Smooth solutions and singularity formation for the inhomogeneous nonlinear wave equation ☆

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ABSTRACT

We study the nonlinear inhomogeneous wave equation in one space dimension: $v_{tt} - T(v, x)_{xx} = 0$. By constructing some "decoupled" Riccati type equations for smooth solutions, we provide a singularity formation result without restrictions on the total variation of the data, which generalizes earlier singularity results of Lax and the first author. We apply these results to compressible Euler flows with a general pressure law and elasticity in an inhomogeneous medium.

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1. Introduction

In this paper, we consider the initial value problems for the second-order quasilinear inhomogeneous wave equation in one space dimension,

$$v_{tt} - T(v, x)_{xx} = 0, \tag{1}$$

where $(x, t) \in \mathbf{R} \times \mathbf{R}^+$, $v(x, t) \in \mathbf{R}$, and $T(v, x)$ is a smooth function satisfying

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$$T_v > 0, \qquad T_{vv} < 0.$$

These assumptions imply that the equation is hyperbolic and genuinely nonlinear, so that solutions exhibit wave-like behavior. Signals propagate in a forward and backward direction with local nonlinear wavespeeds

$$c = \sqrt{T_v} \quad \text{and} \quad -c = -\sqrt{T_v},$$

respectively. Eq. (1) includes a wide variety of interesting physical systems, such as one-dimensional nonlinear elasticity in an inhomogeneous medium [3,4]. In the context of elasticity, x is a material coordinate, v is the strain and T is the elastic stress.

It is well known that solutions of nonlinear hyperbolic equations generally form shock waves in finite time. Shocks form as a result of gradient blowup, which is a consequence of genuine nonlinearity. In this paper, we study smooth solutions, and we are particularly interested in the lifetime of such smooth solutions. We study the dynamical system which governs the growth of gradients, and use this to give estimates on the lifetimes of smooth solutions. This was carried out for the homogeneous nonlinear wave equation, obtained by taking $T = T(v)$, by Lax in [5], and extensions to larger systems (with restrictions on the data) were obtained by John [3] and Liu [7].

It is convenient to write (1) as a system, by setting

$$u(x,t) = \int v_t \, dx, \qquad p(v,x) = -T,$$

which yields the first-order system

$$v_t - u_x = 0, \tag{2}$$

$$u_t + p(v,x)_x = 0, \tag{3}$$

with

$$p_v < 0, \qquad p_{vv} > 0. \tag{4}$$

We make the further assumption that $p = p(v,x)$ is a C^3 function, and that boundedness of derivatives of p and the sign conditions (4) are uniform in x. In particular, p_x and p_{xx} are uniformly bounded in x.

For smooth solutions, Eqs. (2), (3) model compressible inviscid flow for general pressure laws in material coordinates (the Euler equations in a Lagrangian frame). Here p is the pressure, v is the specific volume, and u is the fluid velocity. A general pressure is described by $p = p(v, S)$, but the entropy $S = S(x)$ is stationary as long as the flow is smooth, by the Second Law of Thermodynamics. Moreover, smooth solutions of the 5×5 system of one-dimensional transverse flow in magnetohydrodynamics (MHD), which models a fluid coupled to a transverse (magnetic) vector field, and Eulerian flow in a variable area duct can also be modeled by (2), (3).

For homogeneous isentropic flow, we have $p = p(v)$, and (2), (3) is known as the p-system. This is a simplified 2×2 system which admits a coordinate system of Riemann invariants, and for which Lax proved that any nontrivial data will form a shock wave in finite time [5]. F. John and T.-P. Liu extended Lax's results to general systems of conservation laws of the form

$$u_t + f(u)_x = 0, \tag{5}$$

with $u = (u_1, \ldots, u_n) \in \mathbf{R}^n$, for $n \geqslant 1$ and $(x,t) \in \mathbf{R} \times \mathbf{R}^+$. John proved that gradients will blow up for small compactly supported data, and this was later generalized by Liu, provided the total variation

of the data is small enough [3,7]. There are also some singularity results for multi-dimensional con-
servation laws, subject to a restrictive "null condition" [8,9]. In a recent paper [1], the first author
generalizes the singularity formation results in [5] to the 3×3 compressible Euler equations with
polytropic ideal gas.

In this paper, we give singularity formation results for (2), (3), or equivalently (1). In [3,7], small so-
lutions are expanded asymptotically along integral curves, and wave interactions are treated quadrat-
ically. Here we take a different point of view, considering three wave families, namely forward and
backward waves, as in the p-system, coupled with stationary waves, which carry entropy variations
and inhomogeneity. We are then able to treat all waves without regard to their wave strength.

Our first task is to define the rarefactive and compressive character (R/C character) of the nonlin-
ear (non-stationary) waves in smooth solutions. In a 2×2 system, which is diagonal when expressed
in Riemann invariants, it is clear when a wave is rarefying or compressing. However, in larger systems,
this distinction is not clear as waves of different families generally interact continuously and cannot
be decoupled. The R/C character is a quantitative measure of how much rarefaction or compression
is in the solution at any point.

