On the Matrix Riccati Approach to a Singly Perturbed Regulator Problem

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I. PROBLEM STATEMENT

Consider the time-invariant state regulator problem consisting of the linear state equations

\[ \begin{align*}
    \frac{dx}{dt} &= A_1(\varepsilon)x + A_2(\varepsilon)z + B_1(\varepsilon)u \\
    \frac{dz}{dt} &= A_3(\varepsilon)x + A_4(\varepsilon)z + B_3(\varepsilon)u
\end{align*} \]  

(1.1)
on 0 \leq t \leq 1 (or any other closed, bounded interval), the prescribed initial states

\[ x(0, \varepsilon) \quad \text{and} \quad z(0, \varepsilon), \]  

(1.2)

and the scalar cost functional

\[ J(\varepsilon) = \frac{1}{2} \begin{pmatrix} x(1, \varepsilon)' & z(1, \varepsilon) \end{pmatrix} \pi(\varepsilon) \begin{pmatrix} x(1, \varepsilon) \\
    z(1, \varepsilon) \end{pmatrix} \]

\[ + \frac{1}{2} \int_0^1 \left[ \begin{pmatrix} x(t, \varepsilon)' & z(t, \varepsilon) \end{pmatrix} Q(\varepsilon) \begin{pmatrix} x(t, \varepsilon) \\
    z(t, \varepsilon) \end{pmatrix} + u'(t, \varepsilon)u(t, \varepsilon) \right] dt \]  

(1.3)

which is to be minimized by selection of the control \( u(t, \varepsilon) \). Here \( x, z \), and \( u \) are vectors of dimension \( n, m \), and \( r \), respectively, the prime denotes trans-
position, $\pi$ and $Q$ are symmetric, nonnegative definite matrices having the block forms

$$Q = \begin{pmatrix} Q_1 & Q_2 \\ Q_2' & Q_3 \end{pmatrix} \quad \text{and} \quad \pi = \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix},$$

(1.4)

and $\epsilon$ is a small positive parameter.

We shall seek the asymptotic solution of the optimal control problem (1.1-1.4) as $\epsilon \to 0$ by the matrix Riccati approach. We note that the corresponding time-varying problem has been treated by directly applying the classical calculus of variations and solving the resulting two-point boundary value problem (cf., e.g., O'Malley and Kung [10]). The appropriate Riccati equation has also been previously solved asymptotically (cf., e.g., Yackel and Kokotović [12]) under different hypotheses. It should be observed that the Riccati approach has advantages of computational simplicity, closed loop interpretation, and rather direct extension to the infinite interval problem.

To proceed, let us assume that the matrices $A_i$, $B_i$, $x(0, \epsilon)$, $x(0, \epsilon)$, $\pi_i$, and $Q_i$ all have asymptotic series expansions as $\epsilon \to 0$, e.g.,

$$A_i(\epsilon) \sim \sum_{j=0}^{\infty} A_{ij} \epsilon^j.$$

We would like to obtain the asymptotic expansion of the unique solution under the four hypotheses:

(i) the matrix $A_{40}$ is invertible,

(ii) the eigenvalues of the matrix

$$G = \begin{pmatrix} A_{40} & -B_{30}B_{30}' \\ -Q_{30} & -A_{40}' \end{pmatrix}$$

(1.5)

all have nonzero real parts,

(iii) the matrices $T_{11}$ and $T_{22} - \pi_{20}T_{12}$ are both nonsingular where

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$$

is any nonsingular matrix such that

$$T^{-1}GT = \begin{pmatrix} -A & 0 \\ 0 & A \end{pmatrix}$$

(1.6)

with the eigenvalues of $A$ having positive real parts, and
(iv) the matrix

\[ 2 \equiv Q_{10} - Q_{30}A_{40}^{-1}A_{30} - A_{30}^{-1}A_{40}^{-1}Q_{30} \]

\[ + A_{30}^{-1}Q_{30}A_{40}^{-1}A_{30} - (Q_{20} - A_{40}^{-1}Q_{30}) \]

\[ \cdot A_{40}^{-1}B_{20}(I + B_{20}'A_{40}^{-1}Q_{30}A_{40}^{-1}B_{20})B_{20}'A_{40}^{-1}(Q_{20} - Q_{30}A_{40}^{-1}A_{30}) \]  \( (1.7) \)

is positive semi-definite.

These are the hypotheses of O'Malley and Kung [10]. To use the Riccati approach, we need an additional condition (cf. (2.19)) which will be implied by the simpler assumption that

(v) the matrix

\[ T_{21} - \pi_{30}T_{11} \]

is either nonsingular or identically zero.

The first four hypotheses are discussed by O'Malley and Kung [10]. We note, in particular, that the invertibility of \( A_{40} \) is not necessary (cf. O'Malley [8]) while assumption (iv) follows if either \( Q_{30}^{-1} \) exists, \( Q_{30} = 0 \), \( (B_{20}B_{20}')^{-1} \) exists, or \( B_{20} = 0 \). Hypotheses (v) could probably be eliminated through use of other arguments.

