The Irreducible Tensor Representations of $gl(m|1)$ and Their Generic Homology

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1. INTRODUCTION

In classical representation theory, all irreducible finite dimensional representations of the Lie algebras $gl(n)$ can be constructed using Young symmetrizers as operators on tensor powers of the standard representation of $gl(n)$ and twisting by powers of the determinant character. For the Lie superalgebras $gl(m|n)$, these operations are not sufficient to construct all irreducible finite dimensional representations. In this note we describe how a larger family of tensor operations can be used to construct all irreducible finite dimensional representations of $gl(m|1)$, and then use our constructions to calculate the homology of these and other basic representations under the generic actions of $gl_{1}(m|1)$ and $gl_{-}(m|1)$. This approach, developed in [AW1, AW2] provides a curious link between the representation theory of $gl(m|n)$ and natural tensor complexes used in construction of minimal resolutions associated to generic determinantal ideals.

The family of tensor operations we use consists of the symmetry operators realizing the Pieri maps, combined with trace and evaluation maps on mixed, covariant and contravariant tensors on the standard representation of $gl(m|1)$. From these basic operations, we construct complexes $Z^{\mu}_{(\alpha, \beta)^{\nu}}$.

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of tensor representations of $gl(m|l)$ which contain all finite dimensional irreducible $gl(m|l)$-modules as their cycles. For atypical weights, these complexes are double infinite exact complexes, and their right or left truncations give resolutions which realize over $gl(m|l)$ the character formulas of Bernstein and Leites (BL, L). In Section 4, we exploit the relationship between our constructions and general tensor complexes built from Schur complexes to compute generic homology, over the coordinate algebra of affine $m$-space, of the irreducible modules and Kac modules of $gl(m|l)$.

2. BASIC DEFINITIONS AND NOTATIONS

We are working over an arbitrary field $K$ of characteristic zero. The Lie superalgebra $gl(m|n)$ over $K$ is the $\mathbb{Z}_2$-graded Lie algebra $gl(U)$ of endomorphism of a $\mathbb{Z}_2$-graded vector space $U = U_0 \oplus U_1$ of $\dim U_0 = m$, $\dim U_1 = n$ under the Lie superbracket. We denote $U_0 = F$, $U_1 = G$. The grading $U = U_0 \oplus U_1$ can be viewed as a $\mathbb{Z}$-grading from which the Lie superalgebra $g = gl(U)$ acquires a $\mathbb{Z}$-grading give by

$$
g_0 = gl(F) \times gl(G), \quad g_1 = \operatorname{Hom}(F, G) = F^* \otimes G, \quad g_{-1} = \operatorname{Hom}(G, F) = G^* \otimes F$$

which is consistent with the $\mathbb{Z}_2$-grading. The integral weights for $gl(m|n)$ correspond to pairs $(\alpha, \beta)$ where $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{Z}^m$, $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}^n$ are integral weights for $gL(F), gl(G)$, respectively. The dominant weights $(\alpha, \beta)$ are the pairs where $\alpha$ and $\beta$ are both dominant (i.e., nonincreasing). In particular, non-negative integral dominant weights are pairs of partitions. Also, the dominance partial order on $\mathbb{Z}^m \times \mathbb{Z}^n$ is the product of the dominance orders on $\mathbb{Z}^m$ and $\mathbb{Z}^n$.

We will assume throughout this paper that the odd degree component $U_1 = G$ of our basic superspace $U$ has dimension 1. So the integral weights for $gl(U)$ will correspond to pairs $(\alpha, \beta)$ where $\alpha = (\alpha_1, \ldots, \alpha_m)$ $\in \mathbb{Z}^m$, $\beta \in \mathbb{Z}^1$. Given two pairs $(\alpha, \beta), (\lambda, \mu)$ in $\mathbb{Z}^m \times \mathbb{Z}^1$, the condition that $(\alpha, \beta) \geq (\lambda, \mu)$ in the dominance order means that the inequalities

$$\alpha_1 + \cdots + \alpha_i \geq \lambda_1 + \cdots + \lambda_i \quad \text{for all } i = 1, \ldots, m$$

and

$$\beta \geq \mu$$

hold in $\mathbb{Z}$.
Let $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{Z}^m$, $\beta \in \mathbb{Z}^1$ be a dominant pair of integral weights (i.e., $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_m$). The pair $(\alpha, \beta)$ is called typical if $\alpha_k + m - k \neq \beta$ for all $k = 1, \ldots, m$.

**Definition 1.** Let $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{Z}^m$, $\beta \in \mathbb{Z}^1$ be a dominant pair of integral weights, and suppose that the pair $(\alpha, \beta)$ is atypical, meaning that

$$\alpha_k + m - k = \beta \quad \text{for some } k \in \{1, \ldots, m\},$$

where the integer $k$ is necessarily unique because of the dominance condition, $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_m$. Let $r$ denote the largest positive integer satisfying

$$\alpha_k = \alpha_{k+1} = \cdots = \alpha_{k+r-1},$$

where $r \geq m - k + 1$ by necessity from the range of indices. We define a new pair $(\alpha^{(1)}, \beta^{(1)}) = (\alpha^{(1)}, \beta^{(1)})$ of weights by

$$\alpha^{(1)} = (\alpha_1, \ldots, \alpha_{k-1}, \alpha_k - 1, \ldots, \alpha_{k+r-1} - 1, \alpha_{k+r}, \ldots, \alpha_m)$$

and

$$\beta^{(1)} = \beta - r \in \mathbb{Z}^1. \quad (3)$$

Observing that $(\alpha^{(1)}, \beta^{(1)})$ is a dominant pair of integral weights satisfying the atypicality condition

$$a_{k+r-1}^{(1)} + m - (k + r - 1) = a_{k+r-1}^{(1)} + 1 + m - k - r + 1 = \beta - r = \beta^{(1)}$$

we see that we can define an operation $T$ on the set of atypical pairs of dominant integral weights

$$T(\alpha, \beta) = (\alpha^{(1)}, \beta^{(1)}).$$

Conversely, notice that the inverse operation $T^{-1}$ is also well-defined on atypical pairs. More precisely, if $(\alpha, \beta)$ is an atypical pair with $\alpha_k + m - k = \beta$ then taking $s$ to be the largest positive integer such that

$$\alpha_k = \alpha_{k-1} = \cdots = \alpha_{k-s+1},$$

where $s \geq k$ by necessity, we have $T^{-1}(\alpha, \beta) = (\alpha^{(-1)}, \beta^{(-1)})$ where $\alpha^{(-1)} = (\alpha_1, \ldots, \alpha_{k-s}, \alpha_{k-s+1} + 1, \ldots, \alpha_k + 1, \alpha_{k+1}, \ldots, \alpha_m)$ and $\beta^{(-1)} = \beta + s$. Finally, for any atypical pair $(\alpha, \beta)$ and any integer $n \in \mathbb{Z}$, we define $(\alpha^{(n)}, \beta^{(n)}) = T^n(\alpha, \beta)$.
3. THE TENSOR MODULES OF TYPE $Z$

The Lie superalgebra $gl(m|l) = gl(U)$ acts diagonally on tensor powers $U^\otimes n$ of the superspace $U = F \oplus G$, and these actions commute with the actions of the symmetric groups $\text{Sym}(n)$ on $U^\otimes n$ as supersymmetry operators. The Schur supermodules $S_\alpha U$ associated to a partition $\alpha$ or more generally the Schur supermodules $S_{\lambda/m} U$ associated to skew pairs of partitions are $gl(U)$-submodules of tensor powers of $U$ defined by the actions of classical Young symmetrizers in the group algebras of the symmetric groups (see Sect. 3 of [AW1]). There is also a trace homomorphism $tr: K \to U \otimes U[1]^e$ of $gl(U)$-modules and its dual map $ev: U \otimes U[1]^e \to K$ called evaluation. In terms of a basis $\{f_1, \ldots, f_m\}$ of $F$ and $\{g_1, \ldots, g_l\}$ of $G$ with dual basis $\{f_1^*, \ldots, f_m^*, g_i^*, g_l^*\}$ of $F^*$ and $\{g_i^*, g_l^*\}$ of $G^*$, we have $tr(1) = \sum_{i=1}^m f_i \otimes f_i^* - g_1 \otimes g_1^*$ and $ev(f_i \otimes f_i^*) = \delta_{\beta_i}$, $ev(g_1 \otimes g_1^*) = 0$, $ev(f_i \otimes g_l^*) = 0 = ev(g_1 \otimes f_1^*)$. We are now ready to state the definition of the module $Z_{\alpha,\beta}$ associated to a pair $(\alpha, \beta)$ of dominant non-negative integral weights using a presentation by mixed tensor modules of $gl(U)$.