In stationary solutions (including those with contact discontinuities), which include no compres-
sion or rarefaction, the pressure p and velocity u are constant. We therefore use changes in p (or u)
to define the R/C character of the solution. In doing so, we take the directional derivative *along the
opposite characteristic*, to minimize the effect of waves of the opposite family, see [1]. We use the su-
perscripts "$\backslash$" and "$/$" to denote the directional derivatives along backward and forward characteristics,
respectively, so that

$$\backslash = \partial_t - c\partial_x \quad \text{and} \quad / = \partial_t + c\partial_x, \tag{6}$$

where $c = \sqrt{-p_v}$ is the (local) wavespeed in Lagrangian coordinates, cf. [5].

Definition 1. If the solutions of (2), (3) are smooth in an open set U of the (x,t)-plane and A is a
point in U, then we say the solution is forward (backward) rarefactive at A, if and only if $p^\backslash < 0$
($p' < 0$); it is forward (backward) compressive at A, if and only if $p^\backslash > 0$ ($p' > 0$).

It is convenient to introduce the change of variables

$$h(v,x) \equiv \int_v^{v^*} \sqrt{-p_v}\, dv \quad \text{and} \quad \mu \equiv x, \tag{7}$$

where v^* is a constant or infinity. Then by calculating $p^\backslash$ and p', we introduce equivalent variables α
and β, defined by

$$\alpha \equiv -\frac{p^\backslash}{c^2} = u_x + h_x + \frac{p_\mu}{c},$$

$$\beta \equiv -\frac{p'}{c^2} = u_x - h_x - \frac{p_\mu}{c}.$$

Here $u \pm h$ are the Riemann invariants for the corresponding isentropic p-system. Thus α and β are
direct generalizations of the derivatives of the Riemann invariants. For smooth solutions, we derive
Riccati type ODEs for α and β, which provide a framework for studying smooth solutions and gradient
blowup.

Theorem 1. *In* (2), (3), *smooth solutions satisfy*

$$\alpha' = -\frac{c}{2}\left(\frac{p_\mu}{c}\right)_h (3\alpha + \beta) + \frac{c_h}{2}(\alpha\beta - \alpha^2), \tag{8}$$

$$\beta` = \frac{c}{2}\left(\frac{p_\mu}{c}\right)_h (\alpha + 3\beta) + \frac{c_h}{2}(\alpha\beta - \beta^2), \tag{9}$$

with $c_h > 0$.

Eqs. (8) and (9) can be decoupled by use of an integrating factor. Define

$$y \equiv \sqrt{c}\alpha - I = \sqrt{c}(u+h)_x + \frac{p_\mu}{\sqrt{c}} - I, \quad \text{and} \tag{10}$$

$$q \equiv \sqrt{c}\beta + I = \sqrt{c}(u-h)_x - \frac{p_\mu}{\sqrt{c}} + I, \tag{11}$$

where

$$I = I(h,\mu) \equiv \int_{h_0}^{h} \frac{1}{2}\sqrt{c}\left(\frac{p_\mu}{c}\right)_h dh, \tag{12}$$

and h_0 is a constant.

Theorem 2. *For smooth solutions of* (2), (3), *we have*

$$y' = a_0 + a_1 y - a_2 y^2, \tag{13}$$

$$q` = a_0 - a_1 q - a_2 q^2, \tag{14}$$

where

$$a_0 \equiv -cI_\mu + \frac{1}{2}\sqrt{c}\left(\frac{p_\mu}{c}\right)_h p_\mu - c\left(\frac{p_\mu}{c}\right)_h I - \frac{c_h}{2\sqrt{c}}I^2, \tag{15}$$

$$a_1 \equiv -(2\sqrt{c}I)_h, \tag{16}$$

$$a_2 \equiv \frac{c_h}{2\sqrt{c}} > 0. \tag{17}$$

We note that these are not closed ODEs, since both the directional derivatives and the coefficients are dependent on the underlying solution of (2), (3). Nevertheless, as in [5], we are able to compare them to closed ODEs and derive bounds on the lifespan of smooth solutions. For convenience, we only consider smooth, i.e. C^∞, initial data, although our results also apply to C^2 initial data.