These assumptions seem weaker than the boundary layer controllability and observability assumptions of Yackel and Kokotović [12]. Considering only the special case where \( x, z, \) and \( u \) are scalars (see O'Malley [9]), we observe that boundary layer controllability then implies that \( T_{11} \) is invertible while boundary layer observability implies the existence of \( (T_{22} - \pi_{30}T_{12})^{-1} \). However, the uncontrollable scalar problem where \( B_{20} = 0 \) will have \( T_{12} \) invertible provided \( A_{40} \) is negative-definite, while the unobservable problem where \( Q_{30} = 0 \) will have \( T_{22} - \pi_{30}T_{12} \) invertible provided \( A_{40} \) is negative definite and \( \pi_{30} \) is positive definite. The results of Kucera [5] further indicate that controllability-observability assumptions might often be weakened.

From the familiar Riccati approach (cf. Kalman [4] and Anderson and Moore [2]) which is valid when \( \epsilon > 0 \), we expect the optimal control to be given by

\[ u = - \left( B_1' \frac{B_2'}{\epsilon} \right) k(t, \epsilon) \left( x(t, \epsilon) \right) \]

\( (1.8) \)

where the matrix \( k(t, \epsilon) \) is a symmetric, nonnegative definite solution of the matrix Riccati equation

\[ \frac{dk}{dt} = -k \left( \begin{array}{cc} A_1 & A_2 \\ \frac{A_3'}{\epsilon} & \frac{A_4'}{\epsilon} \end{array} \right) - \left( \begin{array}{cc} A_1' & A_3' \\ \frac{A_2'}{\epsilon} & \frac{A_4'}{\epsilon} \end{array} \right) k + k \left( \begin{array}{c} B_1' \\ \frac{B_2'}{\epsilon} \end{array} \right) \left( \begin{array}{c} B_1' \\ \frac{B_2'}{\epsilon} \end{array} \right) k - Q \]

\( (1.9) \)
and the terminal condition

\[ k(1, \varepsilon) = \pi(\varepsilon). \] (1.10)

We shall proceed to obtain the asymptotic solution for \( k(t, \varepsilon) \) as \( \varepsilon \to 0 \). We will then use this result to obtain asymptotic solutions for the optimal control and the corresponding trajectories.

We note that this singular perturbation problem has been of considerable interest in the recent literature. Its study was motivated by practical problems involving small “parasitics.” Readers unfamiliar with this background are urged to consult the booklet American Society of Mechanical Engineers [1], especially the introductory article by Kokotović. Further progress is reported by Sannuti [11] and by Wilde and Kokotović [13].

II. ASYMPTOTIC SOLUTION OF THE MATRIX RICCATI EQUATION

Because of the singular manner in which the parameter \( \varepsilon \) enters (1.9), we shall seek a solution \( k(t, \varepsilon) \) in the block form

\[
k(t, \varepsilon) = \begin{pmatrix} k_1(t, \varepsilon) & ek_2(t, \varepsilon) \\ ek_2'(t, \varepsilon) & k_3(t, \varepsilon) \end{pmatrix}. \tag{2.1}
\]

Then, (1.9–1.10) becomes equivalent to the singularly perturbed initial value problem

\[
\begin{align*}
dk_1/dt &= -k_1A_1 - A'_1k_1 - k_2A_2 - A'_2k_2 + k_3S_1k_1 + k_2S'k_1 \\
&\quad + k_1S_2k_2 + k_3S_2k_2 - Q_1, & k_1(1, \varepsilon) &= \pi_1(\varepsilon) \\
\varepsilon dk_2/dt &= -k_1A_2 - k_2A_4 - \varepsilon A'_1k_2 - A'_3k_3 + ek_1S_1k_2 + ek_2S'k_2 \\
&\quad + k_1S_2k_3 + k_3S_2k_3 - Q_2, & k_2(1, \varepsilon) &= \pi_2(\varepsilon) \\
\varepsilon dk_3/dt &= -ek_2'k_2 - ek_2'k_2 - A'_3k_3 + e^2k_2'S_1k_2 + ek_3S'k_2 \\
&\quad + ek_2'Sk_3 + k_3S_2k_3 - Q_3, & k_3(1, \varepsilon) &= \pi_3(\varepsilon)
\end{align*}
\tag{2.2}
\]

where \( S_1 = B_1B_1', \quad S = B_2B_2', \) and \( S_2 = B_2B_2' \).

