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Nonlinear Elliptic Problems in Annular Domains

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1. INTRODUCTION

In this paper we consider the question of existence and uniqueness of positive, radially symmetric solutions (or, briefly, p.r.s. solutions) of the equation

$$\Delta u + f(u) = 0 \qquad \text{in} \quad R_1 < |x| < R_0,\ x \in R^N,\ N \geqslant 3 \tag{1.1}$$

subject to one of the following sets of boundary conditions:

$$u = 0 \qquad \text{on } |x| = R_1 \qquad \text{and} \qquad |x| = R_0, \tag{1.2a}$$

$$u = 0 \qquad \text{on } |x| = R_1 \qquad \text{and} \qquad \frac{\partial u}{\partial r} = 0 \qquad \text{on } |x| = R_0, \tag{1.2b}$$

$$\frac{\partial u}{\partial r} = 0 \qquad \text{on } |x| = R_1 \qquad \text{and} \qquad u = 0 \qquad \text{on } |x| = R_0. \tag{1.2c}$$

Here $r = |x|$ and $\partial/\partial r$ denotes differentiation in the radial direction.

We assume that

$$f \in C^1(R),\ f(t) > 0 \qquad \text{for } t > 0 \text{ and } f(0) = 0. \tag{1.3}$$

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This condition on f will be assumed throughout the paper, without further mention. In addition, one or more of the following conditions will be frequently assumed.

(A–1) f is nondecreasing in $(0, \infty)$;

(A–2) $\lim_{t \to \infty} f(t)/t = \infty$;

(A–3) $\lim_{t \to 0} f(t)/t = 0$.

The Dirichlet problem for Eq. (1.1) in general bounded domains has been intensively studied in recent years. When f is superlinear (i.e., it satisfies condition (A–2)) the existence of positive solutions has been proved under various sets of assumptions, always including a restriction on the growth of f at infinity (see e.g., [AR], [BT], [L]). It is known that such a growth condition is, in general, necessary for starlike domains [P]. In the case of the annulus, such a growth condition is not necessary.

When the domain is a ball or an annulus, one may consider in particular radially symmetric solutions, in which case the problems mentioned above reduce to problems in o.d.e. Thus, in terms of the variable

$$\xi = [(N-2)\, r^{N-2}]^{-1} \tag{1.4}$$

Eq. (1.1) obtains the form

$$u''(\xi) + \rho(\xi)\, f(u(\xi)) = 0, \qquad \xi_0 < \xi < \xi_1 \tag{1.1$'$}$$

where

$$\begin{aligned} \rho(\xi) &= [(N-2)\,\xi]^{-k}, \qquad & k &= \frac{2N-2}{N-2}. \\ \xi_i &= [(N-2)\, R_i^{N-2}]^{-1}, \qquad & i &= 0, 1, \end{aligned} \tag{1.5}$$

and the boundary conditions become

$$u(\xi_0) = u(\xi_1) = 0 \tag{1.2a$'$}$$

$$u'(\xi_0) = u(\xi_1) = 0 \tag{1.2b$'$}$$

$$u(\xi_0) = u'(\xi_1) = 0. \tag{1.2c$'$}$$

In this, or other equivalent forms, these problems have been investigated by many authors (see [N], [C1], [Ni], [NN] and the references mentioned there). We note that the Emden–Fowler equation, which has received particular attention, is of the form (1.1)′.

A general existence result for o.d.e. problems as above was first obtained by Nehari [N], assuming that f is a continuous function, positive on $(0, \infty)$, satisfying the condition,

$$\exists\, \delta > 0 \text{ such that } f(t)/t^{1+\delta} \text{ is monotone increasing in } (0, \infty). \tag{1.6}$$

Nehari's result was obtained by a variational method which has many interesting applications. This method is briefly described in the last section of the present paper where it is used in deriving an uniqueness result.

In order to state in a concise form the existence results obtained in the present paper, we shall denote by Ex(a) (resp. Ex(b), Ex(c)) the statement: "Problem (1.1), (1.2a) (resp. (1.2b), (1.2c)) possesses a p.r.s. solution in every annulus $0 < R_1 < |x| < R_0 < \infty$." We have the following results:

I. Assuming (A–3): Ex(a) $\Leftrightarrow$ Ex(b) $\Rightarrow$ Ex(c);

II. Assuming (A–1)–(A–3): Ex(a), Ex(b), Ex(c) are valid.

Furthermore (for each of the problems mentioned above) if we assume (A–3) and a stronger form of (A–1), the existence of a p.r.s. solution in a given annulus implies the existence of p.r.s. solutions in every larger concentric annulus.

Conditions (A–2) and (A–3) are in a sense necessary in II. Thus, if there exists a constant M such that $f(t) < Mt$ on $(0, \infty)$, then (given R_1) there will be no p.r.s. solution of our problems for R_0 near to R_1. On the other hand, if there exists a positive constant c such that $f(t) > ct$ on $(0, \infty)$ then (given R_1) there will be no p.r.s. solution of our problems for R_0 sufficiently large. However, condition (A–1) is not necessary. In fact, using the methods of the present paper, one can establish existence under a condition slightly weaker than (A–1).

The existence results presented in this paper are obtained by a "shooting method" (as in [C1], [CM], [Ni]) combined with comparison results and estimates for eigenvalue problems in ordinary differential equations.

It is known that, under conditions (A–1)–(A–3), problem (1.1), (1.2a) may have more than one p.r.s. solution (see [NN]). In some special cases it was shown that this problem also possesses positive solutions which are not radially symmetric (see [BN], [C2]). Uniqueness results for problems (1.1), (1.2a) and (1.1), (1.2c) were obtained in [C1] under various additional conditions on f. Improved versions of these results and other uniqueness and nonuniqueness results for problem (1.1), (1.2a) were obtained in [Ni] and [NN].

In contrast to the above, uniqueness of p.r.s. solutions of problem (1.1), (1.2b) can be established merely under the assumption (see [M])

(A–1)′ $f(t)/t$ is strictly increasing in $(0, \infty)$.

Our discussion of the uniqueness question may be divided into three parts. First, we provide a proof of the uniqueness result for problem (1.1), (1.2b), different from the proof in [M], which yields additional information on the solutions. For instance, we show that, with R_1 fixed, the solution decreases as R_0 increases. Second we show that some of the uniqueness and nonuniqueness results that were obtained in [NN] for problem (1.1), (1.2a) are also valid for problem (1.1), (1.2c). These observations and some heuristic arguments have led us to the conjecture that uniqueness (in every annulus) in problem (1.1), (1.2a) implies uniqueness (in every annulus) in problem (1.1), (1.2c) and vice-versa. A proof of the first half of this conjecture (assuming that f satisfies (1.6)) is given in the final section of the paper.

2. Existence Theorems: Part I

In this section we shall establish the existence of solutions for problems (1.1), (1.2a) and (1.1), (1.2b). For this purpose we shall examine the family of solutions of the initial value problem

$$\begin{aligned} &u'' + \rho f(u) = 0, && \text{for } \xi < \xi_1, \\ &u(\xi_1) = 0, \quad u'(\xi_1) = -b, && \text{where } b > 0. \end{aligned} \tag{2.1}$$

Here ξ_1 is a positive number that will be kept fixed throughout the present section.