For arbitrarily given positive constants A_i and B_i, we denote by $\mathcal{K}$ the compact set whose interior $\mathcal{K}^o$ is given by

$$|h| < B_1, \qquad A_2 < c < B_2, \qquad A_3 < c_h < B_3,$$

$$|c_\mu| < B_4, \qquad |c_{\mu\mu}| < B_5, \qquad |c_{h\mu}| < B_6, \qquad |p_\mu| < B_7, \qquad |p_{\mu\mu}| < B_8. \tag{18}$$

Theorem 3. *There exists some constant $N > 0$ depending only on $\mathcal{K}$, such that, if the smooth initial data of* (2), (3)*, with* (4)*, satisfies*

$$\big(u(x,0), v(x,0)\big) \in \mathcal{K}^o, \quad \textit{for all } x,$$

and if there exists some x_0 such that

$$y(x_0, 0) < -N \quad or \quad q(x_0, 0) < -N,$$

then there exists $T_* = T_*(\mathcal{K}, N)$ such that either

$$\max\big\{|u_x|, |v_x|\big\} \to \infty \quad \textit{as } t \to T_*,$$

or there is some point (x_b, t_b) with $t_b \leqslant T_*$ such that

$$\big(u(x_b, t_b), v(x_b, t_b)\big) \in \partial \mathcal{K}.$$

This theorem states that gradient blowup occurs at some finite time T_* provided that the solution stays in some compact set. This is necessary to avoid degeneracies such as the vacuum in the equations of gas dynamics. The assumption that the solution does not form a vacuum is implicit in [5], but this is reasonable as a vacuum cannot form in finite time, see [13,14]. In particular, in models of elasticity, there are *a priori* bounds on the state variables which restrict the solution to a compact set.

Blowup of the gradient does not mean that the solution fails to exist: rather, this usually heralds the formation of a shock, and the associated decay of solutions. Generally, solutions are continued as weak solutions which contain shocks. However, in order to study shock propagation our system needs to be in conservation form, and we do not consider those issues here.

Theorem 3 implies that gradients of solutions blow up if the initial compressions are strong enough. When the variation of entropy is mild, N is close to zero, so the shock-free solutions are "almost rarefactive", which is consistent with Lax's singularity formation results in [5]. In [10] and a forthcoming paper [2], examples of solutions containing compressive waves are constructed, but the gradients of those solutions remain finite.

In a series of recent papers [10–12], the possibility of time-periodic solutions in the compressible Euler equations has been demonstrated. A critical feature of this study is how the R/C structure of waves can change across a contact discontinuity. In this paper we check the consistency of our results, which presuppose a smooth entropy field, with those results. By studying the R/C structure and the way in which it can change further, we expect eventually to see time-periodic solutions as in [10, 11] with piecewise smooth entropy, consisting of both contact discontinuities and smooth entropy variations.

Our results apply to a number of systems having structure similar to the inhomogeneous nonlinear wave equation. In a forthcoming paper, we apply these ideas to transverse flow in one-dimensional magnetohydrodynamics (MHD) and inviscid compressible flow in a varying area duct.

The paper is arranged as follows. In Section 2, we give the background of the equations and establish some useful identities. In Section 3, we define rarefactive and compressive waves. In Sections 4 and 5, we prove Theorems 1–3, respectively. Finally, in Section 6, we demonstrate the consistency of R/C structures.

2. Coordinates and background

We focus on Eqs. (2), (3) from now on. We assume that $p(v, x)$ is a smooth function of v and x, satisfying

$$p_v < 0 \quad \text{and} \quad p_{vv} > 0,$$

for all x and $v = v(x, t) \in (0, \infty)$. These conditions imply hyperbolicity and genuine nonlinearity, respectively. The vacuum state corresponds to $v = \infty$; however, since we assume the data remains in a compact set, we will not address questions at vacuum. The local absolute wavespeed is

$$c(v, x) \equiv \sqrt{-p_v} > 0. \tag{19}$$

We make the change of variables

$$h(v, x) \equiv \int_v^{v^*} c \, dv = \int_v^{v^*} \sqrt{-p_v} \, dv,$$

see (7), where $v^* > 0$ is a convenient constant (or ∞ if the integral converges uniformly). Since $p(v, x)$ is smooth, the function $h(v, x)$ is also smooth with respect to v and x. Moreover, since

$$h_v(v, x) = -c < 0, \tag{20}$$

the inverse function $v = v(h, \mu)$ is smooth with respect to h and μ, where we have set

$$\mu \equiv x. \tag{21}$$

For any function $f(v, x)$, we will write

$$f(h, \mu) = f\big(h(v, x), \mu\big) = f(v, x),$$

without ambiguity, and we use the subscript notation

$$f_x = \frac{\partial}{\partial x} f\big(v(x, t), x\big), \qquad f_{\bar{x}} = \frac{\partial}{\partial x} f(v, x), \qquad f_v = \frac{\partial}{\partial v} f(v, x),$$

$$f_h = \frac{\partial}{\partial h} f(h, \mu), \qquad f_\mu = \frac{\partial}{\partial \mu} f(h, \mu),$$

for the various partial derivatives of f.