We shall seek a solution of this nonlinear system by a boundary layer method (cf. O’Malley [7]). Specifically, we shall take the \( k_i \) to be of the form

\[
\begin{align*}
k_1(t, \varepsilon) &= K_1(t, \varepsilon) + \varepsilon l_1(\sigma, \varepsilon) \\
k_2(t, \varepsilon) &= K_2(t, \varepsilon) + l_2(\sigma, \varepsilon) \\
k_3(t, \varepsilon) &= K_3(t, \varepsilon) + l_3(\sigma, \varepsilon)
\end{align*}
\tag{2.3}
\]

where the outer solution \( (K_1, K_2, K_3) \) has an asymptotic series expansion which formally satisfies (2.2) and the boundary layer correction \( (l_1, l_2, l_3) \) has an asymptotic series expansion whose terms tend to zero as \( \sigma = (1 - t)/\varepsilon \) tends to infinity.
The limiting outer solution \((K_{10}, K_{20}, K_{30})\), then, satisfies the reduced system:

\[
dK_{10}/dt = -K_{10}A_{10} - A'_{10}K_{10} - K_{20}A_{20} - A'_{20}K_{20} + K_{10}S_{10}K_{10} + K_{20}S_{20}K_{20} - Q_{10}, \quad K_{10}(1) = \pi_{10}
\]

\[
0 = -K_{10}A_{20} - K_{20}A_{40} - A'_{20}K_{20} + K_{10}S_{10}K_{30} + K_{20}S_{20}K_{30} - Q_{20},
\]

\[
0 = -K_{30}A_{40} - A'_{40}K_{30} + K_{30}S_{20}K_{30} - Q_{30}, \quad (2.4)
\]

The last equation has, as its only symmetric, positive semi-definite solution,

\[
K_{30} = T_{21}T_{11}^{-1}
\]

(cf. hypothesis (iii), Anderson and Moore [2], and Martensson [6] where a proof that \(K_{30}\) is symmetric and positive semi-definite is given). Note that the second equation can be rewritten as

\[
G_{11} = -A_{10} - A'_{10}K_{10} + S_{10}K_{20} + (4 & A_{10} + Q_{10})
\]

where

\[
K_{20} = K_{10}E + (A'_{20}K_{20} + Q_{20}) A^{-1}
\]

(2.7)

(2.5)

where

\[
E = (A_{20} - S_{20}K_{30}) A^{-1}
\]

Note that we could substitute for \(K_{20}\) in the first equation of (2.4) to obtain a matrix Riccati equation for \(K_{10}\). We instead follow the indirect argument of Haddad and Kokotović (1971) who showed that

\[
\dot{K}_{10} = -K_{10}A - A'K_{10} - Q + (K_{10}B - C) R^{-1}(B'K_{10} - C')
\]

where

\[
A = A_{10} - A_{20}A_{40}^{-1}A_{30},
\]

\[
B = B_{10} - A_{20}A_{40}^{-1}B_{30},
\]

\[
C = (Q_{20} - A'_{20}A_{40}^{-1}Q_{30}) A_{40}^{-1}B_{20},
\]

\[
Q = Q_{10} - Q_{20}A_{40}^{-1}A_{30} - A'_{20}A_{40}^{-1}Q_{30} + A_{20}A_{40}^{-1}Q_{30}A_{40}^{-1}A_{30},
\]

and

\[
R = I + B'_{20}A_{40}^{-1}Q_{30}A_{40}^{-1}B_{20}.
\]

Thus, we have the terminal value problem

\[
\dot{K}_{10} = -K_{10}a - a'K_{10} + K_{10}bK_{10} - 2, \quad K_{10}(1) = \pi_{10} \quad (2.8)
\]
where
\[ \alpha = \bar{A} + \bar{B} \bar{R}^{-1} \bar{C}', \]
\[ \delta = \bar{B} \bar{R}^{-1} \bar{B}', \]
and
\[ \mathcal{Q} = Q - CR^{-1}C'. \]

This Riccati equation, however, has a unique symmetric positive semi-definite solution on \( 0 \leq t \leq 1 \) since \( \mathcal{Q} \), by assumption, is positive semi-definite (cf., e.g., Anderson and Moore [2]).

Since the outer solution \( (K_1, K_2, K_3) \) satisfies the system (2.2), its higher order terms will satisfy linear equations obtained from equating higher order coefficients in (2.2). Specifically, the coefficients of \( \epsilon \) imply the system

\[
\frac{dK_{11}}{dt} = -K_{11}A_{10} - A'_{10}K_{11} - K_{21}A_{20} - A'_{20}K_{21}' + K_{11}S_{10}K_{10} + K_{20}S_{20}K_{20}' + K_{31}S_{30}K_{30} + K_{31}S_{31}K_{31}' + \alpha_0
\]

\[
0 = -K_{11}A_{20} - K_{21}A_{30} - A'_{20}K_{31} + K_{11}S_{20}K_{20} + K_{10}S_{10}K_{11} + K_{21}S_{20}K_{21} + \beta_0
\]

\[
0 = -K_{31}A_{40} - A'_{20}K_{31} + K_{31}S_{20}K_{30} + K_{30}S_{20}K_{31} + \gamma_0
\]

where \( \alpha_0, \beta_0, \) and \( \gamma_0 \) are determined by the \( K_{10}'s \). First note that the equation

\[ K_{31} \bar{A} + \bar{A} K_{31} = -\gamma_0 \]

has the unique symmetric solution

\[ K_{31} = \int_0^\infty e^{-\bar{A}s} \gamma_0 e^{-\bar{A}s} ds \]  

(2.10)

since \( \bar{A} \) is positive-definite. Moreover,

\[ K_{21} = K_{11}E + FK_{31}\bar{A}^{-1} - \beta_0\bar{A}^{-1} \]