**Definition 2.** Let $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m$, $\beta \in \mathbb{N}^l$ be a pair of partitions. The $gl(U)$-module $Z_{\alpha,\beta}$ is defined as the cokernel

$$S_{\alpha/1} U \otimes S_{\beta-1} U[1]^e \xrightarrow{\delta(\alpha, \beta)} S_{\alpha} U \otimes S_{\beta} U[1]^e \to Z_{\alpha,\beta} \to 0$$

(4)

of the map $\delta(\alpha, \beta)$ which is the composite of the map

$$S_{\alpha/1} U \otimes K \otimes S_{\beta-1} U[1]^e \xrightarrow{1 \otimes tr \otimes 1} S_{\alpha/1} U \otimes U \otimes U[1]^e \otimes S_{\beta-1} U[1]^e$$

(5)

and the tensor product of Pieri maps

$$S_{\alpha/1} U \otimes U \otimes U[1]^e \otimes S_{\beta-1} U[1]^e \to S_{\alpha} U \otimes S_{\beta} U[1]^e$$

(6)

keeping in mind that Pieri maps in the supersymmetric setting are given by actions of the same elements in the group algebras of the symmetric groups that give the classical versions of the respective Pieri maps.

The following proposition is easily established by direct calculation using the presentations defining the modules of type $Z$.

**Proposition 1.** Let $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m$, $\beta \in \mathbb{N}^l$ be a pair of partitions satisfying the atypicality condition $\alpha_k + m - k = \beta$ for some $k$.

(a) There is a homomorphism $\Theta(\alpha, \beta): Z_{\alpha,\beta}(U) \to Z_{\alpha(1), \beta(1)}(U)$ of $gl(m|1)$-modules induced by an evaluation homomorphism on generators.
\[ S_{\alpha} U \otimes S_{\mu} U[1]^* \rightarrow S_{\alpha(1)} U \otimes S_{\beta(1)} U[1]^* \text{ defined as the composite of Pieri maps} \]
\[ S_{\alpha} U \otimes S_{\mu} U[1]^* \rightarrow S_{\alpha(1)} U \otimes \wedge' U \otimes S_{\mu} U[1]^* \otimes S_{\beta(1)} U[1]^*, \quad (7) \]

followed by evaluation of the middle tensor factors
\[ \wedge' U \otimes S_{\mu} U[1]^* = \wedge' U \otimes \wedge' U^* \overset{e}{\rightarrow} K, \quad (8) \]

where \( r \) is the largest positive integer such that \( \alpha_{k+r-1} = \alpha_k \).

(b) In the case \( \alpha_k > 0 \) and \( \beta \leq r \), the new atypical pair \((\alpha^{(1)}, \beta^{(1)})\) are again a pair of partitions, so that \( \Theta(\alpha^{(1)}, \beta^{(1)}) \) makes sense by part (a) and the composite map \( \Theta(\alpha^{(1)}, \beta^{(1)}) \Theta(\alpha, \beta) : Z_{\alpha, \beta} \rightarrow Z_{\alpha(1), \beta(1)} \rightarrow Z_{\alpha(2), \beta(2)} \) is zero.

We next examine the stable behavior of the modules \( Z_{\alpha, \beta} \) as the weights \( \alpha, \beta \) are twisted by determinant characters. We let \( \det(U) \) denote the one-dimensional supertrace representations \( \wedge^m F \otimes G^* \) of \( \text{gl}(U) = \text{gl}(F \otimes G) = \text{gl}(m1) \). We can express \( \det(U) \) as a product
\[ \det(U) = \det(F) \det(G^*) = \det(F)(\det(G))^{-1} \quad (9) \]
of multiplicative characters. When converted to additive notation, \( \det(F) \) becomes the weight \( 1^m = (1, \ldots, 1) \in \mathbb{Z}^m \) and \( \det(G) \) becomes \( 1 \in \mathbb{Z}^1 \), with \( \det(G^*) \) becoming \(-1\). For any pair of weights \( \alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{Z}^m, \beta \in \mathbb{Z}^1 \), we let \( \alpha \otimes \det \) denote the weight \( \alpha + 1^m = (\alpha_1 + 1, \ldots, \alpha_m + 1) \in \mathbb{Z}^m \) and we let \( \beta \otimes \det = \beta + 1 \in \mathbb{Z}^1 \).

**Proposition 2.** Suppose \( \alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m, \beta \in \mathbb{N}^1 \) are partitions such that \( \alpha_m > 1 \) and \( \beta > m \).

(a) There is a natural isomorphism \( Z_{\alpha \otimes \det, \beta \otimes \det} = Z_{\alpha, \beta} \otimes \det(U) \) of \( \text{gl}(U) \)-modules.

(b) Suppose the pair \((\alpha, \beta)\) satisfies the atypicality condition \( \alpha_k + m - k = \beta \) for some \( k \in \{1, \ldots, m\} \) and we let \( r \) denote the largest non-negative integer such that \( \alpha_k = \alpha_{k+r-1} \). Then the pair \((\alpha \otimes \det, \beta \otimes \det)\) obviously satisfies the atypicality relation with the same \( k \) and \( r \), and the resulting evaluation map
\[ E(\alpha \otimes \det, \beta \otimes \det) : Z_{\alpha \otimes \det, \beta \otimes \det} \rightarrow Z_{(\alpha \otimes \det)^{(1)}, (\beta \otimes \det)^{(1)}} \]
is the map \( E(\alpha, \beta) \) tensored by the identity map on the factor \( \det(U) \).

**Proof.** Tensoring the presentation defining \( Z_{\alpha, \beta} \) with \( \det(U) \) results in giving us the presentation defining \( Z_{\alpha \otimes \det, \beta \otimes \det} \). The second statement is verified similarly by examination of appropriate presentations.
DEFINITION 3. Let \( \alpha \in \mathbb{Z}^m, \beta \in \mathbb{Z}^1 \) be a pair of dominant integral weights.

(a) Using any non-negative integer \( n > \max\{1 - \alpha_m, m - \beta\} \) we can make the definition

\[
Z_{\alpha, \beta}^{st} = (Z_{\alpha \otimes \det^n, \beta \otimes \det^n}) \otimes (\det(U))^{n(-n)}
\]

which is independent of \( n \) by (a) of Proposition 2.

(b) Suppose the pair \((\alpha, \beta)\) is atypical, then using sufficiently large \( n \), we can define an evaluation map

\[
\Theta(\alpha, \beta) : Z_{\alpha, \beta}^{st} \rightarrow Z_{\alpha(1), \beta(1)}^{st}
\]

by tensoring the evaluation map \( E(\alpha \otimes \det^n, \beta \otimes \det^n) \) with \((\det(U))^{n(-n)}\), which is well-defined by (b) of Proposition 2.