We can relate the different partial derivatives as follows: by (20),

$$v_h = -\frac{1}{c}, \quad \text{so that} \quad v_t = v_h h_t = -\frac{1}{c} h_t. \tag{22}$$

Furthermore, since

$$-c^2 = p_v = p_h h_v = -c p_h, \quad \text{we have} \quad p_h = c, \tag{23}$$

and so

$$p_x = p_h h_x + p_\mu = c h_x + p_\mu.$$

Thus, for smooth solutions, (2), (3) can be written as

$$h_t + c u_x = 0, \tag{24}$$

$$u_t + c h_x + p_\mu = 0. \tag{25}$$

Next, differentiating $v = v(h(v, x), \mu)$ with respect to x, and recalling (21) and (22), we get

$$0 = v_h h_{\bar{x}} + v_\mu, \quad \text{so that} \quad v_\mu = -v_h h_{\bar{x}} = \frac{h_{\bar{x}}}{c}.$$

Similarly, for any function $f(v, x)$, we get

$$
\begin{aligned}
f_\mu &= f\big(v(h, \mu), x\big)_\mu \\
&= f_v v_\mu + f_{\bar{x}} \\
&= \frac{f_v}{c} h_{\bar{x}} + f_{\bar{x}},
\end{aligned}
\tag{26}
$$

and

$$f_h = f_v v_h = -\frac{f_v}{c}. \tag{27}$$

It follows that, for smooth solutions, p and c are smooth with respect to h and μ. Moreover, using (26) and (27), any quantities such as the R/C character, ODEs, and singularity formation results in this paper can all be expressed in the variables (v, x) instead of (h, μ).

3. Compressive and rarefactive waves

In this section, we define the rarefactive (R) and compressive (C) characters of (2), (3), which quantitatively indicate the amount of rarefaction or compression in the solutions at any point.

We first consider isentropic flow, for which $p = p(v)$ and our system reduces to the p-system,

$$h_t + c u_x = 0, \tag{28}$$

$$u_t + c h_x = 0. \tag{29}$$

The Riemann invariants $s = u + h$ and $r = u - h$ satisfy the diagonal system

$$s_t + c s_x = 0, \qquad r_t - c r_x = 0,$$

and so are constant along forward and backward characteristics,

$$\frac{dx}{dt} = +c, \qquad \frac{dx}{dt} = -c,$$

respectively. In an isentropic domain, because the system is diagonal, it is clear when waves are compressive or rarefactive: indeed, the amount of compression or rarefaction can be measured by derivatives of the appropriate Riemann invariant. There are several equivalent conditions, and for us it is convenient to consider the change in pressure: if p decreases as we traverse the wave from front to back, the wave is rarefactive (R), while if p increases, it is compressive (C) [1,5,13]. This is consistent with the entropy condition for shocks, which states that the pressure is always larger behind a shock.

When $p = p(v, x)$ explicitly depends on x, we first consider stationary solutions, in which there are no compressive or rarefactive waves, so the R/C characters should vanish. In stationary solutions, the pressure p is constant,

$$p_t = p_v v_t = 0 \quad \text{and} \quad p_x = -u_t = 0,$$

so its directional directives are zero. Physically, in gas dynamics, this means that pressure is not impacted by the variation of entropy, cf. [1]. Thus, by considering the directional derivatives of p along the *opposite* characteristics, we obtain Definition 1: the solution is forward (backward) rarefactive at A, if and only if $p^\backprime < 0$ ($p' < 0$); it is forward (backward) compressive at A, if and only if $p^\backprime > 0$ ($p' > 0$).

We could also use u to define the R/C character, as this is also constant in stationary solutions. We define

$$\alpha \equiv -\frac{p^\backprime}{c^2} \quad \text{and} \quad \beta \equiv -\frac{p'}{c^2}.$$

Lemma 1. *For smooth solutions of* (2), (3), *we have*

$$p' = -cu' \quad \text{and} \quad p^\backprime = cu^\backprime,$$

while also

$$\alpha = u_x + h_x + \frac{p_\mu}{c} \quad \text{and} \quad \beta = u_x - h_x - \frac{p_\mu}{c}. \tag{30}$$

Proof. By (2), (3) and (19),

$$cu' = cu_t + c^2 u_x = -cp_x - p_v v_t = -p',$$

and similarly $p^\backprime = cu^\backprime$. By (23) and (24),

$$\begin{aligned}
-c^2 \beta = p' &= p_t + cp_x \\
&= p_h h_t + c(p_h h_x + p_\mu) \\
&= -c^2 \left(u_x - h_x - \frac{p_\mu}{c} \right),
\end{aligned}$$

and similarly for α, so (30) follows. □

Corollary 1. *The R/C character of a smooth solution is given by*:

$$\begin{aligned}
\text{Forward} \quad &R \quad \text{iff} \quad \alpha > 0, \\
\text{Forward} \quad &C \quad \text{iff} \quad \alpha < 0, \\
\text{Backward} \quad &R \quad \text{iff} \quad \beta > 0, \\
\text{Backward} \quad &C \quad \text{iff} \quad \beta < 0.
\end{aligned}$$

Moreover, provided the solution values remain in $\mathcal{K}$,

$$|\alpha| \text{ or } |\beta| \to \infty \quad \text{iff} \quad |u_x| \text{ or } |v_x| \to \infty. \tag{31}$$

Proof. Clearly, by (30),

$$p^\backprime \gtrless 0 \quad \Leftrightarrow \quad \alpha \lessgtr 0, \quad \text{and} \quad p' \gtrless 0 \quad \Leftrightarrow \quad \beta \lessgtr 0.$$

By Lemma 1,

$$\alpha + \beta = 2u_x, \qquad \alpha - \beta = 2\left(h_x + \frac{p_\mu}{c}\right),$$

and (31) follows since p_μ and c remain finite. $\square$

In an isentropic domain, i.e. $p = p(v)$, it is clear that

$$s_x = \alpha \quad \text{and} \quad r_x = \beta,$$

so we can regard α and β as direct generalizations of the derivatives of the Riemann invariants.