(2.11)

where

\[ F = A'_{20} - K_{10}S_{10} - K_{20}S_{20}. \]

Substituting into the first equation, then, implies the linear equation

\[ \frac{dK_{11}}{dt} = K_{11}G + G'K_{11} + H \]

(2.12)

where

\[ G = -A_{10} + S_{10}K_{10} + S_0K_{20}' + EF \]
and
\[ H = -(\beta F_{y1} - \beta_0) \tilde{A}^{-1} F - F' \tilde{A}'^{-1} (\beta_0' - K_{01} F') + \alpha_0 \]

We therefore obtain a unique symmetric determination of \( K_{11} \) upon specifying its terminal value. Analogously, higher order terms (\( K_{1j}, K_{2j}, K_{3j} \)) of the outer solution can be uniquely obtained successively up to specification of the terminal values \( K_{1j}(1) \). However, (1.10) and (2.3) imply that \( K_{1j}(1, \epsilon) = \pi_1(\epsilon) - \epsilon l_{1j}(0, \epsilon) \), so the value
\[ K_{1j}(1) = \pi_{1j} - l_{1j,1}(0) \]  

(2.13)
is determined successively by the lower order term \( l_{1j,1} \) of the boundary layer correction.

Since the outer solution satisfies the system (2.2), the representation (2.3) implies that the boundary layer correction (\( l_1, l_2, l_3 \)) must satisfy the nonlinear system

\[
\frac{dl_1}{d\sigma} = (dK_1/dt) - (dl_1/dt) = \epsilon(l_1 A_1 + A_1' l_1') + (l_2 A_2 + A_2' l_2')
\]

\[
- [K_1(1 - \epsilon\sigma, \epsilon)(S l_2' + \epsilon S l_2') + (l_2 S' + \epsilon l_1 S_1) K_1(1 - \epsilon\sigma, \epsilon)]
\]

\[
- e^2 l_1 S l_1' = (l_2 S' l_1' + l_1 S l_2') - l_2 S l_2'
\]

\[
\frac{dl_2}{d\sigma} = \epsilon(dK_2/dt) - \epsilon(dl_2/dt) = (l_2 A_2 + A_2' l_2') + (l_1 A_1 + A_1' l_1')
\]

\[
- [K_2(1 - \epsilon\sigma, \epsilon)(S l_3 + \epsilon S l_3') + (l_3 S + \epsilon l_2 S_2) K_2(1 - \epsilon\sigma, \epsilon)]
\]

\[
- (l_3 S' + \epsilon l_2 S_2) K_2(1 - \epsilon\sigma, \epsilon) - (l_2 S_2 + \epsilon l_1 S_1) K_2(1 - \epsilon\sigma, \epsilon)
\]

\[
- l_3 S l_3' = (l_2 S' l_3' + l_1 S l_3) + \epsilon l_1 S l_3
\]

\[
\frac{dl_3}{d\sigma} = \epsilon(dK_3/dt) - \epsilon(dl_3/dt) = (l_3 A_3 + A_3' l_3') + (l_2 A_2 + A_2' l_2')
\]

\[
- [K_3(1 - \epsilon\sigma, \epsilon)(S l_4 + \epsilon S l_4') + (l_4 S + \epsilon l_3 S_3) K_3(1 - \epsilon\sigma, \epsilon)]
\]

\[
- (l_4 S' + \epsilon l_3 S_3) K_3(1 - \epsilon\sigma, \epsilon) - (l_3 S_3 + \epsilon l_2 S_2) K_3(1 - \epsilon\sigma, \epsilon)
\]

\[
- l_4 S l_4' = (l_3 S' l_4' + l_2 S l_4) + \epsilon l_2 S l_4
\]

(2.14)
The terms of the asymptotic expansion for the \( l_i \)'s will be obtained by successively equating coefficients of like powers of \( \epsilon \) in this system. Thus, when \( \epsilon = 0 \), we have

\[
\frac{dl_{10}}{d\sigma} = l_{20} A_{30} + A_{30}' l_{20}' - K_{10}(1) S_{0'} l_{20}' - l_{20} S_{0'} K_{10}(1) - K_{20}(1) S_{20} l_{20}'
\]

\[
- l_{20} S_{20} K_{20}'(1) - l_{20} S_{20} l_{20}'
\]

\[
= l_{20} F' + F l_{30}' - l_{20} S_{20} l_{20}'
\]

\[
\frac{dl_{20}}{d\sigma} = l_{20} A_{40} + A_{40}' l_{30}' - K_{10}(1) S_{0'} l_{30}' - K_{20}(1) S_{20} l_{30}' - l_{20} S_{20} K_{30}
\]

\[
- l_{30} S_{30}' l_{30}'
\]

\[
= - l_{20} (\tilde{A} + S_{20} l_{30}) + F l_{30}
\]

\[
\frac{dl_{30}}{d\sigma} = l_{30} A_{40} + A_{40}' l_{30}' - K_{10}(1) S_{0'} l_{30}' - K_{20}(1) S_{20} l_{30}' - l_{30} S_{20} K_{30}
\]

\[
- l_{30} S_{30}' l_{30}'
\]

\[
= - l_{30} (\tilde{A} + S_{20} l_{30}) + F l_{30}
\]

(2.15)
Further, (2.3) implies the initial values
\[ l_{20}(0) = \pi_2(0) - K_{20}(1) \]
\[ l_{30}(0) = \pi_3(0) - K_{30} = \pi_{30} - T_{21} T_{11}^{-1} = -(T_{21} - \pi_{30} T_{11}) T_{11}^{-1} \]  \hfill (2.16)
while \( l_{10}(0) \) will (presumably) be uniquely determined by the condition that \( l_{10} \to 0 \) as \( \sigma \to \infty \).

