the basis on [53, Theorem 2.7], which is only proved (in [53]) for \( \mathbb{Q} \)-algebras (although this hypothesis is omitted from the statement of [53, Theorem 2.7]). The analogue of [53, Theorem 2.7] for nonzero characteristic is apparently well known to experts (see, e.g., [20, p. 193]), but an explicit proof seems hard to find in the literature. For this reason, we will supply argumentation in the construction to follow.

A prime ideal \( p \) of a ring \( R \) is called localizable if the set
\[
\{ r \in R \mid rs \notin p \text{ and } sr \notin p \text{ whenever } s \notin p \}
\]
is an Ore set in \( R \).

**Theorem 5.2.** Let \( A \) be a commutative noetherian ring with a derivation \( \delta \), and let \( P \subset A \) be a \( \delta \)-prime ideal. Then in the differential operator ring \( R = A[x; \delta] \), the prime ideal \( PR \) is localizable.

**Proof.** As noted earlier, \( PR \subset R \) is a prime ideal. By the Hilbert Basis Theorem for Ore extensions, \( R \) is noetherian; thus, by [47, Theorem 3.16], in order to prove \( PR \subset R \) is localizable it suffices to show that \( PR \) satisfies the second layer condition and that \( \{ PR \} \) is a clique in Spec \( R \).

By [17, Theorem 5.1(a)] and [31, Proposition (8.1.2)], every prime ideal of \( R \) satisfies the second layer condition, so it suffices to show that \( \{ PR \} \) is a clique. By [31, Theorem (8.2.9)], no prime of \( R \) in the same clique as \( PR \) can properly contain \( PR \). Since [17, Lemma 5.2] shows that \( PR \) is an AR-ideal of \( R \), [31, Corollary (5.3.10)] implies that every prime of \( R \) in the same clique as \( PR \) must contain \( PR \). Therefore \( \{ PR \} \) is a clique. \( \square \)

**Lemma 5.3.** There exist local noetherian 2-primal rings that contain a prime ideal that is not completely prime.

**Proof.** Choose a commutative noetherian \( \mathbb{F}_p \)-algebra \( A \) that is a unique factorization domain in which every maximal ideal has height at least 2, with a derivation \( \delta_0 : A \to A \) such that \( \delta_0(\alpha) \notin aA \) for some \( a \in A \) (cf. [33]). Then \( \delta_0(t) \notin tA \) for some irreducible element \( t \in A \). Since the prime ideal \( tA \subset A \) has height 1, we have \( tA \subsetneq M \) for some maximal ideal \( M \subset A \). Choose any element \( c \in M \setminus tA \), and define a derivation \( \delta : A \to A \) by \( \delta(\alpha) = c\delta_0(\alpha) \) for all \( \alpha \in A \).

More generally, we can let \( A \) be any commutative noetherian domain with a derivation \( \delta \), with a maximal ideal \( M \subset A \) such that \( \delta(A) \subset M \), and with an element \( t \in M \) such that \( tA \subset A \) is prime but not a \( \delta \)-ideal and \( \text{char}(A/tA) = p > 0 \).
Put \( R = A[x; \delta] \). Since \( M \subseteq A \) is a (completely) prime \( \delta \)-ideal, we know (e.g., from [32, Lemma 1.4]) that \( MR \subseteq R \) is a completely prime ideal, which by Theorem 5.2 is localizable. Let \( S \) denote the ring of quotients of \( R \) with respect to the Ore set \( R \setminus MR \). We will view \( R \) as a subring of \( S \). By [39, Proposition (10.32)(6)], the local domain \( S \) is noetherian.

Lemma 5.1 implies \( (tA : \delta) = t^pA \subseteq A \). The prime ideal \( (tA : \delta)R = t^pR \subseteq R \) extends to a prime ideal \( (tA : \delta)RS = t^pS \subseteq S \), by [39, Proposition (10.33)(3)]. Clearly \( t \notin t^pS \), so \( t^pS \) is a prime but not completely prime ideal in the local noetherian 2-primal ring \( S \). \( \square \)

Using Lemma 5.3, we will now construct an example that shows that the condition \( 3_{p}\)-perfect cannot be weakened to \( \text{semiperfect} \) in Proposition 3.15.