4. Differential equations for gradients

In this section, we consider the characteristic decompositions of smooth solutions. By considering the directional derivatives of α and β, we derive the ODEs for α and β as stated in Theorem 1.

Proof of Theorem 1. We show (8), since (9) follows in exactly the same way. We have

$$
\begin{aligned}
\alpha' &= \left(u_x + h_x + \frac{p_\mu}{c}\right)_t + c\left(u_x + h_x + \frac{p_\mu}{c}\right)_x \\
&= (u_{xt} + ch_{xx}) + (h_{xt} + cu_{xx}) + \left(\frac{p_\mu}{c}\right)_t + c\left(\frac{p_\mu}{c}\right)_x \\
&= (u_t + ch_x)_x + (h_t + cu_x)_x - c_x h_x - c_x u_x + \left(\frac{p_\mu}{c}\right)_t + c\left(\frac{p_\mu}{c}\right)_x.
\end{aligned}
\tag{32}
$$

By (24), (25),

$$(u_t + ch_x)_x = (-p_\mu)_x = -p_{\mu h} h_x - p_{\mu\mu},$$

and

$$(h_t + cu_x)_x = 0.$$

Thus the right-hand side of (32) is

$$-p_{\mu h} h_x - p_{\mu\mu} - (c_h h_x + c_\mu)(h_x + u_x) + \left(\frac{p_\mu}{c}\right)_h (h_t + ch_x) + c\left(\frac{p_\mu}{c}\right)_\mu,
\tag{33}$$

since $f_x = f_h h_x + f_\mu$ for any function f.

By (30) and (24), we have

$$h_x = \alpha - u_x - \frac{p_\mu}{c} \quad \text{and} \quad h_t = -cu_x,$$

and by (23),

$$c\left(\frac{p_\mu}{c}\right)_h = c_\mu - c_h \frac{p_\mu}{c}.
\tag{34}$$

Thus (33) can be simplified to

$$-p_{\mu h}\left(\alpha - u_x - \frac{p_\mu}{c}\right) - p_{\mu\mu} - \left(c_h\alpha - c_h u_x - c_h\frac{p_\mu}{c} + c_\mu\right)\left(\alpha - \frac{p_\mu}{c}\right)$$
$$+ \left(\frac{p_\mu}{c}\right)_h (c\alpha - 2cu_x - p_\mu) + c\left(\frac{p_\mu}{c}\right)_\mu$$
$$= -c\left(\frac{p_\mu}{c}\right)_h u_x + \left[c_h u_x - c\left(\frac{p_\mu}{c}\right)_h\right]\alpha - c_h\alpha^2.$$

Finally, making the substitution

$$u_x = \frac{\alpha + \beta}{2}$$

yields (8).

By (4), (19) and (20),

$$0 > (\sqrt{-p_v})_v = c_v = c_h h_v = c_h(-c), \tag{35}$$

so $c_h > 0$, and the theorem is proved. □

Corollary 2. *For smooth solutions of* (2), (3), *we have*

$$\alpha' = -\frac{c}{4}\left(\frac{p_{\bar{x}}}{p_v}\right)_v (3\alpha + \beta) - \frac{c_v}{2c}(\alpha\beta - \alpha^2),$$

and

$$\beta^\backprime = \frac{c}{4}\left(\frac{p_{\bar{x}}}{p_v}\right)_v (\alpha + 3\beta) - \frac{c_v}{2c}(\alpha\beta - \beta^2).^2$$

Proof. By (19), (20), (26), (27) and (34), we have

$$c\left(\frac{p_\mu}{c}\right)_h = c_\mu - c_h\frac{p_\mu}{c}$$
$$= c_{\bar{x}} + \frac{c_v}{c^2}p_{\bar{x}}$$
$$= \frac{c}{2}\left(\frac{p_{\bar{x}}}{p_v}\right)_v, \tag{36}$$

and by (35),

$$\frac{c_h}{2} = -\frac{c_v}{2c},$$

and the corollary follows from Theorem 1. □

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2 These equations are the analogues of those analyzed by F. John in [3].

We now make another change of variables and transform (8) and (9) into decoupled differential equations by use of an integrating factor.

Proof of Theorem 2. First, the condition $a_2 > 0$ follows immediately from (35).