The Riccati equation for \( l_{30} \) has the decaying solution
\[ l_{30} (\sigma) = e^{-\alpha \sigma} l_{30}(0) [I + \int (\sigma) l_{30}(0)]^{-1} e^{-\sigma} \]  \hfill (2.17)
where
\[ \int (\sigma) = \int_0^\sigma e^{-\alpha r} S_{20} e^{-\alpha r} dr \]
provided the matrix
\[ l + \int (\sigma) l_{30}(0) \]  \hfill (2.18)
is nonsingular for all \( \sigma \geq 0 \). To relate this condition to our previous hypotheses, we, instead, integrate the equation for \( l_{30} \) less directly. We first note that
\[ r(\nu) = l_{30}(\nu) + K_{30} \]
satisfies the initial value problem
\[ dr/d\sigma = r A_{30} + A_{20} r - r S_{20} r + Q_{30}, \quad r(0) = \pi_{30} \).

We can obtain an explicit solution \( r(\sigma) \) under hypothesis (ii) (cf. equation (15.2-19) of Anderson and Moore [2]). Specifically, we obtain
\[ l_{30}(\sigma)(T_{11} - T_{21} e^{-\alpha \sigma} L e^{-\alpha \sigma}) e^{\alpha \sigma} = (T_{22} - T_{21} T_{11}^{-1} T_{12}) e^{-\alpha \sigma} L \]
where
\[ L = -(T_{22} - \pi_{30} T_{12})^{-1}(T_{21} - \pi_{30} T_{11}) \).

Using hypothesis (v) suppose first that \( T_{21} - \pi_{30} T_{11} \) is nonsingular. Then \( L^{-1} \) exists and, since \( T \) is nonsingular,
\[ T_{11} + T_{12} e^{-\alpha \sigma} L e^{-\alpha \sigma} \]  \hfill (2.19)
is nonsingular for all \( \sigma \geq 0 \). Otherwise, if \( T_{21} = \pi_{30} T_{11}, L = l_{30}(\nu) = 0 \). In either case, we have the decaying solution
\[ l_{30}(\sigma) = (T_{22} - T_{21} T_{11}^{-1} T_{12}) e^{-\alpha \sigma} L e^{-\alpha \sigma} (T_{11} + T_{12} e^{-\alpha \sigma} L e^{-\alpha \sigma})^{-1}. \]  \hfill (2.20)
The invertibility of (2.19) and the existence of \( l_{30}(\sigma) \) will, no doubt, follow under assumptions other than (v).
Knowing $l_{20}$, we can integrate the linear equation for $l_{20}$ to obtain the exponentially decaying solution
\[
l_{20}(\sigma) = \left[ \pi_{20}(0) - K_{20}(1) \right] e^{-\tilde{\alpha}_{20}(\sigma) - s_{20}(\sigma) \sigma} d\sigma
\]
\[
- F \int_{0}^{\sigma} l_{30}(s) e^{-\tilde{\alpha}_{20}(\sigma - s) - s_{20}(\sigma) \sigma} d\sigma.
\]  
(2.21)
The only decaying solution for $l_{10}$ is given by
\[
l_{10}(\sigma) = - \int_{\sigma}^{\infty} \frac{dl_{10}(s)}{d\sigma} ds
\]
and this defines the initial value
\[
K_{11}(1) - \nu_{11} = l_{10}(0)
\]  
(2.22)
needed to completely specify the terms $K_{j1}$ of the outer expansion.  
Higher order terms in the boundary layer correction will satisfy linear equations of the form
\[
\frac{dl_{1j}}{d\sigma} = l_{2j} F' + F' l_{2j} - l_{2j} S_{20} l_{30} - l_{2j} S_{20} l_{3j} + \hat{a}_j(\sigma)
\]
\[
\frac{dl_{2j}}{d\sigma} = -l_{2j}(\tilde{\alpha} + S_{20} l_{30}) + F l_{3j} - l_{2j} S_{20} l_{3j} - \hat{b}_j(\sigma)
\]  
(2.24)
\[
\frac{dl_{3j}}{d\sigma} = -l_{3j}(\tilde{\alpha} + S_{20} l_{30}) - (\tilde{\alpha} + l_{30} S_{20}) l_{3j} + \hat{c}_j(\sigma)
\]
where $\hat{a}_j$, $\hat{b}_j$, and $\hat{c}_j$ are exponentially decaying and known successively.  
Note further that the initial values
\[
l_{2j}(0) = \pi_{2j} - K_{2j}(1)
\]
\[
l_{3j}(0) = \pi_{3j} - K_{3j}(1)
\]  
(2.25)
(cf. (2.3)) are determined by the same order terms of the outer solution.  
Since $-(\tilde{\alpha} + S_{20} l_{30})$ is stable for $\sigma$ sufficiently large, the initial value problems for $l_{3j}$ and $l_{2j}$ can be integrated in turn to provide unique exponentially decaying solutions.  
Finally, the equation for $l_{1j}$ has a unique decaying solution which determines the terminal value
\[
K_{1,j+1}(1) = \pi_{1,j+1} - l_{1j}(0)
\]
needed to specify the next order term of the outer solution.  
In summary, then, we have