2.1. For every $b > 0$, problem (2.1) has an unique solution $u(\cdot, b)$ whose maximal domain of definition in $(0, \xi_1)$ will be denoted by $(\bar{\xi}(b), \xi_1)$. A function u is a solution of (2.1) if and only if it satisfies the integral equation

$$u(\xi) = b(\xi_1 - \xi) - \int_\xi^{\xi_1} (t - \xi)\, \rho(t)\, f(u(t))\, dt, \qquad \xi \leqslant \xi_1. \tag{2.2}$$

From (2.2) it is clear that if u is a positive solution in some interval (α, ξ_1) with $\alpha \geqslant 0$, then

$$u(\xi) \leqslant b(\xi_1 - \xi) \qquad \text{in} \quad (\alpha, \xi_1). \tag{2.3}$$

Therefore if $\alpha > 0$ the above solution can be extended to the left of α. Denote

$$\xi_0(b) = \inf\{\xi_0 > \bar{\xi}(b) \colon u(\xi, b) > 0 \text{ in } (\xi_0, \xi_1)\}.$$

By standard results in o.d.e., $\limsup_{b \to b_0} \bar{\xi}(b) \leqslant \bar{\xi}(b_0)$ for $b_0 > 0$ and the functions $(\xi, b) \to u(\xi, b)$ and $(\xi, b) \to u'(\xi, b)$ are continuously differentiable in the set

$$\{(\xi, b): b > 0, \xi > \bar{\xi}(b)\}.$$

By the implicit function theorem, the function $b \to \xi_0(b)$ is continuously differentiable in the neighborhood of every point $b > 0$ such that $\xi_0(b) > 0$. (Observe that at such a point $u'(\xi_0(b), b) > 0$.)

From Eq. (2.1) and the positivity of f, it is clear that $u(\cdot, b)$ is concave in $(\xi_0(b), \xi_1)$.

2.2. PROPOSITION. *If* $b > 0$ *and* $u(\cdot, b)$ *is defined and positive in* $(0, \xi_1)$ *then* $\lim_{\xi \to 0+} u(\xi, b) = 0$.

Proof. Since u is positive and concave in $(0, \xi_1)$, $\lim_{\xi \to 0+} u(\xi, b)$ exists and is nonnegative. If this limit is positive, there exists $c > 0$ such that $c \leqslant f(u(\xi, b))$ in $(0, \xi')$, where $\xi' = \xi_1/2$. Hence, by (2.2),

$$c \int_{\xi}^{\xi'} (t - \xi)\, \rho(t)\, dt \leqslant b(\xi_1 - \xi) - u(\xi, b), \qquad \forall \xi \in (0, \xi').$$

However, in view of (1.5), this is impossible.

2.3. PROPOSITION. *For every* $b > 0$ *there exists an unique point* $\tau(b)$ *in* $(\xi_0(b), \xi_1)$ *at which* u *attains its maximum over this interval. The function* $b \to \tau(b)$ *is continuously differentiable in* $(0, \infty)$.

Proof. The first statement is obvious. (When $\xi_0(b) = 0$, use 2.2.) The second statement follows from the implicit function theorem.

The next two results provide certain estimates involving $\tau(b)$ and $u_m(b) := u(\tau(b), b)$. We shall need the notation

$$F(t) = \int_0^t f(s)\, ds$$

2.4. PROPOSITION. *For every* $b > 0$,

$$b^2/2\rho(\tau(b)) \leqslant F(u_m(b)) \leqslant b^2/2\rho(\xi_1). \tag{2.4}$$

If $\xi_0(b) > 0$ *and* $a := u'(\xi_0, b)$ *then*

$$a^2/2\rho(\xi_0) \leqslant F(u_m(b)) \leqslant a^2/2\rho(\tau(b)). \tag{2.5}$$

Proof. Let $u(\xi)=u(\xi, b)$ and denote

$$I_1(\xi):=u'(\xi)^2/2+\rho(\xi)\,F(u(\xi)),$$

$$I_2(\xi):=u'(\xi)^2/(2\rho(\xi))+F(u(\xi)).$$

Then $I_1'(\xi)<0$ and $I_2'(\xi)>0$ in $(\xi_0(b), \xi_1)$. These inequalities imply (2.4) and (2.5).

2.5. PROPOSITION. *If f satisfies* (A–1) *then for every $b>0$*

$$u_m(b)>(\xi_1-\tau(b))\,b/2. \tag{2.6}$$

Furthermore,

$$\lim_{b\to\infty} u_m(b)=\infty \tag{2.7}$$

Proof. Let $u=u(\cdot, b)$,. Since $u''(\xi)=-\rho(\xi)\ f(u(\xi))$ is monotone increasing in (τ, ξ_1), the function $-u'$ is concave in this interval. Moreover, $-u'(\xi)>0$ in (τ, ξ_1), $-u'(\xi_1)=b$ and $u'(\tau)=0$. An elementary property of concave functions yields

$$u(\tau)=\int_\tau^{\xi_1}(-u')\,d\xi>(-u'(\xi_1))(\xi_1-\tau)/2=b(\xi_1-\tau)/2.$$

Now suppose that there exists a sequence $\{b_n\}$ with $b_n\to\infty$ such that $\{u_m(b_n)\}$ is bounded. Then, by (2.4), $\rho(\tau(b_n))\to\infty$, so that $\tau(b_n)\to 0$. But this is impossible in view of (2.6).

2.6. For reference we mention the following immediate consequence of Theorem 3′ of [GNN, p. 223].

$$\tau(b)\leqslant\frac{\xi_1+\xi_0(b)}{2}, \qquad \forall b>0. \tag{2.8}$$

Note that Theorem 3′ of [GNN] applies to the solution $u(\cdot, b)$ of Eq. (2.1) in $(\xi_0(b), \xi_1)$ because f is positive and ρ is decreasing.

Next we shall examine the behaviour of $\xi_0(b)$ and $\tau(b)$ as $b\to 0$ and as $b\to\infty$. Our aim is to show that the range of $\xi_0(\cdot)$ and $\tau(\cdot)$ is the entire interval $(0, \xi_1)$. Clearly this will imply the existence of positive solutions to problems (1.1)′, (1.2a)′ and (1.1)′, (1.2b)′.

2.7 LEMMA. *Assume* (A–3). *Then*

(i) $\lim_{b\to 0}\xi_0(b)=0$;

(ii) *If B is a bounded set in $(0, \infty)$ the set $\{\xi_0(b)\colon b\in B\}$ is bounded away from ξ_1.*

Proof. Let b be a positive number such that $\xi_0(b) > 0$ and consider the eigenvalue problem

$$\begin{aligned} &\phi'' + \lambda\rho(\xi)\,\phi = 0 \qquad \text{in} \quad (\xi_0(b), \xi_1) \\ &\phi(\xi_0) = \phi(\xi_1) = 0. \end{aligned} \tag{2.9}$$

Denote by $\lambda_1 = \lambda_1(b)$ its first eigenvalue and let ϕ_1 be a corresponding positive eigenfunction. Then, with $u = u(\xi, b)$ and $g(t) := f(t)/t$ we have

$$0 = \int_{\xi_0}^{\xi_1} \phi_1(u'' + \rho f(u))\, d\xi = \int_{\xi_0}^{\xi_1} (\phi_1'' u + \rho f(u)\,\phi_1)\, d\xi$$

$$= \int_{\xi_0}^{\xi_1} (g(u) - \lambda_1)\,\rho u \phi_1\, d\xi.$$

Hence, $\lambda_1(b) < \sup_{(\xi_0, \xi_1)} g(u(\xi, b)) \leqslant \sup_{0 < t < b\xi_1} g(t)$. Therefore, by (A-3), $\lambda_1(b) \to 0$ as $b \to 0$. Moreover, $\{\lambda_1(b) \colon b \in B\}$ is bounded. These two facts imply the statements of the lemma.

2.8. Lemma. *Assume* (A-3). *Then*

(i) $\lim_{b \to 0} \tau(b) = 0$

(ii) *If B is a bounded set in* $(0, \infty)$, *then the set* $\{\tau(b) \colon b \in B\}$ *is bounded away from* ξ_1.