By (30) and (24),

$$h' = h_t + ch_x = -cu_x + ch_x = -c\left(\beta + \frac{p_\mu}{c}\right),$$

so that

$$\beta = -\frac{h'}{c} - \frac{p_\mu}{c}.$$

Hence, (8) can be written

$$\alpha' = -\frac{c}{2}\left(\frac{p_\mu}{c}\right)_h\left(3\alpha - \frac{h'}{c} - \frac{p_\mu}{c}\right) + \frac{c_h}{2}\alpha\left(-\frac{h'}{c} - \frac{p_\mu}{c}\right) - \frac{c_h}{2}\alpha^2.$$

We move the terms including h' to the left-hand side, so

$$\alpha' - \frac{1}{2}\left(\frac{p_\mu}{c}\right)_h h' + \frac{c_h}{2c}\alpha h' = \frac{1}{2}\left(\frac{p_\mu}{c}\right)_h p_\mu + \left(-\frac{3}{2}c\left(\frac{p_\mu}{c}\right)_h - \frac{c_h}{2c}p_\mu\right)\alpha - \frac{c_h}{2}\alpha^2. \tag{37}$$

Now, by (23),

$$-\frac{3}{2}c\left(\frac{p_\mu}{c}\right)_h - \frac{c_h}{2c}p_\mu = -c\left(\frac{p_\mu}{c}\right)_h - \frac{p_{\mu h}}{2} = -c\left(\frac{p_\mu}{c}\right)_h - \frac{c_\mu}{2},$$

and, since $\mu' = c$, we have

$$\sqrt{c}\alpha' + \frac{c_h}{2\sqrt{c}}\alpha h' + \frac{\sqrt{c}}{2}c_\mu\alpha = (\sqrt{c}\alpha)'.$$

Thus, multiplying (37) by $\sqrt{c}$ and simplifying, we get

$$(\sqrt{c}\alpha)' - \frac{1}{2}\sqrt{c}\left(\frac{p_\mu}{c}\right)_h h' = \frac{1}{2}\sqrt{c}\left(\frac{p_\mu}{c}\right)_h p_\mu - c\sqrt{c}\left(\frac{p_\mu}{c}\right)_h \alpha - \frac{c_h\sqrt{c}}{2}\alpha^2. \tag{38}$$

By (10),

$$\alpha = \frac{y + I}{\sqrt{c}},$$

where I is defined in (12) and satisfies

$$I' = I_h h' + cI_\mu.$$

Using these in (38), we get

$$y' = a_0 + a_1 y - a_2 y^2, \tag{39}$$

where a_0, a_1 and a_2 are defined in (15)–(17).

The derivation of the differential equation along backward characteristics,

$$q^{\backprime} = a_0 - a_1 q - a_2 q^2, \tag{40}$$

where q is defined in (11), is similar, and the proof is complete. □

Corollary 3. *A singularity (gradient blowup) forms if and only if*

$$|y| \text{ or } |q| \to \infty \quad \text{iff} \quad |u_x| \text{ or } |v_x| \to \infty,$$

provided the solution takes values in the compact set $\mathcal{K}$.

Proof. By (10) and (11),

$$y + q = \sqrt{c}(\alpha + \beta),$$
$$y - q = \sqrt{c}(\alpha - \beta) - 2I.$$

By (34), (12) and compactness, I remains finite, and the result follows. □

Because the coefficients a_0, a_1, and a_2 in Theorem 2 don't include derivative terms v_x, u_x, v_t or u_t, they are lower order when compared to y and q. Using (26) and (27), these coefficients can be expressed in terms of v and $\bar{x}$ rather than h and μ, as in Corollary 2.

In the p-system, $p = p(v)$ and $p_\mu = 0$, so that $a_0 = a_1 = 0$. In this case, (13) and (14) become

$$y' = -a_2 y^2, \qquad q^{\backprime} = -a_2 q^2, \tag{41}$$

which are exactly the ODEs derived in [5,6] for the isentropic homogeneous case.

5. Formation of singularity

We now consider the formation of singularities, which take the form of the blowup of gradients u_x and/or v_x, and correspond to shock formation in conservative systems.

We will study Eqs. (13), (14) as a dynamical system, even though this is not a pure system of ODEs. First, consider the ODE

$$\dot{\xi} = \psi_\pm(\xi),$$

where $\psi_\pm$ is defined by

$$\psi_\pm(\xi) \equiv a_0 \pm a_1 \xi - a_2 \xi^2, \tag{42}$$

with $a_2 > 0$, and the a_i are treated as constants. The equilibria, if they exist, are the roots of the quadratic equation

$$a_0 \pm a_1 \xi - a_2 \xi^2 = 0,$$

and we have $\dot{\xi} > 0$ between the roots, and $\dot{\xi} < 0$ otherwise. A typical phase line is shown in Fig. 1.