DEFINITION 4. Let \( \alpha \in \mathbb{N}^m, \beta \in \mathbb{N}^1 \) be a pair of partitions. The presentation (4) of the module \( Z_{\alpha, \beta} \) extends back naturally to a chain complex

\[
C_{\alpha, \beta}(U) : \ldots \rightarrow \bigoplus_{|\gamma|=i} S_{a/\gamma} U \otimes S_{b/\gamma} U^*[1] \rightarrow \ldots \\
\rightarrow \otimes S_{\beta/\gamma} U^*[1] \rightarrow S_a U \otimes S_b U^*[1],
\]

where the sum in degree \( i \) is over partitions \( \gamma \) of weight \( i \). The first differential of \( C_{\alpha, \beta}(U) \) is the map \( \delta_\gamma(\alpha, \beta) \) described in (5)–(6), and the higher differentials \( \delta_i(\alpha, \beta) \) are defined as \( \delta_i(\alpha, \beta) \) was in (5)–(6) using trace and Pieri maps. Moreover, these constructions make sense for any pair of partitions \( \alpha = (\alpha_1, \ldots, \alpha_p) \in \mathbb{N}^p, \beta = (\beta_1, \ldots, \beta_q) \in \mathbb{N}^q \) where \( p, q \) are arbitrary positive integers, and can be applied to finite dimensional \( \mathbb{Z}_2 \)-graded vector space \( M = M_0 \oplus M_1 \) in place of \( U \), so that the complexes \( C_{\alpha, \beta}(M) \) are defined for an arbitrary pair \( \alpha, \beta \) of partitions.

PROPOSITION 3. Let \( \alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m, \beta \in \mathbb{N}^1 \) be a pair of partitions.

(a) \( H_0(C_{\alpha, \beta}(U)) = Z_{\alpha, \beta} \) as \( \text{gl}(U) \)-modules.

(b) For a typical pair \( (\alpha, \beta) \), \( H_i(C_{\alpha, \beta}(U)) = 0 \) for all \( i \neq 0 \), and the \( \text{gl}(F) \times \text{gl}(G) \)-character of \( H_0(C_{\alpha, \beta}(U)) \) is \( \wedge(F^* \otimes G) \otimes S_\alpha F \otimes S_\beta G^* \).

(c)–(e) Suppose \( (\alpha, \beta) \) is atypical with \( \alpha_i + m - k = \beta \). Then \( H_i(C_{\alpha, \beta}(U)) = 0 \) unless \( i = 0 \) or \( i = \alpha_k - 1 \). Moreover, the \( \text{gl}(F) \times \text{gl}(G) \)-characters of the nonvanishing homology can be described as follows.
(c) If \( \alpha_k \leq 2 \) then the character of \( H_0(C_{a, \beta}(U)) \) is
\[
\wedge (F^* \otimes G) \otimes S^{F} F \otimes S^{G*} F
\]
and the character of \( H_{\alpha_k - 1}(C_{a, \beta}(U)) \) is the same as that of
\[
(X_{(a_1, \ldots, a_{k-1}, a_k - 1, \ldots, a_m - 1, -1)}(1)) \otimes \det(U)^{\beta \cdot a_k - 1}.
\]

(d) If \( \alpha_k = 1 \) then the character of \( H_0(C_{a, \beta}(U)) \) is the sum of the characters of (11) and (12).

(e) If \( \alpha_k = 0 \) then the character of \( H_0(C_{a, \beta}(U)) \) is the difference of the characters of (11) and (12).

Proof. Note that (a) is just a reminder of the definition of \( Z_{a, \beta} \) in (4).
To prove the rest, we start by recalling that the definition of \( C_{a, \beta}(U) \) was formulated in a manner to be applicable to any finite dimensional vector superspace \( M = M_0 \oplus M_1 \) in place of our basic space \( U = U_0 + U_1 = F \oplus G \). Since \( \dim(G) = 1 \), the complex \( C_{a, \beta}(U) \) has the form
\[
\cdots \to \bigoplus_{|\gamma| = 1} S_{\alpha / (i)} U \otimes S_{\beta - 1} U^*[1] \to \cdots
\]
\[
\to S_{\alpha / l} U \otimes S_{\beta - 1} U^*[1] \to S_{\alpha} U \otimes S_{\beta} U^*.
\]
Now, recall that
\[
S_{\alpha / (i)} U = \bigoplus_{j=0}^{m} S_{(\alpha / (i)) / (l)} F \otimes S_j G
\]
and
\[
S_{\beta} U^*[1] = \bigoplus_{j=0}^{m} S_{\beta - 1} G^* \otimes \wedge^j F^*.
\]
Therefore
\[
S_{\alpha / (i)} U \otimes S_{\beta - 1} U^*[1] = \bigoplus_{0 \leq s, t \leq m} S_{(\alpha / (i)) / (l)} F
\]
\[
\otimes S_j G \otimes S_{\beta - 1} G^* \otimes \wedge^t F^*.
\]

Consider the trace map acting from \( S_{\alpha / (i)} U \otimes S_{\beta - 1} U^*[1] \) to \( S_{\alpha / (i - 1)} U \otimes S_{\beta - 1} U^*[1] \). The trace decomposes into the components \( tr(F) \) and \( tr(G) \) sending \( K \), respectively, to \( F \otimes F^* \) and \( G \otimes G^* \). This defines on the complex (13) the structure of a double complex.

Let us consider the typical \( tr(F) \)-strand of the double complex. It looks like
\[
\cdots \to S_{(\alpha / (i - 2)) / (l')} F \otimes \wedge^{r - 2} F^* \to S_{(\alpha / (i - 1)) / (l')} F \otimes \wedge^{r - 1} F^*
\]
\[
\to S_{(\alpha / (i)) / (l')} F \otimes \wedge F^* \to 0.
\]
Since $S_{(\alpha)/(i)} F = S_{(\alpha)/(i) \oplus (i')} F = S_{(i')} F$ this strand is a direct sum of complexes of type

$$C_{\alpha, \beta}(F): \cdots \to \bigotimes_{|\gamma|=i} S_{\alpha_i, \ldots, \alpha_i, \gamma} F \otimes \wedge^{\beta_i, \ldots, \beta_i} F^* \to \cdots$$

$$\to S_{\alpha/(i)} F \otimes \wedge^{\beta/(i)} F^* \to S_{\alpha} F \otimes \wedge^{\beta} F^* \to 0.$$  \hspace{1cm} (16)

The proof of Proposition 1 follows from the following lemma (by a spectral sequence argument), which calculates the homology of complexes of type (16) (notice that you have to apply the proposition below with $m+1$ instead of $m$).

**LEMMA.** Let $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m$, $\beta \in \mathbb{N}^1$ be a pair of partitions and let us assume that $\dim(F) = m - 1$.

(a) If $(\alpha, \beta)$ is typical, i.e., $\alpha_k + m - k \neq \beta$, for $k = 1, \ldots, m$ then $C_{\alpha, \beta}(F)$ is exact.

(b) If for some $k$, $\alpha_k + m - k = \beta_1$, then the only nonzero homology of $C_{\alpha, \beta}(F)$ occurs in place $\alpha_k$ and equals $S_{\alpha_1, \ldots, \alpha_k, 1, \ldots, \alpha_m} F$.

**Proof.** The proof is done by a counting argument, because only Pieri’s formulas are involved.

**Note.** If $\alpha \in \mathbb{N}^m$, $\beta \in \mathbb{N}^1$ is any pair of partitions satisfying the atypicality condition $\alpha_k + m - k = \beta$ then the Euler characteristic of the complex $C_{\alpha, \beta}(U)$ is the sum of (11) and $(-1)^{\alpha_k - 1}$ times (12).