**Theorem 1.** Under hypotheses (i–v), the terminal value problem (1.9–1.10) for the Riccati gain $k(t, \epsilon)$ has a unique solution of the form
\[
k(t, \epsilon) = \begin{pmatrix}
k_1(t, \epsilon) & e k_2(t, \epsilon) \\
e k_2'(t, \epsilon) & e k_3(t, \epsilon)
\end{pmatrix}
\]
where for each integer $I \geq 0$

\[ k_i(t, \epsilon) = K_0(t) + \sum_{i=0}^{I} \left[ K_{i+1}(t) + I_{i-1} \left( \frac{1-t}{\epsilon} \right) \right] \epsilon^i + O(\epsilon^{I+1}) \]

and

\[ \begin{align*} k_j(t, \epsilon) &= \sum_{i=0}^{J} \left[ K_{ji}(t) + l_{ji} \left( \frac{1-t}{\epsilon} \right) \right] \epsilon^i + O(\epsilon^{J+1}), \\
&j = 2 \text{ and } 3, \text{ on } 0 \leq t \leq 1 \text{ where the } l_{ni} \to 0 \text{ as } \sigma = (1-t)/\epsilon \to \infty. \end{align*} \]

The proof of the theorem follows from well-known results for singularly perturbed initial value problems (cf., e.g., O'Malley [7]).

III. ASYMPTOTIC DETERMINATION OF THE OPTIMAL TRAJECTORIES AND THE OPTIMAL CONTROL

Using the representation (2.1) for the Riccati gain and the control law (1.8), we find that the optimal control is given by

\[ u(t, \epsilon) = -(B_1'k_1 + B_2'k_2)x - (\epsilon B_1'k_2 + B_2'k_3)z. \quad (3.1) \]

Then, using the asymptotic decomposition (2.3) for the Riccati gains and substituting for $u$ in the state equations (1.1), we have

\[ \begin{align*} dx/dt &= [\bar{A}_1(t, \epsilon) + a_1(\sigma, \epsilon)]x + [\bar{A}_2(t, \epsilon) + a_2(\sigma, \epsilon)]z, \\
\epsilon(dx/dt) &= [\bar{A}_3(t, \epsilon) + a_3(\sigma, \epsilon)]x + [\bar{A}_4(t, \epsilon) + a_4(\sigma, \epsilon)]z \end{align*} \quad (3.2) \]

where

\[ \begin{align*} \bar{A}_1(t, \epsilon) &= A_1 - S_1K_1(t, \epsilon) - SK_2'(t, \epsilon), \\
\bar{A}_2(t, \epsilon) &= A_2 - \epsilon S_1K_2(t, \epsilon) - S_2K_3(t, \epsilon), \\
\bar{A}_3(t, \epsilon) &= A_3 - S'K_1(t, \epsilon) - S_2K_2'(t, \epsilon), \\
\bar{A}_4(t, \epsilon) &= A_4 - \epsilon S'K_2(t, \epsilon) - S_2K_3(t, \epsilon), \\
a_1(\sigma, \epsilon) &= -\epsilon S_1l_1(s, \epsilon) - S_1l_2'(s, \epsilon), \\
a_2(\sigma, \epsilon) &= -\epsilon S_1l_2(s, \epsilon) - S_2l_3(s, \epsilon), \\
a_3(\sigma, \epsilon) &= -\epsilon S'l_1(s, \epsilon) - S_2l_2'(s, \epsilon), \\
a_4(\sigma, \epsilon) &= -\epsilon S'l_2(s, \epsilon) - S_2l_3'(s, \epsilon). \end{align*} \]

We must solve the linear system (3.2) subject to the initial conditions (1.2). Since the differential order of (3.2) is less when $\epsilon = 0$ than for $\epsilon > 0$, we can anticipate nonuniform convergence at $t = 0$. Further, since the $a_i$'s feature
exponential decay away from \( t = 1 \), we can also expect nonuniform convergence there. Thus, we shall seek a solution to the initial value problem for (3.2) of the form
\[
X(t, \epsilon) = X(t, \epsilon) + \epsilon m_1(\kappa, \epsilon) + \epsilon n_1(\sigma, \epsilon)
\]
\[
z(t, \epsilon) = Z(t, \epsilon) + m_2(\kappa, \epsilon) + n_2(\sigma, \epsilon)
\]
where the \( m_i \to 0 \) as \( \kappa = t/\epsilon \to \infty \) and the \( n_i \to 0 \) as \( \sigma \to \infty \). This is the form of the optimal trajectories determined in O'Malley [8] and O'Malley and Kung [10].