Proof. From (2.3) and (2.4),

$$b^2/2\rho(\tau(b)) \leqslant F(u_m(b)) \leqslant F(b\xi_1), \qquad \forall b > 0.$$

By (A-3)

$$F(t)/t^2 \leqslant t^{-1} \int_0^t f(s)\, s^{-1}\, ds \to 0 \qquad \text{as} \quad t \to 0.$$

Consequently

$$1/\rho(\tau(b)) \leqslant 2F(b\xi_1)/b^2 \to 0 \qquad \text{as} \quad b \to 0.$$

This implies (i). Statement (ii) follows from 2.7(ii) and 2.6.

2.9. Lemma. *Assume* (A-1), (A-2). *Then*

(i) $\lim_{b \to \infty} \tau(b) = \xi_1$,

(ii) $\lim_{b \to \infty} \xi_0(b) = \xi_1$.

Proof. In view of 2.6, statement (ii) follows from (i). To prove (i), suppose there exists a sequence $\{b_n\}$ with $b_n \to \infty$ such that $\xi_1 - \tau(b_n) > \alpha > 0$ for all n. Then by (2.6),

$$u_m(b_n) \geqslant \alpha b_n/2, \qquad n = 1, 2, \ldots. \tag{2.10}$$

Setting $g(s) := f(s)/s$,

$$F(t) \geqslant \int_{t/2}^{t} g(s)\, s\, ds \geqslant h(t) \cdot 3t^2/8$$

where $h(t) := \inf_{(t/2, \infty)} g(s)$. Hence, by (2.10),

$$F(u_m(b_n)) \geqslant h(\alpha b_n/2)\, 3\alpha^2 b_n^2/32.$$

From this inequality and (2.4) we obtain

$$h(\alpha b_n/2) \leqslant 16/(3\alpha^2 \rho(\xi_1)), \qquad n = 1, 2, \ldots.$$

By (A–2), $h(t) \to \infty$ as $t \to \infty$. Therefore $\{b_n\}$ must be bounded, in contradiction to our assumption.

2.10. Lemma. *Assume* (A–3). *Let* $E = \{b > 0 : \xi_0(b) > 0\}$ *and suppose that* E *is not empty. (Note that* E *is an open set.) If* J *is a connected component of* E, *say* (b', b''), *then* $\lim_{b \to b'+} \xi_0(b) = 0$.

Proof. If $b' = 0$ the result follows from 2.7(i). If $b' > 0$ then (since $b' \notin E$) $\xi_0(b') = 0$, so that $u(\cdot, b') > 0$ in $(0, \xi_1)$. Now, if the conclusion of the lemma does not hold, it is easily seen (using 2.7(ii)) that $u(\cdot, b')$ must vanish at some point in $(0, \xi_1)$, thus arriving at a contradiction.

As a consequence of Lemmas 2.7–2.10 we have

2.11. Theorem. *Assuming* (A–1), (A–2), (A–3), *each of the problems* (1.1), (1.2a) *and* (1.1), (1.2b) *has at least one positive radially symmetric solution for all* R_1, R_0 *such that* $0 < R_1 < R_0 < \infty$.

2.12. *Remark.* An examination of the proof shows that the following result also holds.

Assume (A–3). Consider problems (1.1), (1.2a) and (1.1), (1.2b). If for one of these problems it is known that it has a positive, radially symmetric solution in every annulus $0 < R_1 < |x| < R_0 < \infty$, then the same is true w.r.t. the other problem.

For instance, consider the case when the existence result is known for (1.1), (1.2b). Then there exists a sequence of positive numbers $\{b_n\}$ such that $\tau(b_n) \to \xi_1$ and hence, by 2.6, $\xi_0(b_n) \to \xi_1$. Let E be as in 2.10 and let J_n

be the connected component of E which contains b_n. Then, since $\xi_0(\cdot)$ is continuous on J_n, 2.10 implies that the range of $\xi_0(\cdot)$ over J_n contains the interval $(0, \xi_0(b_n))$. Since $\xi_0(b_n) \to \xi_1$, the range of $\xi_0(\cdot)$ over E is $(0, \xi_1)$.

3. Existence Theorems: Part II

Here we deal with the existence of positive solutions of problem (1.1), (1.2c). We shall establish the following result.

3.1. Theorem. *Suppose that problem* (1.1), (1.2a) *has a positive, radially symmetric solution for every* R_1, R_0 *such that* $0 < R_1 < R_0 < \infty$. *In addition assume* (A–3). *Then problem* (1.1), (1.2c) *has a positive radially symmetric solution for every* R_1, R_0 *as above.*

3.2. For the proof of the theorem we consider the family of solutions of the initial value problem

$$\begin{aligned} &v'' + \rho f(v) = 0 && \text{for } \xi_0 < \xi, \\ &v(\xi_0) = 0, \qquad v'(\xi_0) = a, && \text{where } a > 0. \end{aligned} \tag{3.1}$$

Here ξ_0 is a positive number that will be kept fixed throughout the present section. For every $a > 0$, problem (3.1) has an unique solution $v(\cdot, a)$ whose maximal domain of definition in (ξ_0, ∞) will be denoted by $(\xi_0, \bar{\xi}(a))$. A function v is a solution of (3.1) if and only if it satisfies

$$v(\xi) = a(\xi - \xi_0) - \int_{\xi_0}^{\xi} (\xi - t)\, \rho(t)\, f(v(t))\, dt. \tag{3.2}$$

Therefore, if v is a positive solution in some interval (ξ_0, β) we have

$$v(\xi) \leqslant a(\xi - \xi_0) \qquad \text{in} \quad (\xi_0, \beta). \tag{3.3}$$

Hence if $\beta < \infty$, the above solution can be extended to the right of β.

If $v(\cdot, a)$ vanishes somewhere in $(\xi_0, \bar{\xi}(a))$—which certainly happens when $\bar{\xi}(a) < \infty$—we denote by $\xi_1(a)$ the first zero of $v(\cdot, a)$ to the right of ξ_0. We denote by D the set of points $a > 0$ for which $\xi_1(a)$ is defined.

Since $v(\cdot, a)$ is concave in every interval in which it is positive, it is clear that $a \in D$ if and only if there exists a point $\tau_0(a)$ such that $v'(\tau_0(a), a) = 0$. This point is unique. By standard results in o.d.e., the set D is open and $\xi_1(\cdot)$, $\tau_0(\cdot)$ are in $C^1(D)$.

3.3. Lemma. *Under the assumptions of the theorem, there exists a sequence* $\{a_n\}$ *in* D *such that* $\xi_1(a_n) \to \xi_0$ *and* $a_n(\xi_1(a_n) - \xi_0) \to \infty$.

Proof. Given $\xi_1 > \xi_0$ let $u = u(\xi; \xi_0, \xi_1)$ be a positive solution of the problem

$$\begin{aligned} &u'' + \rho f(u) = 0 \qquad \text{in} \quad (\xi_0, \xi_1), \\ &u(\xi_0) = u(\xi_1) = 0. \end{aligned} \tag{3.4}$$

Denote by $\lambda_1 = \lambda_1(\xi_0, \xi_1)$ the first eigenvalue of the problem

$$\begin{aligned} &\phi'' + \lambda \rho \phi = 0 \qquad \text{in} \quad (\xi_0, \xi_1), \\ &\phi(\xi_0) = \phi(\xi_1) = 0. \end{aligned} \tag{3.5}$$

As in the proof of 2.7 we obtain

$$\lambda_1(\xi_0, \xi_1) \leqslant \sup_{\xi_0 \leqslant \xi \leqslant \xi_1} g(u(\xi; \xi_0, \xi_1)) \tag{3.6}$$

where $g(t) := f(t)/t$. Since $\lambda_1(\xi_0, \xi_1) \to \infty$ when $\xi_1 \to \xi_0+$ and g is bounded in bounded sets (see (A–3)), we deduce that $\sup_{\xi_0 \leqslant \xi \leqslant \xi_1} u(\xi; \xi_0, \xi_1) \to \infty$ as $\xi_1 \to \xi_0$. Choose a sequence $(\xi_{1,n}\}$ such that $\xi_{1,n} \to \xi_0+$ and a corresponding sequence of solutions $\{u_n\}$ of (3.4) (with $\xi_1 = \xi_{1,n}$). Then, in view of the previous argument and (3.3), the lemma holds with $a_n := u_n'(\xi_0)$.