**Example 5.4.** A local noetherian Type X ring. Let \( S \) be a local noetherian 2-primal ring with a prime but not completely prime ideal \( p \subseteq S \). Define

\[
R = \left\{ \begin{bmatrix} x & y + p \cr 0 & z \end{bmatrix} \in \begin{bmatrix} S & S/p \cr 0 & S \end{bmatrix} \mid x = z \right\}.
\]

By [6, Propositions 2.2, 2.5(ii)], the ring \( R \) is 2-primal. Since \( S \) is local and noetherian, it follows easily that \( R \) has both properties as well.

Let

\[
a = \begin{bmatrix} 0 & 1 + p \\
0 & 0 \end{bmatrix} \in R.
\]

Then

\[
\text{ann}^R(aR) = \left\{ \begin{bmatrix} x & y + p \\
0 & x \end{bmatrix} \in R \mid x \in p \right\}.
\]

Thus, \( R/\text{ann}^R(aR) \cong S/p \), which is a prime ring with zero-divisors; therefore, \( R \) is a local noetherian 2-primal ring that does not satisfy (PS I).

6. **Appendix: On the size of \((p : \delta)\)**

Given a commutative ring \( A \) with a derivation \( \delta \) and a prime non-\( \delta \)-ideal \( p \subseteq A \), Lemma 5.1 tells us, in the case where \( p \) is a principal ideal, the size of \( (p : \delta) \) in terms of the powers of \( p \). If \( p \) is required only to be finitely generated rather than principal, we observe that the "obvious generalization" of Lemma 5.1 fails:

**Example 6.1.** Let \( k \) be a field, let \( A = k[x, y] \), and let \( p = (x, y) \subseteq A \). Define the derivation \( \delta = \frac{\partial}{\partial x} \) on \( A \); so \( p \) is not a \( \delta \)-ideal.
If \( k \cong A/p \) has characteristic 0, then
\[
(p : \delta) = (y) \neq (0) = \bigcap_{m=1}^{\infty} p^m.
\]

If \( k \cong A/p \) has characteristic \( p > 0 \), then
\[
(p : \delta) = (x^p, y) \neq (x^p, x^{p-1}y, x^{p-2}y^2, \ldots, y^p) = p^p.
\]

Thus, in either case, the analogue of the equation at the end of Lemma 5.1, where each occurrence of \( t^i A \) is replaced by \( p^i \), is invalid.

Nevertheless, given any ideal \( I \) in any (not necessarily commutative) ring \( A \) with a derivation \( \delta \), because
\[
\delta^k(I^m) \subseteq I^{m-k}
\]
for all positive integers \( k < m \), we always have \( (I : \delta) \supseteq \bigcap_{m=1}^{\infty} I^m \). So one might say the ideal \( (tA : \delta) \) achieves its lower bound in the characteristic 0 case of Lemma 5.1.

One might initially speculate (in light of Example 6.1) that \( p^p \) is a lower bound on \( (p : \delta) \) in the case where \( \text{char}(A/p) = p > 0 \). But this is not the case, as the following example shows.

**Example 6.2.** Let \( p \in \mathbb{N} \) be prime, let \( A = F_p[x_1, \ldots, x_p] \), let \( p = (x_1, x_p) \subset A \), and define a derivation \( \delta : A \to A \) by
\[
\delta(w) = \frac{\partial w}{\partial x_p} + \sum_{i=1}^{p-1} \frac{\partial w}{\partial x_i}
\]
for all \( w \in A \). Then \( p^p \not\subseteq (p : \delta) \), since \( \delta^{2p-1}(x_1x_p^{p-1}) = -1 \notin p \).