Whenever there are real roots, say $\xi_1 \leqslant \xi_2$, the region $\{\xi > \xi_1\}$ is an invariant region for the ODE. In particular, if $a_0 > 0$, then the roots are of opposite signs and the region $\{\xi > 0\}$ is invariant. Moreover, the region $\{\xi < \xi_1\}$ is also invariant, and solutions that originate in this interval have only a finite time of existence: it is this that drives the growth and blowup of gradients in the full system (39), (40).
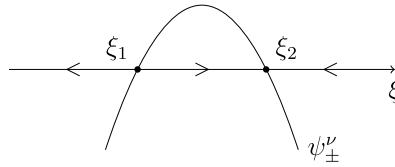
Fig. 1. Phase line for $\dot{\xi} = \psi_{\pm}^{\nu}(\xi)$.

For the p-system, we have $p_{\mu} = 0$, so $a_0 = a_1 = 0$ and $I = 0$. In this case, $y = \sqrt{c}s_x$ and $q = \sqrt{c}r_x$ are (multiples of) the gradients of the Riemann invariants. By the above discussion, the regions $\{y > 0\}$, $\{q > 0\}$ are invariant domains for the system (41), as are the regions $\{y < 0\}$, $\{q < 0\}$. Thus if y or q is negative somewhere, the negative quadratic functions in (41) drive blowup of y or q in finite time, as shown by Lax in [5]. From our point of view this is a trivial case, for which we have uniform estimates. In general, the forward and backward waves interact nonlinearly with the varying stationary background, and we do not expect uniformity. In fact, interactions can cause waves to change their R/C character, as demonstrated in [10] and the upcoming paper [2]. We expect that a complete analysis of the dynamics of (13), (14) will yield a rich variety of new and unexpected phenomena.

We now prove the breakdown results of Theorem 3 by studying (39), (40) as a dynamical system. Our aim is to describe sufficient conditions which imply that the gradient blows up in finite time.

Proof of Theorem 3. Fix a constant $0 < \nu \ll 1$, and define

$$\psi_{\pm}^{\nu}(\xi) \equiv a_0 \pm a_1 \xi - (1 - \nu) a_2 \xi^2, \tag{43}$$

so that our ODEs (39), (40) can be written

$$y' = \psi_{+}^{\nu}(y) - \nu a_2 y^2 \quad \text{and} \quad q' = \psi_{-}^{\nu}(q) - \nu a_2 q^2. \tag{44}$$

Now let $N = N(\nu) < 0$ be a uniform lower bound for the (real) roots of $\psi_{\pm}^{\nu}$, or $N = 0$ if there are no real roots. Then since $a_2 > 0$, we have

$$\psi_{\pm}^{\nu}(\xi) \leqslant 0 \quad \text{for every } \xi \leqslant N.$$

Now suppose there is some x_0 such that the data satisfies

$$y_0 = y(x_0, 0) < N.$$

Then for the forward characteristic emanating from $(x_0, 0)$, we have

$$y' \leqslant -\nu a_2 y^2,$$

so that the solution satisfies

$$\frac{1}{y(t)} \geqslant \frac{1}{y(0)} + \nu \int_0^t a_2 \, dt, \tag{45}$$

where the integral is taken along the forward characteristic. Since $y(0) < 0$ and a_2 is uniformly positive, there is some finite T_* such that the right-hand side of (45) vanishes, and so we obtain

$$y(t) \to -\infty \quad \text{as } t \to T_*.$$

A similar calculation holds for q.

Finally, we calculate the lower bound N. The roots of $\psi_{\pm}^{\nu}$ solve the quadratic equations

$$\psi_{\pm}^{\nu}(\xi) = a_0 \pm a_1 \xi - (1 - \nu)a_2 \xi^2 = 0$$

and so are

$$\xi = \frac{\pm a_1 \pm \sqrt{\Delta}}{2(1 - \nu)a_2}, \quad \text{where } \Delta = a_1^2 + 4(1 - \nu)a_0 a_2.$$

The minimum of these is clearly

$$-\frac{|a_1| + \sqrt{\Delta}}{2(1 - \nu)a_2},$$

and the lower bound N is obtained by maximizing the ratios $|a_1|/a_2$ and a_0/a_2, so we look for upper bounds for a_0 and $|a_1|$, and a lower bound for a_2.

From the expressions (15)–(17) and (12), we obtain the bounds provided c and c_h are bounded away from zero, while the quantities

$$h, \ c, \ c_h, \ p_\mu, \ p_{\mu h}, \ p_{\mu\mu}, \ p_{\mu hh} \quad \text{and} \quad p_{\mu\mu h}$$

remain finite, and recalling (23), the bound follows.

Using (26) and (27), the bounds can also be expressed by the bounds on the derivatives of p (and c) with respect to ν and $\bar{x}$. □

When $a_0 > 0$, the equilibria are on opposite sides of the origin, so the stable invariant region includes the values $\xi_1 < \xi \leqslant 0$. If the data can be set up in such a way that y_0 and q_0 lie in the interval (ξ_1, ∞) for all x_0, then the corresponding solution would have finite gradients for all time, and these would be nontrivial shock-free solutions. This topic is part of the authors' ongoing research.