3.4. Lemma. *Assume that D is not empty. Let (a', a'') be a connected component of D.*

(i) *If $a' = 0$ and* (A–3) *holds or $a' > 0$, then*

$$\lim_{a \to a'+0} \tau_0(a) = \infty. \tag{3.7}$$

(ii) *If $a'' < \infty$ then*

$$\lim_{a \to a''-0} \tau_0(a) = \infty. \tag{3.7$'$}$$

Proof. It is easily verified that (3.7) holds when $a' > 0$ and that (3.7)′ holds when $a'' < \infty$. Accordingly, we describe only the proof of (3.7) in the case $a' = 0$, assuming that (A–3) holds.

Given a number $\tau > \xi_0$, consider the eigenvalue problem

$$\begin{aligned} &\psi'' + \mu \rho \psi = 0 \qquad \text{in} \quad (\xi_0, \tau), \\ &\psi(\xi_0) = 0, \qquad \psi'(\tau) = 0. \end{aligned} \tag{3.8}$$

Suppose that v is a solution of,

$$\begin{aligned} &v'' + \rho f(v) = 0 \qquad \text{in} \quad (\xi_0, \tau) \\ &v(\xi_0) = 0, \qquad v'(\tau) = 0. \end{aligned} \tag{3.9}$$

Let $\mu_1 = \mu_1(\xi_0, \tau)$ be the first eigenvalue of (3.8). Then, as in 2.7, we obtain

$$\sup_{\xi_0 \leqslant \xi \leqslant \tau} g(v(\xi)) \geqslant \mu_1(\xi_0, \tau). \tag{3.10}$$

Now suppose that there exists a sequence $\{a_n\} \in D$ such that $a_n \to 0+$ and $\{\tau_0(a_n)\}$ is bounded. By (3.3) and the definition of $\tau_0(a)$ we have

$$\sup_{\xi_0 \leqslant \xi \leqslant \xi_1(a)} v(\xi, a) \leqslant a(\tau_0(a) - \xi_0), \qquad \forall a \in D. \tag{3.11}$$

Hence,

$$\sup_{(\xi_0, \xi_1(a_n))} v(\xi, a_n) \to 0 \qquad \text{as} \quad n \to \infty$$

Therefore, by (3.10) and (A–3),

$$\lim_{n \to \infty} \mu_1(\xi_0, \tau_0(a_n)) = 0.$$

But this is impossible when $\{\tau_0(a_n)\}$ is bounded.

3.5. *Proof of Theorem* 3.1. Let $\{a_n\}$ be as in 3.3 and let J_n be the connected component of D containing the point a_n. Then, by 3.4 the range of $\tau_0(\cdot)$ over J_n contains the interval $(\xi_1(a_n), \infty)$ and by 3.3, $\xi_1(a_n) \to \xi_0$. Therefore we conclude that the range of $\tau_0(\cdot)$ is (ξ_0, ∞) and this implies the statement of the theorem.

4. Existence Results: Part III

Suppose that f satisfies (A–1) and (A–3) but not (A–2) and that

$$g(t) := f(t)/t \text{ is bounded.} \tag{4.1}$$

Then for each of the three problems treated here, there are no positive radially symmetric solutions in sufficiently thin annuli. This is easily seen using estimates (3.6) and (3.10).

In this section we consider all three problems, without assuming (A–2), and prove the following result:

4.1. Theorem. *Suppose that f satisfies* (A–3) *and*

(A–1)′ $g(t) = f(t)/t$ *is strictly increasing.*

For each of the problems mentioned above, if there exists a positive radially symmetric solution in a certain annulus $0 < R_1 < |x| < R_0 < \infty$, *then such a* *solution exists in every annulus centered at the origin, which contains* $R_1 < |x| < R_0$.

The theorem is proved through a series of lemmas.

4.2. LEMMA. *Under the assumptions described in* 4.1 *the following statements hold*:

(i) *Problems* (1.1)′, (1.2a)′ *and* (1.1)′, (1.2b)′ *have positive solutions in every interval* $\bar{\xi}_0 < \xi < \xi_1$ *with* $\bar{\xi}_0 \in (0, \xi_0]$.

(ii) *Problem* (1.1)′, (1.2c)′ *has a positive solution in every interval* $\xi_0 < \xi < \bar{\xi}_1$ *with* $\xi_1 < \bar{\xi}_1$.

Proof. We prove the statement concerning problem (1.1)′, (1.2a)′. The other statements are proved in a similar way.

Let $\{u(\cdot, b): b > 0\}$ be the family of solutions of (2.1). By assumption there exists b^* such that $\xi_0(b^*) = \xi_0$. Let $E = \{b > 0: \xi_0(b) > 0\}$ and let E^* be the connected component of E which contains b^*. E^* is an open interval, say, (b', b''). By Lemma 2.10 we have

$$\lim_{b \to b'+} \xi_0(b) = 0.$$

Since $\xi_0(\cdot)$ is continuous on E^*, it follows that its range over E^* contains the interval $(0, \xi_0]$.

4.3. LEMMA. *Let* u_1 *and* u_2 *be two positive solutions of* $u'' + \rho g(u)\, u = 0$ *in* (ξ_0, ξ_1) *such that* $u_2 > u_1$ *in* (ξ_0, ξ_1). *If g satisfies* $(A\text{-}1)'$, *then the following situations are impossible*:

(i) $u_1(\xi_0) = u_1(\xi_1) = 0$,

(ii) $u_1'(\xi_0) = u_1(\xi_1) = 0$, *and* $u_2'(\xi_0) \leqslant 0$,

(iii) $u_1(\xi_0) = u_1'(\xi_1) = 0$, *and* $u_2'(\xi_1) \geqslant 0$.

Proof. Integration by parts yields

$$\int_{\xi_0}^{\xi_1} (u_1'' u_2 - u_2'' u_1)\, d\xi = u_1' u_2 - u_2' u_1 \Big|_{\xi_0}^{\xi_1} \tag{4.2}$$

In view of our assumptions,

$$\int_{\xi_0}^{\xi_1} (u_1'' u_2 - u_2'' u_1)\, d\xi = \int_{\xi_0}^{\xi_1} [g(u_2) - g(u_1)]\, \rho u_1 u_2\, d\xi > 0. \tag{4.3}$$

However, in each of the cases (i)–(iii), the right-hand side of (4.2) is non-positive, thus contradicting (4.3).

4.4. LEMMA. *Under the assumptions of* 4.1, *the following statements hold*:

(i) *Given* $\bar{\xi}_1 > \xi_1$ *there exists* $\xi_0' \in (\xi_0, \bar{\xi}_1)$ *such that the problems* (1.1)′, (1.2a)′ *or* (1.1)′, (1.2b)′ *have positive solutions in the interval* $(\xi_0', \bar{\xi}_1)$.

(ii) *Given* $\bar{\xi}_0 \in (0, \xi_0)$ *there exists* $\xi_1' \in (\bar{\xi}_0, \xi_1)$ *such that problem* (1.1)′, (1.2c)′ *has a positive solution in the interval* $(\bar{\xi}_0, \xi_1')$.

Proof. Again we prove statement (i) for problem (1.1)′, (1.2a)′. The other statements are proved in a similar way.