**4. THE COMPLEXES $Z^i_{(\alpha, \beta)}$ AND THEIR CYCLES**

We begin by introducing our notation for Kac modules which are $\mathfrak{g}$-modules induced up from finite dimensional simple modules of distinguished parabolic subalgebras of $\mathfrak{g}$. There are two distinguished $\mathbb{Z}$-gradings $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ on the Lie superalgebra $\mathfrak{g} = \mathfrak{gl}(m|1)$, one of them being the $\mathbb{Z}$-grading in (1) which we shall keep, and the other being its opposite grading which switches $\mathfrak{g}_{-1}$ with $\mathfrak{g}_1$. The two distinguished parabolic subalgebras of $\mathfrak{g}$ are given by

$$\mathfrak{p}_+ = \mathfrak{g}_0 \oplus \mathfrak{g}_1$$

and

$$\mathfrak{p}_- = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0.$$ \hspace{1cm} (17)

For any pair $\alpha \in \mathbb{Z}^n$, $\beta \in \mathbb{Z}^1$ of dominant integral weights, let $S_{\alpha, \beta}$ denote the irreducible $\mathfrak{gl}(F) \times \mathfrak{gl}(G)$ module of highest weight $(\alpha, \beta)$. If $\alpha, \beta$ are partitions then $S_{\alpha, \beta} = S_{\alpha} F \otimes S_{\beta} G$; otherwise we can use the
notation $\alpha \otimes \det, \beta \otimes \det$ from Section 3 to write

$$S_{\alpha, \beta} = S_{\alpha \otimes \det}(F) \otimes S_{\beta \otimes \det}(G) \otimes (\Lambda^m F \otimes G)^{\otimes n}$$

$$= S_{(\alpha_1 + \ldots + \alpha_m + n)} F \otimes S_{\beta + n}(G) \otimes (\Lambda^m F \otimes G)^{\otimes n}$$

(18)

for any $n$ large enough so that $\alpha_1 + \ldots + \alpha_m + n, \beta + n$ are all non-negative. Let us extend the $\mathfrak{g}_0\sigma$-module action on $S_{\alpha, \beta}$ trivially to a $\mathfrak{p}_-$-action and call the resulting $\mathfrak{p}_-$-module $V^0_{\alpha, \beta}$. We let $V_{\alpha, \beta}$ denote the Kac module

$$V_{\alpha, \beta} = U(gl(m|1)) \otimes_{U(p_-)} V^0_{\alpha, \beta}$$

(19)

induced up from the simple $\mathfrak{p}_-$-module $W^0_{\alpha, \beta}$. Similarly, we can extend the $\mathfrak{g}_0\sigma$-module action on $S_{\alpha, \beta}$ trivially to a $\mathfrak{p}_+$-action and call the resulting $\mathfrak{p}_+$-module $W^0_{\alpha, \beta}$. The opposite Kac module $W_{\alpha, \beta}$ can then be defined as

$$W_{\alpha, \beta} = U(gl(m|1)) \otimes_{U(p_+)} W^0_{\alpha, \beta}.$$  

(20)

As $gl(F) \otimes gl(G)$-modules, we have natural isomorphisms

$$V_{\alpha, \beta} = \Lambda (F^* \otimes G) \otimes S_{\alpha, \beta}$$

(21)

and

$$W_{\alpha, \beta} = \Lambda (F \otimes G^*) \otimes S_{\alpha, \beta}.$$  

(22)

Remark. (1) Let $\nu: gl(m|1) \to gl(m|1)$ be the Lie superalgebra automorphism that sends each matrix in $gl(m|1)$ to the negative of its supertranspose. Notice that $\nu: g \to g$ reverses the $\mathbb{Z}$-grading $g = g_{-1} \oplus g_0 \oplus g_1$ of $g = gl(m|1)$, i.e., $\nu$ exchanges $g_1$ with $g_{-1}$, and similarly reverses weights. Given any $g$-module $M$, we let $M^{(\nu)}$ denote the $g$-module resulting from twisting the $g$-action on $M$ by the automorphism $\nu$. Using this notation, we can describe the relationship between the two families $V_{\alpha, \beta}$ and $W_{\alpha, \beta}$ of $gl(m|1)$-modules as

$$V_{(\alpha_1, \ldots, \alpha_m), \beta}^{(\nu)} = W_{(-\alpha_1, \ldots, -\alpha_m), -\beta}.$$  

(23)

(2) From a special case of a theorem of Kac ([K]), we know that the $gl(m|1)$-module $V_{\alpha, -\beta}$ is irreducible if and only if the pair $(\alpha, \beta)$ is typical.

(3) From (ii) of the corollary to Proposition 3, we see that $gl(m|1)$-modules $Z^0_{\alpha, \beta}$ and $V_{\alpha, -\beta}$ have the same $gl(F) \times gl(G)$-character $\Lambda (F^* \otimes G) \otimes S_{\alpha, -\beta}$.  


(4) It follows from (2) and (3) that the $gl(m|1)$-module $Z^{\varphi}_{\alpha, \beta}$ is irreducible if and only if the pair $(\alpha, \beta)$ is typical. In particular, for a typical pair $(\alpha, \beta)$ the $gl(m|1)$-modules $Z^{\varphi}_{\alpha, \beta}$ and $V_{\alpha, \beta}$ are isomorphic. Although this is not true in general, it does follow easily from Proposition 5 in Section 4 that there is always an isomorphism

$$Z^{\varphi}_{(a_1, \ldots, a_m), \beta} \cong W_{(a_1-1, \ldots, a_m-1), m-\beta}$$  \hspace{1cm} (24)$$

of $gl(m|1)$-modules.

(5) Let $(\alpha_1, \ldots, \alpha_m) \in \mathbb{Z}^m$, $\beta \in \mathbb{Z}^1$ be an atypical pair of dominant integral weights satisfying the atypicality condition $\alpha_k + m - k = p$. From (4) we know that the $gl(m|1)$-module $Z^{\varphi}_{\alpha, \beta}$ is not irreducible. The $gl(F) \times gl(G)$-character of $Z^{\varphi}_{\alpha, \beta}$ does decompose nicely into two complementary parts. For convenience, we can twist by powers of $det(U)$ to reduce to the case where $\beta \geq m$ and $\alpha_i \geq m$ for all $i = 1, \ldots, m$, so that the $gl(F) \times gl(G)$-character of $Z^{\varphi}_{\alpha, \beta}$

$$\Lambda(F \otimes G) \otimes S_\alpha F \otimes S_\beta G^*$$

$$= \Lambda^m F \otimes S_\alpha G \otimes (F \otimes G) \otimes S_\alpha F \otimes S_\beta G^*,$$  \hspace{1cm} (25)$$
can be decomposed using Cauchy’s and Pieri’s formulas into a direct sum of irreducible $gl(F) \times gl(G)$-modules of the form $S_\lambda F \otimes S_\mu G^*$ where $\lambda, \mu$ are partitions. The pairs $(\lambda, \mu)$ appearing in the decomposition can be sorted into the following types according to the size of $\lambda_k$:

- type (1) are pairs $(\lambda, \mu)$ where $\lambda_k = \alpha_k$, \hspace{1cm} (26)$$
- type (2) are pairs $(\lambda, \mu)$ where $\lambda_k = \alpha_k - 1$. \hspace{1cm} (27)$$