6. Generalized R/C structure

The R/C structure at a single contact discontinuity or entropy jump for the compressible Euler equation is fully analyzed in [10]. This is an analysis of how the R/C character changes when a wave crosses a jump discontinuity, where the system is a p-system on either side of the jump. Using this R/C structure on an entropy jump, a class of time-periodic solutions in compressible Euler equations with polytropic ideal gas has been studied in a series of papers [10–12]. In this section, we show that the R/C characters we have defined for smooth solutions are consistent with those results. Furthermore, by the study of R/C structures in the generalized Euler equations (2), (3) with smooth and piecewise smooth entropy fields, we expect eventually to see a large class of time-periodic or quasi-periodic solutions as in [10,11] with both piecewise smooth and smooth entropy profiles.

We first consider how the R/C character can change at a fixed point in a smooth solution.

Lemma 2. *If $(\frac{p_{\bar{x}}}{p_\nu})_\nu < 0$ (or equivalently $(\frac{p_\mu}{c})_h < 0$), the backward R/C character can only change from R to C (resp. C to R), if the solution is forward R (resp. C); the forward R/C character can only change from C to R (resp. R to C), if the solution is backward R (resp. C). If $(\frac{p_{\bar{x}}}{p_\nu})_\nu > 0$ (or $(\frac{p_\mu}{c})_h > 0$), all the above R/C character changes will only happen in the opposite direction.*

Proof. Suppose the backward R/C character changes at (x_0, t_0), then $\beta = 0$ at (x_0, t_0). Then by (9), we have

$$\beta^\backprime = \frac{c}{2} \left(\frac{p_\mu}{c} \right)_h \alpha,$$

which has the sign of $\alpha(\frac{p_\mu}{c})_h$. Thus, the backward wave changes from C ($\beta < 0$) to R ($\beta > 0$) at (x_0, t_0), if and only if α and $(\frac{p_\mu}{c})_h$ have the same sign. Thus if $(\frac{p_\mu}{c})_h < 0$, then $\alpha < 0$, and the crossing forward wave is necessarily compressive by Corollary 1. All other cases follow similarly, and using Corollary 2 for the equivalence of derivative conditions completes the proof. $\quad\square$

In order to compare the R/C structure for smooth solutions to that of a single entropy jump, we recall the relevant argument from [10]. Discontinuities in weak solutions are governed by the Rankine–Hugoniot jump conditions, which are

$$\sigma[v] = -[u] \quad \text{and} \quad \sigma[u] = [p], \tag{46}$$

plus a third equation for energy conservation. Here σ is the speed of the discontinuity, and brackets denote the jump $[f] = f_R - f_L$ in f across the discontinuity. An entropy jump has zero speed, $\sigma = 0$, so that (46) reduce to $[u] = 0 = [p]$, that is

$$u_R = u_L \quad \text{and} \quad p_R = p_L.$$

Across the entropy jump, the R/C structure changes as follows, see [10]:

Lemma 3. *For $\frac{c_R}{c_L} < 1$, the backward R/C character can only change from R to C (resp. C to R), if the solution is forward R (resp. C); the forward R/C character can only change from C to R (resp. R to C), if the solution is backward R (resp. C). For $\frac{c_R}{c_L} > 1$, all the above R/C character changes will only happen in the opposite direction.*

Comparing these lemmas, we see that the condition $\frac{c_R}{c_L} < 1$ for an entropy jump should be consistent with the condition $(\frac{p_{\bar{x}}}{p_v})_v < 0$ for smooth solutions. Recalling that the change in pressure at an entropy jump is zero, we can assume $p_x = 0$, and treat the condition $\frac{c_R}{c_L} < 1$ as $c_x < 0$. Thus it suffices to show that, if $p_x = 0$, then

$$\left(\frac{p_{\bar{x}}}{p_v}\right)_v < 0 \quad \Leftrightarrow \quad c_x < 0. \tag{47}$$

Since $f_x = f_v v_x + f_{\bar{x}}$, if $p_x = 0$, we have

$$v_x = -\frac{p_{\bar{x}}}{p_v},$$

so that also

$$c_x = c_v v_x + c_{\bar{x}} = -c_v \frac{p_{\bar{x}}}{p_v} + c_{\bar{x}} = \frac{-c_v p_{\bar{x}} + c_{\bar{x}} p_v}{p_v}.$$

On the other hand,

$$\left(\frac{p_{\bar{x}}}{p_v}\right)_v = \frac{p_{\bar{x}v} p_v - p_{\bar{x}} p_{vv}}{(p_v)^2} = -\frac{2c(c_{\bar{x}} p_v - p_{\bar{x}} c_v)}{(p_v)^2},$$

where we recall $p_v = -c^2$. Comparing these, it follows that

$$\left(\frac{p_{\bar{x}}}{p_v}\right)_v = -\frac{2c}{p_v} c_x = \frac{2}{c} c_x,$$

which proves (47). Thus our continuous R/C character is consistent with that of a single entropy jump.

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