For $b > 0$, let $u(\cdot, b)$ be the solution of (2.1) with ξ_1 as previously defined and let $\bar{u}(\cdot, b)$ be the solution of (2.1) with ξ_1 replaced by $\bar{\xi}_1$. Denote

$$a^* := u'(\xi_0, b^*) \qquad \text{(differentiation w.r.t. } \xi)$$

with b^* as in 4.2. By Proposition 2.4,

$$a^* \geqslant b^*. \tag{4.4}$$

Let $\bar{\xi}_0(b)$ be defined as in 2.1 and $\bar{\tau}(b)$ and $\bar{u}_m(b)$ as in 2.3 w.r.t. $\bar{u}(\cdot, b)$. By Proposition 2.5, $\lim_{c \to \infty} \bar{u}_m(c) = \infty$. Choose $c > 0$ such that

$$\bar{u}_m(c) > a^* \bar{\xi}_1. \tag{4.5}$$

We claim that

$$\bar{\xi}_0(c) \geqslant \xi_0. \tag{4.6}$$

Suppose that this is not the case. We define a function v in $(\xi_0(c), \bar{\xi}_1)$ as follows:

$$v(\xi) = \begin{cases} \bar{u}_m(\xi - \bar{\xi}_0(c))/(\bar{\tau} - \bar{\xi}_0(c)) & \text{in} \quad (\bar{\xi}_0(c), \bar{\tau}) \\ \bar{u}_m(\xi - \bar{\xi}_1)/(\bar{\tau} - \bar{\xi}_1) & \text{in} \quad (\bar{\tau}, \bar{\xi}_1) \end{cases} \tag{4.7}$$

where $\bar{u}_m = \bar{u}_m(c)$ and $\bar{\tau} = \bar{\tau}(c)$. Since $\bar{u}(\cdot, c)$ is concave, we have

$$\bar{u}(\xi, c) \geqslant v(\xi) \qquad \text{in} \quad (\xi_0(c), \bar{\xi}_1). \tag{4.8}$$

By (4.5) and the negation of (4.6)

$$v\left(\frac{\xi_0 + \xi_1}{2}\right) > a^* \left(\frac{\xi_1 - \xi_0}{2}\right).$$

Hence,

$$v(\xi) > v^*(\xi) \qquad \text{in} \quad (\xi_0, \xi_1), \tag{4.9}$$

where

$$v^*(\xi) := \begin{cases} a^*(\xi - \xi_0) & \text{in} \quad (\xi_0, (\xi_1 + \xi_0)/2) \\ a^*(\xi_1 - \xi) & \text{in} \quad ((\xi_1 + \xi_0)/2, \xi_1). \end{cases}$$

By (4.4) and the concavity of $u(\cdot, b^*)$,

$$u(\xi, b^*) < v^*(\xi) \qquad \text{in} \quad (\xi_0, \xi_1). \tag{4.10}$$

Now, by (4.8)–(4.10),

$$\bar{u}(\xi, c) > u(\cdot, b^*) \qquad \text{in} \quad (\xi_0, \xi_1).$$

But, by Lemma 4.3(i) this is impossible. Therefore (4.6) holds and statement (i) is proved for the problem (1.1)′, (1.2a)′.

5. Uniqueness Results: Part I

It is known that under appropriate conditions on f, R_0, R_1 problem (1.1), (1.2a) has at most one positive radially symmetric solution (see [C1], [Ni], [NN]). For instance, this is the case if f satisfies condition (A–1)′ (see 4.1) and $R_0/R_1 \leqslant (N-1)^{1/(N-2)}$ (see [NN, Theorem 1.7]). It can be shown that a similar result holds for problem (1.1), (1.2c). A proof will be given below.

On the other hand, it is known (see [NN, Theorem 1.10]) that for functions f of the form $f(t) = t^p + \varepsilon t^q$ $(1 < p < (N+2)/(N-2) < q < \infty$, $0 < \varepsilon)$ problem (1.1), (1.2a) has at least three positive radially symmetric solutions, provided that (for R_0 fixed) ε and R_1 are sufficiently small. It is easy to see that the proof of [NN] yields also the same result for problem (1.1), (1.2c).

These observations raise the question of a possible relation between uniqueness for problem (1.1), (1.2a) and (1.1), (1.2c). This relation will be discussed in Section 6.

In contrast to the above, it can be shown that, if f satisfies condition (A–1)′, problem (1.1), (1.2b) possesses at most one positive radially symmetric solution, for all R_0, R_1. This uniqueness result was established in [M]. We supply here a different proof which yields additional information on the solutions (see Theorem 5.1 and Lemmas 5.2–5.4).

5.1. Theorem. *Suppose that f satisfies the condition*

$$\text{(A–1)}'' \quad f'(t) \geqslant g(t) := f(t)/t, \; 0 < t < \infty,$$

and the inequality is strict for a sequence $t_n \to 0^+$. Then problem (1.1), (1.2b) *possesses at most one positive, radially symmetric solution. Further, suppose* *that* u^* (*resp.* u^{**}) *is a solution of this type in the annulus* $R_1 < |x| < R^*$ (*resp.* $R_1 < |x| < R^{**}$). *If* $R^* < R^{**}$, *then* $u^{**} < u^*$ *everywhere in* $R_1 < |x| < R^*$.

Remark. Note that (A–1)′ implies (A–1)″.

The proof of the theorem is based on several lemmas.

5.2. LEMMA. *Let* $\{u(\cdot, b): b > 0\}$ *be the family of solutions of* (2.1). *Denote* $w := \partial u/\partial b$. *Then for every* $b > 0$,

$$w(\xi, b) > 0 \qquad \text{for all } \xi \in [\tau(b), \xi_1), \tag{5.1}$$

with $\tau(b)$ *as in* 2.3.

Proof. The function w satisfies the conditions

$$w(\xi_1, b) = 0, \qquad w'(\xi_1, b) = -1. \tag{5.2}$$

Therefore $w(\cdot, b)$ is positive in some left hand neighborhood of ξ_1. If $w(\cdot, b) > 0$ in the whole interval $(\xi_0(b), \xi_1)$ there is nothing more to prove. Otherwise, let $\alpha = \alpha(b)$ denote the largest value in $(\xi_0(b), \xi_1)$ for which $w(\cdot, b)$ vanishes. We have to show that $\alpha(b) < \tau(b)$.

Note that $w(\cdot, b)$ satisfies,

$$\begin{aligned} &w'' + \rho f'(u)\, w = 0 && \text{in} \quad (\alpha, \xi_1). \\ &w(\alpha) = w(\xi_1) = 0 \text{ and } w > 0 && \text{in} \quad (\alpha, \xi_1). \end{aligned} \tag{5.3}$$

for $\alpha = \alpha(b)$. Furthermore, if $v = -(du/d\xi)(\cdot, b)$,

$$\begin{aligned} &v'' + \rho f'(u)\, v = \rho' f(u) < 0 && \text{in} \quad (\tau, \xi_1), \\ &v > 0 && \text{in} \quad (\tau, \xi_1], \end{aligned} \tag{5.4}$$

for $\tau = \tau(b)$. Now if $\tau(b) \leqslant \alpha(b)$, (5.3) and (5.4) lead to a contradiction.

5.3. LEMMA. *Assume* (A–1)″. *Let* $w(\cdot, b)$ *be defined as in* 5.2 (*with* $b > 0$). *Then there exists a point* $\gamma(b)$ *in* $(\tau(b), \xi_1)$ *such that*

$$w'(\xi, b) \begin{cases} < 0 & \text{in} \quad (\gamma(b), \xi_1) \\ = 0 & \text{for} \quad \xi = \gamma(b) \\ > 0 & \text{in} \quad [\tau(b), \gamma(b)). \end{cases} \tag{5.5}$$

Proof. Since w satisfies (5.3) it follows that w is concave in $[\tau(b), \xi_1]$. Therefore it is sufficient to show that $w'(\tau(b), b) > 0$.