Clearly, if \( A \) is any ring (not necessarily commutative) with a derivation \( \delta \), then for all \( a_1, a_2, \ldots, a_\ell \in A \) and for all \( k \in \mathbb{N} \),
\[
\delta^k(a_1a_2\cdots a_\ell) = \sum_{i_1+i_2+\cdots+i_\ell = k} \binom{k}{i_1, i_2, \ldots, i_\ell} \delta^{i_1}(a_1)\delta^{i_2}(a_2)\cdots\delta^{i_\ell}(a_\ell),
\]
where \( \binom{k}{i_1, i_2, \ldots, i_\ell} \) is the usual multinomial coefficient. It follows from Eqs. (3) and (4) that if \( A \) is commutative, \( p \subset A \) is a prime non-\( \delta \)-ideal such that \( \text{char}(A/p) = p > 0 \), and \( k \in \mathbb{N} \), then the equation \( \delta^k(p^p) \not\subseteq p \) implies that \( k \geq 2p - 1 \). Note that in this statement \( 2p - 1 \) cannot be replaced by anything larger, as shown by Example 6.2.

When the prime \( p \) is finitely generated, we can still find a lower bound on \( (p : \delta) \), in terms of the number of generators of \( p \), in the characteristic \( p \) case. Suppose \( A \) is a
commutative ring with a derivation $\delta$, and $p \subset A$ is a prime non-$\delta$-ideal with $\text{char}(A/p) = p > 0$. If $p$ can be generated by $m$ elements, then

$$(p : \delta) \supseteq p^{m(p-1)+1}$$

since every element of the ideal $p^{m(p-1)+1}$ is a sum of elements each having a factor of the form $x^p$ where $x$ is a generator of $p$, and $\delta^k(x^pA) \subseteq pA + p^k \subseteq p$ for all $k \geq 0$.

When the prime $p$ is not finitely generated, $(p : \delta)$ might not be bounded below by any power of $p$ (regardless of the characteristic of $A/p$):

**Example 6.3.** Let $k$ be a field, and put

$$A = k[x_1, x_2, x_3, \ldots], \quad p = (x_1, x_2, x_3, \ldots) \subset A.$$ Define a derivation $\delta : A \to A$ by

$$\delta(w) = \frac{\partial w}{\partial x_1} + \sum_{i=1}^{\infty} x_i \frac{\partial w}{\partial x_i + 1}$$

for all $w \in A$. Fix any $m \in \mathbb{N}$; we will show that $(p : \delta) \supsetneq p^m$. If $\text{char} k > 0$ then let $p = \text{char} k$; if $\text{char} k = 0$ then let $p$ be any positive prime integer. Put

$$\alpha = \prod_{i=0}^{m-1} (x_i^p)^{p-1} \in p^m.$$ We apply Eq. (4) with $k = p^m - 1$ and $\ell = m(p - 1)$ and

$$a_r(p-1)+1 = a_r(p-1)+2 = \cdots = a_{(r+1)(p-1)-1} = x^p$$

for each $r \in \{0, 1, 2, \ldots, m - 1\}$.

In the expansion of $\delta^k(\alpha)$ via Eq. (4), every term of the summation is in $p$ except for the term

$$C(\delta(x_1))^{p-1}(\delta^p(x_1))^{p-1}(\delta^{p^2}(x_1))^{p-1}\cdots(\delta^{p^{m-1}}(x_{p^{m-1}}))^{p-1} = C$$

where

$$C = \frac{(p^m - 1)!}{(1!)^{p-1}(p!)^{p-1}([p^2]!)^{p-1}\cdots([p^{m-1}]!)^{p-1}} \in \mathbb{N}.$$ Using the fact that $p^{(p^m-1)/(p-1)}$ is the highest power of $p$ that divides $(p^m)!$, we see that $p \nmid C$. (In fact, since $p^a | p^{(p^m-1)/(p-1)} \equiv (-1)^n (\text{mod } p)$ for each $n \in \mathbb{N}$, we have

$$\frac{(p^m - 1)!}{(1!)^{p-1}(p!)^{p-1}([p^2]!)^{p-1}\cdots([p^{m-1}]!)^{p-1}} \equiv (-1)^m (\text{mod } p).$$
a slight generalization of Wilson’s Theorem.) We have shown that \( \delta^k(\alpha) \notin p; \) therefore, there exists no \( m \in \mathbb{N} \) for which \( (p : \delta) \supseteq p^m \).

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