Within \((0, 1)\), the solution will be asymptotically given by the outer solution \((X(t, \epsilon), Z(t, \epsilon))\) which we assume has an asymptotic series expansion in \( \epsilon \). Thus, the outer solution within \((0, 1)\) must satisfy the system (3.2) with the \( a_i \)'s neglected and its leading term will satisfy the reduced system
\[
dX_0/dt = \bar{A}_{10}(t) X_0 + \bar{A}_{20}(t) Z_0
\]
\[
0 = \bar{A}_{30}(t) X_0 + \bar{A}_{40}(t) Z_0 .
\]
Since \( \bar{A}_{40}(t) = -\bar{A} \) is invertible, we have
\[
Z_0(t) = \bar{A}^{-1}(A_{30} - S_0 K_{10} - S_{20} K_{20}) X_0(t)
\]
and there remains the linear initial value problem
\[
dX_0/dt = [(A_{10} - S_{10} K_{10} - S_{20} K_{20})
\]
\[
+ (A_{20} - S_0 K_{30}) \bar{A}^{-1}(A_{30} - S_0 K_{10} - S_{20} K_{20})] X_0(t),
\]
\[
X_0(0) = x(0, 0).
\]
Higher order terms in the outer expansion will satisfy nonhomogeneous forms of (3.5) subject to the successively determined initial condition
\[
X_j(0) = x_j(0) - m_{1,j-1}(0).
\]
The Fredholm alternative and the solvability of the reduced problem for \((X_0, Z_0)\) implies that each \((X_j, Z_j)\) can be uniquely determined, in turn, up to specification of \( X_j(0) \).

Since the \( a_i(\sigma) \)'s are asymptotically negligible near \( t = 0 \), the representation (3.3) implies that the initial boundary layer correction \((\epsilon m_1, \epsilon m_2)\) must satisfy the linear system
\[
dm_1/d\kappa = \epsilon \bar{A}_1(\epsilon \kappa, \epsilon) m_1 + \bar{A}_2(\epsilon \kappa, \epsilon) m_2
\]
\[
dm_2/d\kappa = \epsilon \bar{A}_3(\epsilon \kappa, \epsilon) m_1 + \bar{A}_4(\epsilon \kappa, \epsilon) m_2 .
\]
Thus the leading terms satisfy
\[ \frac{\partial m_{30}}{\partial \kappa} = (A_{20} - S_0 K_{20}) m_{20} \]
\[ \frac{\partial m_{30}}{\partial \kappa} = -\tilde{\Lambda} m_{20} \]
and since (3.3) implies
\[ m_{2j}(0) = z_j(0) - Z_j(0) \]
for each \( j \geq 0 \), we have the decaying solutions
\[ m_{30}(\kappa) = e^{-\tilde{\Lambda} \kappa} [z_0(0) - Z_0(0)] \]
and
\[ m_{10}(\kappa) = -(A_{20} - S_0 K_{20}) \tilde{\Lambda}^{-1} e^{-\tilde{\Lambda} \kappa} [z_0(0) - Z_0(0)]. \]
Note that we’ve determined the initial value \( m_{30}(0) \) needed to specify the next terms of the outer expansion (cf. (3.8)). Higher order terms of this boundary layer correction can be obtained analogously.

At \( t = 1 \), the initial boundary layer correction is asymptotically negligible, so the terminal boundary layer correction \((\epsilon n_1, n_2)\) must satisfy
\[ \frac{\partial n_2}{\partial \sigma} = -[(\tilde{A}_1(1 - \epsilon \sigma, \epsilon) + a_1(\sigma, \epsilon)) n_1 \]
\[ + [\tilde{A}_2(1 - \epsilon \sigma, \epsilon) + a_2(\sigma, \epsilon)] n_2 + a_1(\sigma, \epsilon) X(1 - \epsilon \sigma, \epsilon) \]
\[ + a_2(\sigma, \epsilon) Z(1 - \epsilon \sigma, \epsilon)] \]
\[ = -[\tilde{A}_3(1 - \epsilon \sigma, \epsilon) + a_3(\sigma, \epsilon)] n_1 + [\tilde{A}_4(1 - \epsilon \sigma, \epsilon) + a_4(\sigma, \epsilon)] n_2 \]
\[ + a_3(\sigma, \epsilon) X(1 - \epsilon \sigma, \epsilon) + a_4(\sigma, \epsilon) Z(1 - \epsilon \sigma, \epsilon)]. \]