Keep in mind that since $dim(G) = 1$, the $gl(G)$-character $S_\mu G^*$ of $S_\lambda F \otimes S_\mu G^*$ appearing in (26) or (27) can easily be determined from $\lambda$. In the proof of Theorem 1 we shall see that each type determines a composition factor of $Z^{\varphi}_{\alpha, \beta}$. Moreover, let us define a new pair $(\alpha, \beta)^\vee = (\alpha^\vee, \beta^\vee)$ of weights $\alpha^\vee = (\alpha_1^\vee, \ldots, \alpha_m^\vee) \in \mathbb{Z}^m$ and $\beta^\vee \in \mathbb{Z}^1$ by the rules

$$\alpha_i^\vee = \begin{cases} 
\alpha_i & \text{if } \alpha_i = \alpha_k \\
\alpha_i - 1 & \text{if } \alpha_i \neq \alpha_k 
\end{cases}$$  \hspace{1cm} (28)$$

and

$$\beta^\vee = \beta - (\alpha_1 + \cdots + \alpha_m) + (\alpha_1^\vee + \cdots + \alpha_m^\vee).$$  \hspace{1cm} (29)$$

It is easy to see that this pair $(\alpha^\vee, \beta^\vee)$ is the lowest weight pair (under dominance) among the pairs $(\lambda, \mu)$ of type (1) appearing in the decomposition of (25) over $gl(F) \times gl(G)$.\n
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IDENFINTION 5. (a) Let \( \alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{Z}^m \), \( \beta \in \mathbb{Z}^l \) be an atypical pair of dominant integral weights. From the results given in Section 2, we can define a doubly infinite cochain complex \( Z_{(\alpha, \beta)^+}^{\text{st}} \) of \( \mathfrak{sl}(U) \)-modules having the form

\[
\cdots \rightarrow Z_{(\alpha, \beta)^+}^{\text{st}}(-1) \xrightarrow{\partial^1} Z_{(\alpha, \beta)^+}^{\text{st}} \xrightarrow{\partial^0} Z_{(\alpha, \beta)^+}^{\text{st}}(-1) \rightarrow \cdots,
\]

where \( (\alpha, \beta)^{(n)} = (\alpha^{(n)}, \beta^{(n)}) \) and the \( n \)th differential

\[
\partial^n : Z_{(\alpha, \beta)^+}^{\text{st}} \rightarrow Z_{(\alpha, \beta)^+}^{\text{st}}(n+1)
\]

is the map \( \Theta(\alpha^{(n)}, \beta^{(n)}) \) for all \( n \in \mathbb{Z} \). Also, we define \( X_{(\alpha, \beta)} \) to be the module of cycles of \( Z_{(\alpha, \beta)^+}^{\text{st}} \) in degree 0, that is to say \( X_{(\alpha, \beta)} \) is the kernel of

\[
\Theta(\alpha, \beta) : Z_{(\alpha, \beta)^+}^{\text{st}} \rightarrow Z_{(\alpha, \beta)^+}^{\text{st}}(0).
\]

Notice that the cycles of \( Z_{(\alpha, \beta)^+}^{\text{st}} \) in degree \( n \) are the modules \( X_{(\alpha, \beta)^+}^{n} \) which we shall prove to be irreducible.

(b) For a typical pair \( (\alpha, \beta) \) of weights, we define \( Z_{(\alpha, \beta)^+}^{\text{st}} \) to be the complex \((0 \rightarrow Z_{(\alpha, \beta)^+}^{\text{st}} \rightarrow 0)\) with zero differentials and concentrated in degree 0.

**Theorem 1.** Let \( \alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{Z}^m \), \( \beta \in \mathbb{Z}^l \) be any pair of dominant integral weights such that \( (\alpha, \beta) \) is atypical. Then (a) the complex \( Z_{(\alpha, \beta)^+}^{\text{st}} \) is exact and (b) the \( \mathfrak{gl}(m|1) \)-module \( X_{(\alpha, \beta)} \) is irreducible.

**Proof.** From the definition of the complexes \( Z_{(\alpha, \beta)^+}^{\text{st}} \) it is sufficient to prove exactness at the places

\[
Z_{\alpha^{(-1)}, \beta^{(-1)}}^{\text{st}} \xrightarrow{\partial^1} Z_{\alpha, \beta}^{\text{st}} \xrightarrow{\partial^0} Z_{\alpha^{(1)}, \beta^{(1)}}^{\text{st}},
\]

of degree 0. Moreover, we can twist by a high enough power of \( \text{det}(U) \) to assume without loss of generality that the weights \( \alpha^{(-1)}, \alpha^{(-1)}, \beta^{(-1)}, \beta, \beta^{(1)} \) are all partitions. By (c) of Proposition 3, we have an \( \mathfrak{gl}(F) \times \mathfrak{gl}(G) \)-module isomorphism

\[
Z_{\alpha, \beta}^{\text{st}} = \wedge (F^{*} \otimes G) \otimes S_\alpha F \otimes S_\beta G^{*}
\]

which we decompose into a direct sum of irreducible representations using Cauchy’s and Pieri’s formulas. By twisting with a high enough power of \( \text{det}(U) \), we can assume that all irreducible \( \mathfrak{gl}(F) \times \mathfrak{gl}(G) \)-modules appearing in the decomposition of \( (25) \) are of the form \( S_\lambda F \otimes S_\mu G^{*} \) where \( \lambda, \mu \) are partitions.
Suppose that the atypicality condition satisfied by the pair $(\alpha, \beta)$ is $\alpha_k + m - k = \beta$. Recall that the representations $S_\mu F \otimes S'_\mu G^*$ occurring in the $\mathfrak{gl}(F) \times \mathfrak{gl}(G)$ decomposition of $Z_{a, \beta}^{st}$ can be sorted into two complementary parts described in (26) and (27). This sorting process applies to any module $Z_{a, \beta}^{st}$ with the index $k$ depending of course on $a$, $b$. Comparing these sortings for $Z_{a, \beta}^{st}$, it is easy to see that the representations of type 1 occurring in the decomposition of $Z_{a, \beta}$ are precisely the representations of type 2 for $Z_{a, \beta}^{st}$. By the same principle, the type 2 representations in $Z_{a, \beta}$ are precisely the type 1 representations for $Z_{a, \beta}^{st}$. This counting tells us that proving exactness is the same as proving that

$$\ker(\partial^0) = \text{sum of terms in } Z_{a, \beta}^{st} \text{ of type (1)}$$

or equivalently that

$$\text{in } (\partial^0) = \text{sum of terms in } Z_{a, \beta}^{st} \text{ of type (2)}.$$  

The counting process by itself gives us an inclusion $\supseteq$ in (32), and the reverse inclusion $\subseteq$ in (33). Exactness would follow from verifying that the appropriate component maps of the differential $\partial^0$ are nonzero. However, we can use part (b) to reduce this verification to the property of $\partial^0$ being an obviously nonzero map.

We will now proceed to prove (a) and (b) together using reverse induction on the index $k$ of the condition $\alpha_k + m - k = \beta$. Let us examine the initial case $k = m$. We have $\alpha_m = \beta$ and it is easy to see in this case that the sum of the terms in (32) is the $g_0$-character of

$$S_{(a_1 - \beta, \ldots, a_m - \beta)}(U) \otimes \det(U) \otimes \beta$$

which we know to be an irreducible $gl(U)$-module by a theorem of Berele–Regev ([BR]). Since $\text{im}(\partial^{-1})$ is a $gl(U)$-module whose $g_0$-character is contained in (34), it follows that $\text{im}(\partial^{-1})$ is an irreducible $gl(U)$-module.

Now we observe that the pair $(\alpha^{(1)}, \beta^{(1)})$ satisfies the atypicality condition $\alpha^{(1)}_m = \alpha_m - 1 = \beta - 1 = \beta^{(1)}$ with the same index $k = m$, the above discussion applied to $(\alpha^{(1)}, \beta^{(1)})$ in place of $(\alpha, \beta)$ gives us that $\text{im}(\partial^0)$ is an irreducible $gl(U)$-module satisfying the isomorphism in (32). It follows that (30) is exact and that $X_{a, \beta}$, which is $\ker(\partial^0)$ by definition, is an irreducible $gl(U)$-module satisfying the equality in (31).