In view of (A–1)″, (5.1) and (5.3), $w = w(\cdot, b)$ satisfies

$$\begin{aligned} &w'' + \rho g(u)\, w \leqslant 0 && \text{in} \quad (\tau(b), \xi_1), \\ &w(\xi_1) = 0 \text{ and } w > 0 && \text{in} \quad (\tau(b), \xi_1), \end{aligned} \tag{5.6}$$

and the first inequality is strict at some points of $(\tau(b), \xi_1)$. On the other hand, $u = u(\cdot, b)$ satisfies

$$\begin{aligned} &u'' + \rho g(u)\, u = 0 && \text{in} \quad (\tau(b), \xi_1), \\ &u(\xi_1) = 0,\ u'(\tau(b)) = 0,\ u > 0 && \text{in} \quad [\tau(b), \xi_1). \end{aligned} \tag{5.7}$$

Now (5.6) and (5.7) imply that

$$0 = (uw' - u'w)(\xi_1) > (uw' - u'w)(\tau(b)) = (uw')(\tau(b))$$

and hence $w'(\tau(b), b) > 0$.

5.4. Lemma. *Let $\tau(b)$ and $u_m(b)$ be as in 2.3. Then,*

$$\tau'(b) > 0 \qquad \textit{in} \quad (0, \infty) \tag{5.8}$$

and

$$u_m'(b) > 0 \qquad \textit{in} \quad (0, \infty) \tag{5.9}$$

Proof. Since $u'(\tau(b), b)$ is identically zero in $(0, \infty)$ we have

$$u''(\tau(b), b)\, \tau'(b) + w'(\tau(b), b) = 0, \qquad \forall b > 0.$$

Furthermore, $u''(\tau(b), b) < 0$ and (by Lemma 5.3) $w'(\tau(b), b) > 0$ for every $b > 0$. This implies (5.8). Next we have

$$\frac{du_m(b)}{db} = \frac{d}{db}\, u(\tau(b), b) = w(\tau(b), b) \qquad \forall b > 0.$$

In view of Lemma 5.2, this implies (5.9).

5.5. *Proof of Theorem* 5.1. By 2.3 and (5.8) the function $b \to \tau(b)$ is strictly monotone increasing in $(0, \infty)$. Therefore problem (1.1)′, (1.2b)′ possesses at most one positive solution. This implies the first statement of the theorem.

Now, consider the family of solutions $\{u(\cdot, b): b > 0\}$ and let b^*, b^{**} be the values of the parameter corresponding to the solutions u^*, u^{**} mentioned in the theorem. Thus, $\tau(b^*) = R^{*(2-N)}/(N-2)$ and $\tau(b^{**}) = R^{**(2-N)}/(N-2)$. Since $\tau(\cdot)$ is continuous in $(0, \infty)$ it follows that the range of this function contains the interval $[\tau(b^{**}), \tau(b^*)]$. By (5.8), $b^{**} < b^*$ and $\tau(b) < \tau(b^*)$ for every $b \in (0, b^*)$. Hence, by Lemma 5.2,

$$w(\xi, b) > 0 \qquad \text{for} \quad \forall b \in [b^{**}, b^*], \forall \xi \in [\tau(b^*), \xi_1).$$

Clearly, this implies the second statement of the theorem.

Next we discuss the uniqueness for thin annuli.

5.6. THEOREM. *Let f satisfy* (A–1)′. *Consider the problem*

$$\begin{aligned} &\Delta u + f(u) = 0 \ \textit{in} \ \tilde{R} < |x| < R_0 \\ &\frac{\partial u}{\partial r} = 0 \ \textit{on} \ |x| = \tilde{R} \qquad \textit{and} \qquad u = 0 \ \textit{on} \ |x| = R_0. \end{aligned} \tag{5.10}$$

If

$$\tilde{R} \geqslant \tfrac{1}{2}[(N-1)^{-1/(N-2)} + 1]\, R_0 \tag{5.11}$$

then problem (5.10) *possesses at most one positive radially symmetric solution.*

Before giving the proof of the theorem, we make some preliminary observations. In terms of the variable ξ introduced in Section 1, problem (5.10) can be rewritten in the form

$$\begin{aligned} &v''(\xi) + \rho(\xi)\, f(v(\xi)) = 0 \qquad \text{in} \quad \xi_0 < \xi < \tilde{\tau} \\ &v(\xi_0) = v'(\tilde{\tau}) = 0, \end{aligned} \tag{5.12}$$

where $\tilde{\tau} := [(N-2)\, \tilde{R}^{N-2}]^{-1}$ and $\xi_0 = [(N-2)\, R_0^{N-2}]^{-1}$. If v is a positive solution of (5.12), it can be extended to an interval $[\xi_0, \xi_1]$ so that the extension is a positive solution of the equation in (ξ_0, ξ_1) and $v(\xi_1) = 0$. (The extension will also be denoted by v.) Further, by [GNN, Theorem 2], $\tilde{R} < (R_0 + R_1)/2$, where $R_1 := [(N-2)\, \xi_1]^{-1/(N-2)}$. By the assumptions of the theorem, $\tilde{R} > R_0/2$ so that $R_1 > 2\tilde{R} - R_0 > 0$. Thus,

$$\xi_1 < [2\tilde{\tau}^{-1/(N-2)} - \xi_0^{-1/(N-2)}]^{2-N} =: A. \tag{5.13}$$

A simple computation shows that (5.11) implies

$$(\xi - \xi_0)\, \rho'(\xi) + 2\rho(\xi) > 0 \qquad \text{for} \quad \xi_0 \leqslant \xi < A, \tag{5.14}$$

with A as in (5.13).

We now prove the following lemma, which is an adaptation of a result of Ni and Nussbaum [NN, Theorem 2.4].

5.7. LEMMA. *Assume* (A–1)′. *Consider problem* (5.12) *and assume that* (*with* A *as in* (5.13)) $A>0$ *and* (5.14) *holds. Under these assumptions problem* (5.12) *possesses at most one positive solution.*

Remark. In view of our previous observations, Theorem 5.6 is an immediate consequence of the lemma.

Proof. Assume that (5.12) has a positive solution. Consider the family of solutions of problem (3.1), $\{v(\cdot, a): a>0\}$ and put $z := \partial v/\partial a$. Let a^* be such that $v(\cdot, a^*)$ solves (5.12) (i.e., $\tilde{\tau} = \tau_0(a^*)$) and let $\xi_1 = \xi_1(a^*)$. By (A–1)′ and Sturm's comparison theorem $z(\xi, a^*)$ vanishes at least once in (ξ_0, ξ_1). Let γ be its first zero greater than ξ_0. Put $y = (\xi - \xi_0)\, v'$. Then

$$\frac{d}{d\xi}(y'z - yz') = -[(\xi - \xi_0)\, \rho'(\xi) + 2\rho(\xi)]\, zf(v). \tag{5.15}$$

By (5.13) the right-hand side of (5.15) is negative in (ξ_0, γ). Since $z(\xi_0) = z(\gamma) = 0$, integration of (5.15) from ξ_0 to γ yields

$$-(\gamma - \xi_0)\, z'(\gamma)\, v'(\gamma) < 0,$$

Further, since $z'(\gamma) < 0$, it follows that $v'(\gamma) < 0$ and hence,

$$\gamma > \tilde{\tau} \tag{5.16}$$

Let $\omega \in (\xi_0, \gamma)$ be such that $z'(\omega) = 0$. We claim that $\omega < \tilde{\tau}$. Indeed, using (A–1)′ we obtain

$$0 > \int_{\xi_0}^{\omega} (f(v) - vf'(v))\, \rho z = \int_{\xi_0}^{\omega} (z''v - v''z)\, d\xi = -v'(\omega)\, z(\omega)$$

which implies that $v'(\omega) > 0$. Thus $\omega < \tilde{\tau}$ and hence, in view of (5.16),

$$z'(\tau_0(a^*), a^*) < 0. \tag{5.17}$$

Since $v'(\tau_0(a), a)$ is identically zero,

$$\begin{aligned} 0 &= v''(\tau_0(a), a)\, \tau_0'(a) + z'(\tau_0(a), a) \\ &= -\rho(\tau_0(a))\, f(v(\tau_0(a), a))\, \tau_0'(a) + z'(\tau_0(a), a), \end{aligned}$$

which together with (5.17) shows that $\tau_0'(a^*) < 0$.