Thus, when \( \epsilon = 0 \), we have
\[ \frac{\partial n_{10}}{\partial \sigma} = -[(\tilde{A}_{20}(1) + a_{20}(\sigma)) n_{20} + a_{10}(\sigma) X_{0}(1) + a_{20}(\sigma) Z_{0}(1) \]
\[ \frac{\partial n_{20}}{\partial \sigma} = -[(\tilde{A}_{40}(1) + a_{40}(\sigma)) n_{20} + a_{20}(\sigma) X_{0}(1) + a_{40}(\sigma) Z_{0}(1)]. \]
Noting that \( \tilde{A}_{40}(1) + a_{40}(\sigma) = -\tilde{A} + a_{40}(\sigma) \) is stable for \( \sigma \) large and that the \( n_{10} \to 0 \) as \( \sigma \to \infty \), we integrate to obtain the unique exponentially decaying solutions
\[ n_{30}(\sigma) = \int_{0}^{\infty} e^{\tilde{\Lambda}(\sigma - s)} \int_{s}^{\sigma} a_{40}(\sigma) [a_{30}(s) X_{0}(1) + a_{40}(s) Z_{0}(1)] \, ds \]
and
\[ n_{10}(\sigma) = -\int_{\sigma}^{\infty} \frac{dn_{10}(s)}{d\sigma} \, ds. \]
Higher order terms in this boundary layer correction are also uniquely determined recursively.
Substituting the expansions (2.3) for the Riccati gains and the expansions (3.3) for the optimal trajectories into the control law (3.1), we find an asymptotic representation of the optimal control in the form

$$u(t, \epsilon) = U(t, \epsilon) + v(\kappa, \epsilon) + \omega(\sigma, \epsilon)$$

(3.15)

where $U$, $v$, and $\omega$ have asymptotic series expansion in $\epsilon$ with the terms of $v(\omega)$ tending to zero as $\kappa(\sigma)$ tends to infinity. Substituting this expansion and those for the optimal trajectories into the cost (1.3) implies that the optimal cost $J^*(\epsilon)$ is of the form

$$J^*(\epsilon) = \frac{1}{2} \lambda(\epsilon) + \frac{1}{2} \int_0^1 L_2(t, \epsilon) \, dt$$

$$+ \frac{\epsilon}{2} \int_0^\infty L_2(\kappa, \epsilon) \, d\kappa + \frac{\epsilon}{2} \int_0^\infty L_3(\sigma, \epsilon) \, d\sigma$$

where $\lambda$ and the $L_i$'s have asymptotic series expansion with integrable coefficients. Thus, $J^*(\epsilon)$ also has an asymptotic series expansion in $\epsilon$.

Summarizing, then, we state

**Theorem 2.** Under hypotheses (i-v), the problem (1.1-1.4) has a unique asymptotic solution for $\epsilon$ sufficiently small such that for every integer $N \geq 0$, the optimal control, the corresponding trajectories, and the optimal cost satisfy

$$u(t, \epsilon) = \sum_{l=0}^N \left[ U_l(t) + v_l(\kappa) + \omega_l(\sigma) \right] \epsilon^l + O(\epsilon^{N+1})$$

$$x(t, \epsilon) = X_0(t) + \sum_{l=1}^N \left[ X_l(t) + m_{1,l-1}(\kappa) + n_{1,l-1}(\sigma) \right] \epsilon^l + O(\epsilon^{N+1})$$

$$z(t, \epsilon) = \sum_{l=0}^N \left[ Z_l(t) + m_{2l}(\kappa) + n_{2l}(\sigma) \right] \epsilon^l + O(\epsilon^{N+1})$$

and

$$J^*(\epsilon) = \sum_{l=0}^N J_l^* \epsilon^l + O(\epsilon^{N+1}).$$

The expansions are uniformly valid for $0 \leq t \leq 1$ and the functions of $\kappa = t/\epsilon$ and $\sigma = (1 - t)/\epsilon$ decay to zero as the appropriate variable tends to infinity.

We recall that this result was previously obtained by O'Malley and Kung [10] without hypothesis (v). We've now shown that it can also be obtained through the more flexible matrix Riccati method. The theorem could...
be proved by showing that the expansions obtained here agree with those found previously. In particular, then, we would have

\[ u(t, \varepsilon) = U_0(t) + O(\varepsilon), \quad 0 < t < 1 \]
\[ x(t, \varepsilon) = X_0(t) + O(\varepsilon), \quad 0 < t < 1 \]
\[ z(t, \varepsilon) = Z_0(t) + O(\varepsilon), \quad 0 < t < 1 \]

and

\[ J^*(\varepsilon) = J_0^* + O(\varepsilon) \]

where \( U_0(t) \) is the optimal control; \( X_0(t) \) and \( Z_0(t) \), the corresponding trajectories; and \( J_0^* \), the optimal cost, for the reduced problem

\[
\begin{align*}
\frac{dX_0}{dt} &= A_3(t, 0) X_0 + A_4(t, 0) Z_0 + B_3(t, 0) U_0 \\
 sol &= A_3(t, 0) X_0 + A_4(t, 0) Z_0 + B_3(t, 0) U_0 \\
X_0(0) &= x(0, 0)
\end{align*}
\]

with

\[
J(0) = \frac{1}{2} X_0'(1) Q_0(0) X_0(1)
\]
\[ + \frac{1}{2} \int_0^1 \left[ (X_0(t))' Q(t, 0) (X_0(t)) + U_0'(t) U_0(t) \right] dt \]

to be minimized.

**REFERENCES**