Now we consider the case $\alpha_k + m - k = \beta$ with $k < m$, and assume by induction that $X_{\lambda, \mu}$ is irreducible for any atypical pair $(\lambda, \mu)$ satisfying $\lambda_j + m - j = \mu$ with $j > k$. Let $\mu$ be an irreducible $gl(U)$-submodule of $\text{im}(\partial^{-1})$. We know that the $g_0$-character of $M$ can contain only terms of
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type (2) for $Z_{a,\beta}^{\prime}$ or equivalently terms of type (1) in $Z_{a,\beta}^{\prime\prime}$. Therefore it the highest weight $(\lambda, \mu)$ of $M$ is not $(\alpha, \beta)$ then from the type (1) condition $\lambda_k = \alpha_k$, we have for any $i \leq k$

$$\lambda_i + m - i \geq \alpha_k + m - i \geq \alpha_k + m - k = \beta > \mu$$

which implies that $\lambda_i + m - j = \mu$ can happen only for $j > k$. The pair $(\lambda, \mu)$ cannot be typical because then $M$ would be a Kac module which is impossible as $Z_{a,\beta}^{\prime\prime}$ cannot properly contain a Kac module. So $(\lambda, \mu)$ has to be atypical with $\lambda_i + m - j = \mu$ for some $j > k$. By induction, $M$ has to be the irreducible $gl(U)$-module $X_{\lambda,\mu}$ we have

$$X_{\lambda,\mu} = M \subseteq \text{im}(\tilde{\partial}^{(-1)}) \subseteq \ker(\tilde{\partial}^0) = X_{a,\beta}$$

so we just need to show $(\lambda, \mu) = (\alpha, \beta)$. Suppose that $(\lambda, \mu) \neq (\alpha, \beta)$. Then $\lambda < \alpha$ and $\mu < \beta$ under dominance orders. Let us compare the lowest weight $(\lambda, \mu)$ of type (1) in $Z_{a,\beta}^{\prime\prime}$ with the lowest weight $(\lambda, \mu)$ in the irreducible module $X_{\lambda,\mu}$. Recall that $(\lambda, \mu)$ satisfies the type (1) condition $\lambda_k = \alpha_k$ and the inequality $j > k$ implies that $\lambda_k \geq \lambda_j$. So we have $\lambda_j \leq \alpha_k$. If $\lambda_k = \alpha_k$ then the condition $\lambda < \alpha$ implies $\lambda^\vee < \alpha^\vee$ which is impossible because all weights of type (1) in $Z_{a,\beta}^{\prime\prime}$ must dominate $(\alpha^\vee, \beta^\vee)$. On the other hand, $\lambda_j < \alpha_k$ is also impossible because it would imply that $\lambda_j^\vee = \lambda_j - 1 = \alpha_k - 1 < \alpha_k$ and $(\lambda^\vee, \mu^\vee)$ is supposed to be a weight of type (1) in $Z_{a,\beta}^{\prime\prime}$. Therefore we see that the supposition $(\lambda, \mu) \neq (\alpha, \beta)$ yields a contradiction as desired.

**COROLLARY.** For any pair $(\alpha, \beta)$ of dominant integral weights, the cycles in degree zero of $Z_{a,\beta}^{\prime\prime}$ are irreducible $gl(m|1)$-modules, and all finite-dimensional $gl(m|1)$-modules are up to isomorphism of this form.

5. THE GENERIC HOMOLOGY OF $gl(m|1)$-MODULES $X_{a,\beta}$ AND $Z_{a,\beta}^{\prime\prime}$

We start by describing two functors from graded $gl(m|1)$-modules to (graded) free complexes with linear differentials over polynomial rings in $m$ variables. Let $M = \oplus_{\ell \in \mathbb{Z}} M_{\ell}$ be a $gl(m|1)$-module with a $\mathbb{Z}$-grading which is compatible with the $\mathbb{Z}$-grading of $g = gl(m|1)$ in (1). Any element $u \in g_{-1} = \text{Hom}(G, F)$ acts as a differential of degree $-1$ on the graded vector space $M$. The generic version of this construction lives over the coordinate algebra $A^- = K[g - 1]$ of the affine space $g_{-1}$. The polynomial ring $A^-$ on $m$ variables is the symmetric algebra $\text{Sym}(g^\ast) = \text{Sym}(G \otimes F^\ast)$, and the generic map $\varphi: A^- \otimes G \to A^- \otimes F$ is a homomorphism of
free $A$-modules given by the generic $m \times 1$ matrix whose entries are $f^*_i \otimes g$ relative to bases $\{g\}$ and $\{f_1, \ldots, f_m\}$. The generic map $\varphi$ is an element of degree $-1$ in $A^- \otimes gl(m|1)$ and hence acts on $A^- \otimes M$ as a differential of degree $-1$. The resulting complex

$$ \cdots \to A^- \otimes M_i \to A^- \otimes M_{i+1} \to \cdots $$

of free $A^-$-modules will be denoted by $M^-$. Similarly, we define a complex $M^+$ over the symmetric algebra $A^+ = \text{Sym}(g^*_i) = \text{Sym}(F \otimes G^*)$ using the action of the generic map $\psi: A^+ \otimes F \to A^+ \otimes G$ on $A^+ \otimes M$. Since $\psi$ is an element of degree 1 in $A^+ \otimes gl(m|1)$, the result is a cochain complex

$$ \cdots \to \psi A^+ \otimes M_i \to \psi A^+ \otimes M_{i+1} \to \cdots $$

of free $A^+$-modules which we denote as $M^+$.

**EXAMPLE.** Recall that the standard $gl(m|1)$-module $U = U_0 \oplus U_1 = F \oplus G$ is $\mathbb{Z}$-graded. The complexes $U^-$ and $U^+$ are the homomorphism $\varphi$ and $\psi$, respectively, viewed as complexes of length one. More explicitly, $U^-$ is $0 \to U_1^\varphi \to U_0^- \to 0$ where $U_1^\varphi = A^- \otimes G$, $U_0^- = A^- \otimes F$ and $U^+$ is $0 \to U_0^\psi \to U_1^\psi \to 0$ where $U_0^\psi = A^+ \otimes F$, $U_1^\psi = A^+ \otimes G$. If $M$ is a tensor power $U^m$ then $M^-$ and $M^+$ are tensor powers of $\varphi$ and $\psi$, respectively. More generally, $S_{\alpha/\mu}(U)^-$ is the Schur complex $S_{\alpha/\mu}(\varphi)$ and $S_{\lambda/\mu}(U)^+$ is $S_{\lambda/\mu}(\psi)$. Similarly, $S_{\alpha/\mu}(U[1]^*)^-=S_{\alpha/\mu}(\varphi[1]^*)$ and $S_{\lambda/\mu}(U[1]^*)^+=S_{\lambda/\mu}(\psi[1]^*)$, keeping in mind that $(U[1]^*)_i = U_0^\varphi = F^\alpha$ and $(U[1]^*)_0 = U_1^\psi = G^\alpha$.

**Proposition 4.** If $M = Z_{\alpha,\beta}^m$ then $H_i(M^-) = 0$ for $i \neq 0$ and $H_0(M^-) = S_{\alpha,-\beta}$. In other words, $(Z_{\alpha,\beta}^m)^-$ is a free $A^-$-resolution of $S_{\alpha,-\beta}$.