Now let D be as in 3.2 and denote $D(\tilde{\tau}) := \{a \in D: \tau_0(a) = \tilde{\tau}\}$. By the previous part of the proof $\tau_0'(a) < 0$ everywhere in $D(\tilde{\tau})$. Consequently, every connected component of D contains at most one point of $D(\tilde{\tau})$. (Otherwise it would follow that there exists $\tilde{a} \in D(\tau)$ with $\tau_0'(\tilde{a}) \geq 0$.) Furthermore, if J is a *bounded*, connected component of D, then

$J \cap D(\tilde{\tau}) = \varnothing$. (Otherwise, in view of Lemma 3.4, it would follow that J contains at least two points of $D(\tilde{\tau})$.) Therefore $D(\tilde{\tau})$ contains at most one point, i.e., (5.12) possesses at most one positive solution.

6. Uniqueness Results: Part II

The observations mentioned at the beginning of Section 5 and some heuristic arguments lead us to the following conjecture. Under appropriate conditions on f (e.g., (1.6)) uniqueness of positive, radially symmetric solutions for problem (1.1), (1.2a) implies uniqueness in the same sense for (1.1), (1.2c) and vice-versa.

At present we are able to establish only the first half of this conjecture. Its proof is the subject of this section.

6.1. Under assumption (1.6) it is known (see [N]) that problem (1.1)′, (1.2a)′ possesses a positive, variational solution, i.e., a solution which also solves a certain variational problem which is described below. (Without loss of generality we may and shall assume that f is an odd function.)

Consider the functional,

$$H[v] = \int_{\xi_0}^{\xi_1} \left(\tfrac{1}{2} v f(v) - F(v)\right) \rho \, d\xi \tag{6.1}$$

where $0 < \xi_0 < \xi_1 < \infty$, and F is defined as in 2.3. Let K denote the set $\{v\}$ of absolutely continuous functions in $[\xi_0, \xi_1]$ such that $v(\xi_0) = v(\xi_1) = 0$, v is not identically zero and

$$\int_{\xi_0}^{\xi_1} v'^2 \, d\xi = \int_{\xi_0}^{\xi_1} v f(v) \, \rho \, d\xi. \tag{6.2}$$

The variational problem referred to above is the problem of minimizing H over K. Nehari has shown that this variational problem has a positive solution which must also satisfy

$$\begin{aligned} &v'' + \rho(\xi) f(v) = 0 \qquad \text{in} \quad (\xi_0, \xi_1) \\ &v(\xi_0) = v(\xi_1) = 0. \end{aligned} \tag{6.3}$$

6.2. Consider the eigenvalue problem

$$\begin{aligned} &\phi'' + \lambda \rho f'(v) \phi = 0 \qquad \text{in} \quad (\xi_0, \xi_1) \\ &\phi(\xi_0) = \phi(\xi_1) = 0 \end{aligned} \tag{6.4}$$

where v is a positive solution of (6.3). Denote by $\lambda_k = \lambda_k(v; \xi_0, \xi_1)$ the kth eigenvalue of (6.4). Later on we shall prove,

6.3. LEMMA. *Assume* (A–1)″ (*as in* 5.1). *If v is a variational solution of* (6.3) *then*

$$\lambda_2(v; \xi_0, \xi_1) \geq 1. \tag{6.5}$$

6.4. Consider problem (3.1) with ξ_0 a fixed positive number. In the sequel we shall use the notation of Section 3.

Suppose that D is not empty and that $\tau_0'(a) < 0$ for every $a \in D$. Then, in view of 3.4, D must be a half line (a', ∞). Therefore the strict monotonicity of $\tau_0(\cdot)$ implies that problem

$$\begin{aligned} &v'' + \rho f(v) = 0 \qquad \text{in} \quad (\xi_0, \beta) \\ &v(\xi_0 = v'(\beta) = 0 \end{aligned} \tag{6.6}$$

has at most one positive solution for every $\beta > \xi_0$. Thus nonuniqueness is possible only if $\tau_0'(a) \geq 0$ for some $a \in D$. We shall prove

6.5. LEMMA. *Assume* (A–1)″. *Suppose that for some $\tilde{a} \in D$, $\tau_0'(\tilde{a}) \geq 0$. Consider* (6.4) *with $\xi_1 = \xi_1(\tilde{a})$ and $v = v(\cdot, \tilde{a})$. Then*

$$\lambda_2(v; \xi_0, \xi_1) < 1. \tag{6.7}$$

Before we turn to the proof of 6.3, 6.5, let us note that as a consequence of these two lemmas we obtain the result mentioned at the beginning of the section, namely,

6.6. THEOREM. *Suppose that f satisfies* (1.6). *If problem* (1.1)′, (1.2a)′ *has an unique positive solution for each $\xi_0, \xi_1 \in (0, \infty)$, then problem* (1.1)′, (1.2c)′ *has the same uniqueness property.*

Proof. By Nehari's result, the unique positive solution of (6.3) is a variational solution. Therefore (6.5) holds for every positive solution v of (6.3). On the other hand, if there exists an interval (ξ_0, β) for which (6.6) has more than one positive solution, then by 6.4 the assumptions of Lemma 6.5 hold and we obtain (6.7) in contradiction to (6.5).

6.7. *Proof of Lemma* 6.3. Let $\lambda_i = \lambda_i(v; \xi_0, \xi_1)$ and let ϕ_1, ϕ_2 be corresponding eigenfunctions with $\phi_1 > 0$. Denote: $P(\xi) := f'(v(\xi))\, \rho(\xi)$. Using (6.4) with λ_1, ϕ_1 and λ_2, ϕ_2 we obtain

$$\begin{aligned} &\int_{\xi_0}^{\xi_1} \phi_i'^2 \, d\xi = \lambda_i \int_{\xi_0}^{\xi_1} \phi_i^2 P d\xi, \\ &\int_{\xi_0}^{\xi_1} \phi_1' \phi_2' \, d\xi = \lambda_i \int_{\xi_0}^{\xi_1} \phi_1 \phi_2 P d\xi, \qquad \text{for} \quad i = 1, 2. \end{aligned} \tag{6.8}$$

Hence,

$$\int_{\xi_0}^{\xi_1} \phi_1 \phi_2 P d\xi = \int_{\xi_0}^{\xi_1} \phi_1' \phi_2' \, d\xi = 0. \tag{6.9}$$

Therefore if ψ is a linear combination of ϕ_1, ϕ_2 it will satisfy

$$\int_{\xi_0}^{\xi_1} \psi'^2 \, d\xi \leqslant \lambda_2 \int_{\xi_0}^{\xi_1} \psi^2 P d\xi. \tag{6.10}$$

Next we observe that for every sufficiently small $t \geqslant 0$, there exists $\delta(t)$ (which depends continuously on t and satisfies $\delta(0) = 0$) such that the function

$$\psi_t = \delta(t)\, \phi_1 + t\phi_2 \tag{6.11}$$

satisfies

$$\int_{\xi_0}^{\xi_1} (\psi_t' + v')^2 \, d\xi = \int_{\xi_0}^{\xi_1} (\psi_t + v)\, f(\psi_t + v)\, \rho d\xi. \tag{6.12}$$

To verify this statement set

$$F(\delta, t) := \int_{\xi_0}^{\xi_1} (V'^2 - Vf(V)\, \rho)\, d\xi$$

where $V = \delta\phi_1 + t\phi_2 + v$. Note that, by (6.3), $F(0, 0) = 0$. Further,

$$\frac{\partial F}{\partial \delta}(0, 0) = \int_{\xi_0}^{\xi_1} [2v'\phi_1' - (f(v) + vf'(v))\, \phi_1 \rho]\, d\xi.$$

Again by (6.3),

$$\int_{\xi_0}^{\xi_1} (v'\phi_i' - f(v)\, \phi_i \rho)\, d\xi = 0, \qquad i = 1, 2. \tag{6.13}$$

Hence,

$$\frac{\partial F}{\partial \delta}(0, 0) = \int_{\xi_0}^{\xi_1} (f(v) - vf'(v))\, \phi_1 \rho d\xi < 0,$$

by the assumption on f. Thus our statement follows by the implicit function theorem.