Proof. We can twist by a high enough power of $\det(U)$ to make $\alpha$ and $\beta$ partitions so that $S_{\alpha,-\beta} = S_{\alpha} F \otimes S_{\beta} G^*$. Since $Z_{\alpha,\beta}^m$ is a complex of length $M$ over $A^-$, by the acyclicity lemma of Peskine–Szpiro (PS) it is enough to show acyclicity after inverting one of the variables, i.e., one of the entries of the generic matrix $\varphi$. By $gl(F) \times gl(G)$-equivariance (and universality under base change) this is equivalent to specializing the generic map $\varphi: A^- \otimes G \to A^- \otimes F$ to an arbitrary nonzero map $\varphi: G \to F$ (i.e., an arbitrary nonzero $\varphi \in g_{-\beta}$). We will work with total complex of $C_{\alpha,\beta}(U)$ with the differentials being given by the action of $\varphi$. We can write $\varphi$ in the canonical form $\tilde{\varphi}: G \to G \otimes F^1$ where $\tilde{\varphi}$ maps $G$ to $G$ by the identity map. From the results on Schur complexes in [ABW], we know $S_{\alpha,\beta}(\tilde{\varphi})$ is homotopically equivalent to $S_{\alpha,\beta}(F^1)$ and $S_{\beta-1}(\varphi[1]^*)$ is homotopy equivalent to $\wedge^{\beta-1} F^* \otimes [\beta-i]$. Acyclicity of $Z_{\alpha,\beta}^m$ now follows
from Proposition 3, provided we keep track of the extra homology of 
\( C_{\alpha,\beta}(U) \) which occurs in case (c) of Proposition 3. This extra homology of 
\( C_{\alpha,\beta}(U) \) is in degree \( \alpha_k - 1 \) and is equal to a Schur supermodule

\[
S_{(a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_m-1, 0)}(U)
\]  

(38)
tensored by \( \text{det}(U^*) = \wedge^m F^* \otimes G \). When we replace \( U \) in (38) by the map \( \varphi \), the resulting Schur complex is homotopy equivalent to (38) with \( U \)
replaced by \( F^1 \), which cancels out \( M_{\alpha_i}(C_{\alpha,\beta}(F^1)) \) in the derived category.

Since the module of chains of \( Z_{\alpha,\beta}^i \) in degree \( i \) is

\[
A^* \otimes \wedge^i(F^* \otimes G) \otimes S_\alpha F \otimes S_\beta G^*
\]  

(39)
\( Z_{\alpha,\beta}^i \) has the same terms and hence the same Euler characteristic as the
Koszul resolution of the trivial \( A^* \)-module \( S_\alpha F \otimes S_\beta G^* \). Therefore the
acyclicity of \( Z_{\alpha,\beta}^i \) implies that \( H_0(Z_{\alpha,\beta}^i) = S_\alpha F \otimes S_\beta G^* \), or in fact that \( Z_{\alpha,\beta}^i \)
is the Koszul resolution of \( S_\alpha F \otimes S_\beta G^* \).

**Theorem 2.** Let \( \alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{Z}^m, \beta \in \mathbb{Z}^1 \) be an atypical pair of
weights satisfying the atypicality condition \( \alpha_k + m - k = \beta \). Write \((\alpha_1, \ldots, \alpha_m)\) in the form \((\alpha_1, \ldots, \alpha_s)\) where \( \alpha_1 > \cdots > \alpha_s \). Then the nonzero
homology modules of \( X_{\alpha,\beta}^- \) are \( H_0(X_{\alpha,\beta}) \) and \( H_{\mu_1 + \cdots + \mu_i}(X_{\alpha,\beta}^-) \) for 
\( i = 1, \ldots, s \). Letting \( u \) denote the sum \( u_1 + \cdots + u_i \) for \( i = 1, \ldots, s \), we can
describe the nonzero homology groups of \( X_{\alpha,\beta}^- \) as \( \text{gl}(F) \times \text{gl}(G) \)-modules as follows.

(a) In the case \( k = m \), the only nonzero homology group is \( H_0(X_{\alpha,\beta}^-) \)
and is given by

\[
H_0(X_{\alpha,\beta}^-) = \bigoplus_{n \geq 0} S_{(a_1, \ldots, a_{m-1}, a_m-1-n)} n+1-\beta.
\]  

(40)

For (b)–(d), assume \( k < m \),

(b) \( H_0(X_{\alpha,\beta}^-) = \bigoplus_{n=0}^{\alpha_k-k} S_{(a_1, \ldots, a_{k-1}, a_k-n, a_{k+1}, \ldots, a_m)} n-\beta \),

(c) \( H_{\mu_1}(X_{\alpha,\beta}^-) = \bigoplus_{n=0}^{\alpha_k-\mu_1-1-\alpha_k} S_{(a_1, \ldots, a_{\mu_1-1}, a_{\mu_1-1-n-1}, a_k+n, a_k+n-1, \ldots, a_m)} n+\mu_1-\beta \),

(d) \( H_{\mu_2}(X_{\alpha,\beta}^-) = \bigoplus_{n=0}^{\alpha_k-\mu_2-1-\alpha_k} S_{(a_1, \ldots, a_{\mu_2-1}, a_{\mu_2-1-n-1}, a_{\mu_2+n}, \ldots, a_m)} n+\mu_2-\beta \).

Notice that the formula in (40) can be viewed as a degenerate case of (d),
and that all the homology modules except (d) and (40) are of finite length.

**Example.** Before starting the proof of the theorem, let us illustrate the
complicated notion with an example: Take \( \alpha = (2, 1, 1, 0) \), \( \beta = 3 \), where
$m = 4$. Since $\alpha_2 + 4 - 2 = 1 + 4 - 2 = 3 = \beta$, this pair is atypical with $k = 2$. We have $(\alpha_2, \alpha_3, \alpha_4) = (1^2, 0^1)$ so that $s = 2, u_1 = 2, u_2 = 1$. The
theorem states that the nonzero homology modules are

$$
H_0(X_{(2,1,1,0), 3}) = S_{(2,1,1,0), -3} = S_{(2,1,1,0)} F \otimes S_3 G^\#,
$$

$$
H_1(X_{(2,1,1,0), 3}) = S_{(2,0,0,0), -1} = S_{(2,0,0,0)} F \otimes S_1 G^\#,
$$

$$
H_2(X_{(2,1,1,0), 3}) = \bigoplus_{n \geq 1} S_{(2,0,0, -n), n} 
$$

$$
= \bigoplus_{n \leq 1} \left( \Lambda^n F \right) \otimes S_{(\alpha + 2, 2, 2, 0)} F^\# \otimes S_\alpha G.
$$

Proof of Theorem 2. First of all we may assume that $\alpha$ and $\beta$ are
partitions because twisting by $\det(U)$ commutes with $g_{-1}$ homology. From
Theorem 1, we have a right resolution

$$
0 \to X_{\alpha, \beta} \to Z^\alpha_{\alpha, \beta} \to Z^\beta_{(\alpha, \beta^{(1)})} \to Z^\beta_{(\alpha, \beta^{(2)})} \to \cdots. \tag{41}
$$

We begin with the case $k = m$. In this case we have to prove $Z^\alpha_{\alpha, \beta}$ is
acyclic and then calculate its Euler characteristic to be (40). Since $Z^\alpha_{\alpha, \beta}$ is
a complex of length at most $m$ over the polynomial ring $A^-$ in $m$
variables, we can again use the acyclicity lemma of Peskine–Szpiro. As in
the proof of Proposition 4, it is enough to prove acyclicity after specializing
the generic differential $\varphi: A^- \otimes G \to A^- \otimes F$ to an arbitrary nonzero
element $\varphi: G \to F$ of $g_{-1}$ acting as a differential on $X_{\alpha, \beta}$. In canonical
form, $\varphi$ is a map $G \to G \otimes F^1$ sending $G$ identically to $G$. We can see
without difficulty that under this differential, $X_{\alpha, \beta}$ has terms occurring in
pairs and hence is a split acyclic complex. Therefore by the acyclicity
lemma, the complex $Z^\alpha_{\alpha, \beta}$ is acyclic, and the character of its only homology
group can be calculated, using Proposition 4, from the Euler characteristic
of the right resolution in (41) which is easily seen to give the formula in
(40).