Next we note that from (6.1), (6.2), (6.12)

$$H[\psi_t + v] = H[v] + \int_{\xi_0}^{\xi_1} (\tfrac{1}{2}\psi_t'^2 + \psi_t' v')\, d\xi$$
$$- \int_{\xi_0}^{\xi_1} (f(v)\, \psi_t + \tfrac{1}{2} f'(v)\, \psi_t^2)\, \rho d\xi + O(t^3).$$

Hence, by (6.13) and (6.10), (recall that $P(\xi) = f'(v(\xi))\, \rho(\xi)$)

$$H[\psi_t + v] - H[v] = \frac{1}{2} \int_{\xi_0}^{\xi_1} (\psi_t'^2 - f'(v)\, \psi_t^2)\, \rho d\xi + O(t^3)$$
$$< \frac{1}{2} (\lambda_2 - 1) \int_{\xi_0}^{\xi_1} \psi_t^2 P d\xi + O(t^3).$$

Since, by (6.12), $\psi_t + v \in K$ and v is a variational solution, we have $H[v] \leqslant H[\psi_t + v]$. Therefore, P being positive, we deduce that $\lambda_2 \geqslant 1$ as claimed.

6.8. *Proof of Lemma* 6.5. Denote $z(\cdot, a) := (\partial v/\partial a)(\cdot, a)$. Then $z(\cdot, a)$ satisfies

$$\begin{aligned} &z'' + \rho f'(v)\, z = 0 \qquad \text{in} \quad (\xi_0, \xi_1(a)) \\ &z(\xi_0) = 0,\ z'(\xi_0) = 1 \end{aligned} \tag{6.14}$$

where $v = v(\cdot, a)$. In what follows we set $a = \tilde{a}$, $v(\xi) = v(\xi, \tilde{a})$, $z(\xi) = z(\xi, \tilde{a})$, $\xi_1 = \xi_1(\tilde{a})$ and $\tau = \tau_0(\tilde{a})$. We also denote

$$\begin{aligned} &p(\xi) := \rho(\xi)\, g(v(\xi)), \qquad \text{where } g(t) = f(t)/t \\ &P(\xi) := \rho(\xi)\, f'(v(\xi)). \end{aligned} \tag{6.15}$$

By our assumption on f, $P(\xi) \geqslant p(\xi)$ with strict inequality at some points in every neighborhood of ξ_0 and ξ_1.

Note that

$$z'(\tau) \geqslant 0. \tag{6.16}$$

Indeed $v'(\tau_0(a), a) = 0$ for all $a \in D$. Hence,

$$v''(\tau_0(a), a)\, \tau_0'(a) + z'(\tau_0(a), a) = 0, \qquad \forall a \in D.$$

Since $\tau_0'(\tilde{a}) \geqslant 0$ and $v'' < 0$ everywhere in $(\xi_0, \xi_1(a))$, we obtain (6.16).

We claim that z must change sign at least once in (ξ_0, τ). Suppose this is not the case. Then $z > 0$ in (ξ_0, τ) so that

$$\begin{aligned} &z'' + pz \leqslant 0 \quad \text{in} \quad (\xi_0, \tau) \\ &z(\xi_0) = 0 \quad \text{and} \quad z'(\tau) \geqslant 0, \end{aligned} \tag{6.17$_1$}$$

and the first inequality is strict in some subinterval of (ξ_0, τ).

On the other hand,

$$\begin{aligned} &v'' + pv = 0 \quad \text{in} \quad (\xi_0, \tau) \\ &v(\xi_0) = v'(\tau) = 0. \end{aligned} \tag{6.17$_2$}$$

Since both v and z are positive in (ξ_0, τ), $(6.17)_1$ and $(6.17)_2$ lead to a contradiction.

Next we observe that z' must vanish at least twice in $(\xi_0, \tau]$. This follows from the previous statement and the fact that $z'(\xi_0) = 1$, $z'(\tau) \geqslant 0$.

Now we claim that z must vanish at least twice in (ξ_0, ξ_1). Suppose that this is not the case. Then, in view of the previous statements, $z < 0$ and $z' > 0$ in (τ, ξ_1). Therefore $\tilde{z} = -z$ satisfies

$$\tilde{z}'' + P\tilde{z} = 0, \qquad \tilde{z} > 0 \text{ and } \tilde{z}' < 0 \text{ in } (\tau, \xi_1)$$

while

$$v'' + pv = 0 \text{ in } (\tau, \xi_1) \qquad \text{and} \qquad v'(\tau) = v(\xi_1) = 0.$$

In view of the relation between p and P this again leads to a contradiction.

Denote by $\bar{\xi}$ the second zero of z to the right of ξ_0. Then z is an eigenfunction of the problem

$$\begin{gathered} \psi'' + \lambda P \psi = 0 \quad \text{in} \quad (\xi_0, \bar{\xi}) \\ \psi(\xi_0) = \psi(\bar{\xi}) = 0 \end{gathered}$$

corresponding to the eigenvalue $\lambda = 1$ and this is the second eigenvalue of the problem. Since $\bar{\xi} < \xi_1$, the monotonicity of eigenvalues with respect to the domain implies (6.7). This completes the proof of the lemma.

References

[AR] A. Ambrosetti and P. H. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Funct. Anal.* **14** (1973), 349–381.

[BN] H. Brezis and L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, *Comm. Pure Appl. Math.* **34** (1983).

[BT] H. Brezis and E. L. Turner, On a class of superlinear elliptic problems, *Comm. Partial Differential Equations* **2** (1977), 601–614.

[C1] C. V. Coffman, On the positive solutions of boundary value problems for a class of nonlinear differential equations, *J. Differential Equations* **3** (1967), 92–111.

[C2] C. V. Coffman, A nonlinear boundary value problem with many positive solutions, *J. Differential Equations* **54** (1984), 429–437.

[CM] C. V. Coffman and M. Marcus, Superlinear elliptic Dirichlet problems in almost spherically symmetric domains, *Arch. Rat. Mech. Anal.* **96** (1986), 167–197.

[GNN] B. Gidas, W.-M. Ni, and L. Nirenberg, Symmetry and related properties via the maximum principle, *Comm. Math. Phys.* **68** (1979), 209–243.

[L] P. L. Lions, On the existence of positive solutions of semilinear elliptic equations, *SIAM Rev.* **24** (1982), 441–467.

[M] R. M. Moroney, Note on a theorem of Nehari, *Proc. Amer. Math. Soc.* **13** (1962), 407–410.

[N] Z. Nehari, On a class of second order differential equations, *Trans. Amer. Math. Soc.* **95** (1960), 101–123.

[Ni] W.-M. Ni, Uniqueness of solutions of nonlinear Dirichlet problems, *J. Differential Equations* **50** (1983), 289–304.

[NN] W.-M. Ni and R. Nussbaum, Uniqueness and non-uniqueness for positive radial solutions of $\Delta u + f(u, r) = 0$, *Comm. Pure Appl. Math.* **38** (1985), 67–108.

[P] S. I. Pohozaev, Eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$, *Soviet Math. Dokl.* **5** (1965), 1408–1411.