Now, in the general case we make use of the short exact sequences,

$$
0 \to X_{\alpha, \beta} \to Z^\alpha_{\alpha, \beta} \to X_{\alpha^{(1)}, \beta^{(1)}} \to 0,
$$

arising from Theorem 1. Proceeding by reverse induction on $k$, and a
secondary induction on $\alpha_k - \alpha_{k+1}$, utilizing the long exact homology
sequences associated to the short exact sequences,

$$
0 \to X^\alpha_{\alpha, \beta} \to Z^\alpha_{\alpha, \beta} \to X^\beta_{\alpha^{(1)}, \beta^{(1)}} \to 0, \tag{42}
$$
combined with Proposition (4) reduces the theorem to the initial case \( k < m \) taken care of above.

Let us illustrate the compound procedures in the proof with the example \( \alpha = (2, 1, 1, 0), \beta = 3 \) mentioned earlier. We have two short exact sequences

\[
0 \rightarrow X_{(2,1,1,0),3}^{\text{st}} \rightarrow Z_{(2,0,0,0),1}^{\text{st}} \rightarrow X_{(2,0,0,0),1}^{\text{st}} \rightarrow 0
\]

and

\[
0 \rightarrow X_{(2,0,0,0),1}^{\text{st}} \rightarrow Z_{(2,0,0,0),1}^{\text{st}} \rightarrow X_{(2,0,0,0),1}^{\text{st}} \rightarrow 0
\]

which reduce the statement about the homology of \( X_{(2,1,1,0),3}^{\text{st}} \) to that of \( X_{(2,0,0,0),1}^{\text{st}} \). For convenience, we twist by \( \det(U) = \Lambda^4 F \otimes G^* \) to convert \( X_{(2,0,0,0),1}^{\text{st}} \) to \( X_{(3,1,0,0),0}^{\text{st}} \) with non-negative indices. Since \( X_{(3,1,0,0),0}^{\text{st}} \) is in the Schur supermodule \( S_{(2,1)}(U) \) the corresponding chain complex \( X_{(3,1,0,0),0}^{\text{st}} \) is the Schur complex \( S_{(3,1)}(\varphi) \) on the generic map \( \varphi: A^* \otimes G \rightarrow A^* \otimes F \), and the acyclicity of \( S_{(3,1)}(\varphi) \) is established using the acyclicity lemma.

Finally we will briefly indicate how to apply what we have done with generic \( \mathfrak{g}_{-1} \) homology to compute the generic \( \mathfrak{g}_1 \) homology of \( X_{\alpha,\beta} \) and of \( Z_{\alpha,\beta}^{\text{st}} \).

**Proposition 5.** Let \( \alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{Z}^m, \beta \in \mathbb{Z}^1 \) be an atypical pair of dominant integral weights satisfying the atypicality condition \( \alpha_k + m - k = \beta \). Define a new atypical pair \( (\alpha, \beta)^{\vee} = (\alpha^{\vee}, \beta^{\vee}) \) where \( \alpha^{\vee} = (\alpha^{\vee}_1, \ldots, \alpha^{\vee}_m) \) is given by

\[
\alpha^{\vee}_i = \begin{cases} 
\alpha_i & \text{if } \alpha_i = \alpha_k \\
\alpha_i - 1 & \text{if } \alpha_i \neq \alpha_k
\end{cases}
\]  

(44)

and

\[
\beta^{\vee} = \beta - (\alpha_1 + \cdots + \alpha_m) + (\alpha^{\vee}_1 + \cdots + \alpha^{\vee}_m).
\]  

(45)

Then we have an \( \mathfrak{g}(F) \times \mathfrak{g}(G) \)-module homomorphism

\[
H_i(X_{\alpha,\beta}^+) \cong H_i(X^{\vee, -\alpha^{\vee}}, -\beta^{\vee})^*,
\]  

(46)

where \(- \alpha^{\vee} = (-\alpha^{\vee}_m, \ldots, -\alpha^{\vee}_1)\).

**Proof.** Let \( \eta: \mathfrak{g}(m|1) \rightarrow \mathfrak{g}(m|1) \) be the Lie superalgebra automorphism sending each matrix to the negative of its supertranspose discussed in remark (1) of Section 4. Using the covariant functor \( M \rightarrow M^{\vee} \) on \( \mathfrak{g}(m|1) \)-modules described in that remark, we have natural \( \mathfrak{g}(F) \times \mathfrak{g}(G) \)-module
isomorphisms
\[ H_i(M^+) \cong (H_i(M^{\nu-}))^\mathbb{C}. \tag{47} \]

Since the irreducible $\mathfrak{gl}(m|1)$-module $X_{\alpha, \beta}$ has $(\alpha^\vee, \beta^\vee)$ as the lowest dominant weight pair in its $\mathfrak{gl}(F) \times \mathfrak{gl}(G)$-character, it follows from the effect of $\nu$ on weights that $(X_{\alpha, \beta}^{\nu})^\vee$ is isomorphic to $X_{(-\alpha^\vee, -\beta^\vee)}$ as a $\mathfrak{gl}(m|1)$-module.

**Proposition 6.** Let $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{Z}^m$, $\beta \in \mathbb{Z}^1$ be a pair of dominant integral weights.

(a) In the case $(\alpha, \beta)$ is typical, $H_i(Z_{\alpha, \beta}^+) = 0$ unless $i = 0$, and $H_0(Z_{\alpha, \beta}^+) = S_{(\alpha_1 - 1, \ldots, \alpha_m - 1, m - \beta)}$.

(b) In the case $(\alpha, \beta)$ is atypical, let $\alpha_k + m - k = \beta$ be the atypicality relation satisfied by $(\alpha, \beta)$, and let $r$ be the largest positive integer such that $\alpha_k + r - 1 = \alpha_k$. Then we have natural isomorphisms

\[ H_i(Z_{\alpha, \beta}^+) = H_i(X_{\alpha, \beta}^+) \oplus H_{i-1}(X_{\alpha_1^\vee, \beta_1^\vee}^+) \]

which combined with Propositions 4 and 5 allow us to deduce the generic $\mathfrak{g}_{-1}$ homology of $Z_{\alpha, \beta}^+$.

**Proof.** In the case (a), the $\mathfrak{gl}(m|1)$-module $Z_{\alpha, \beta}^+$ is isomorphic to the Kac module $V_{\alpha, -\beta}$ and we know $V_{\alpha, -\beta}^+$ is the Koszul resolution of the trivial $A^+$-module $S_{(\alpha_1 - 1, \ldots, \alpha_m - 1, m - \beta)}$.

In the case (b), from Theorem 1 we have a short exact sequence from (42),

\[ 0 \to X_{\alpha, \beta} \to Z_{\alpha, \beta}^+ \to X_{\alpha_1^\vee, \beta_1^\vee}^+ \to 0, \tag{43} \]

over $\mathfrak{g} = \mathfrak{gl}(m|1)$ which respects the natural $\mathbb{Z}$-gradings if we shift the $\mathbb{Z}$-grading on $X_{\alpha_1^\vee, \beta_1^\vee}^+$ by $r$. It is easy to check this short exact sequence splits over the positive parabolic subalgebra $p_+ = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, and hence we have $Z_{\alpha, \beta}^+ = X_{\alpha, \beta}^+ \oplus X_{\alpha_1^\vee, \beta_1^\vee}^+[r]$ as complexes.

**References**


IRREDUCIBLE TENSOR REPRESENTATIONS


