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The Solution to the Matrix Equation $AV + BW = EVJ + R$

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Abstract—This note considers the solution to the generalized Sylvester matrix equation $AV + BW = EVJ + R$, where A, B, E, and R are given matrices of appropriate dimensions, J is an arbitrary given Jordan matrix, while V and W are matrices to be determined. A general parametric solution for this equation is proposed, based on the Smith form reduction of the matrix $[A - sE \quad B]$. The solution possesses a very simple and neat form, and does not require the eigenvalues of matrix J to be known. An example is presented to illustrate the proposed solution.

Keywords—Generalized Sylvester matrix equation, Parametric solution, R-controllability, Smith form reduction.

1. INTRODUCTION

Consider the following generalized Sylvester matrix equation

$$AV + BW = EVJ, \tag{1.1}$$

where $A, E \in R^{n\times n}$, $B \in R^{n\times r}$ are some given matrices, $J \in C^{p\times p}$ is a given Jordan matrix, and $V \in C^{n\times p}$, $W \in C^{r\times p}$ are to be determined. This matrix equation has found applications in many problems in linear systems theory, such as eigenstructure assignment [1–5], observer design [6], control of systems with input constraints [7], and robust fault detection [8,9]. For this equation, the general complete parametric solution has been explored in [1,2,10].

When dealing with complicated linear systems, such as large scale systems with interconnections, second- or higher-order linear systems [11], linear systems with certain partitioned structures or extended models, and linear systems with input constraints [7], we sometimes naturally encounter matrix equations in the following generalized form of (1.1),

$$AV + BW = EVJ + R, \tag{1.2}$$

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where $R \in R^{n\times p}$ is some given matrix. Very recently, the author and his coauthors proposed complete parametric solutions to matrix equation (1.2) [3,11]. In [11], a complete parametric solution to this matrix equation is proposed for the special case $E = I$ and J being diagonal based on right-coprime factorization. The solution is applied to eigenstructure assignment in second-order linear systems. In [3], a complete parametric solution to this equation is presented based on the Smith form reduction of the matrix $[A - sE \quad B]$. However, the solution is not in an explicit closed form, but is iterative. The purpose of this paper is to derive, for matrix equation (1.2), simpler and general complete parametric solutions in direct closed forms.

In the following sections, we use $f^{(l)}(x_0)$ to represent the l^{th} derivative of function $f(x)$ at $x = x_0$. Moreover, we adopt the convention that $f^{(l)}(x) = 0$ for l negative.

2. PRELIMINARIES

First, let us state the following simple fact.

FACT 2.1. Let

$$J = \text{Blockdiag}\,[\,J_1 \quad J_2 \quad \cdots \quad J_q\,] \tag{2.1}$$

and partition the matrices V, W, and R, correspondingly as

$$\begin{aligned} V &= [\,V_1 \quad V_2 \quad \cdots \quad V_q\,], \\ W &= [\,W_1 \quad W_2 \quad \cdots \quad W_q\,], \\ R &= [\,R_1 \quad R_2 \quad \cdots \quad R_q\,], \end{aligned} \tag{2.2}$$

where the matrices V_i, W_i, and R_i are in consistent dimensions with the Jordan block J_i. Then, (1.2) can be decomposed equivalently into the following set of equations:

$$AV_i + BW_i = EV_iJ_i + R_i, \qquad i = 1, 2, \ldots, q. \tag{2.3}$$

The above fact states that equation (1.2), with matrix J being a Jordan matrix, can be decomposed into a series of matrix equations in the same form as (1.2), but with matrix J being a Jordan block. Therefore, without loss of generality, we impose the following assumption, as in [10].

ASSUMPTION A1. *The matrix J is a Jordan block of order p with eigenvalue σ.*

In accordance with the above assumption, we can write

$$\begin{aligned} V &= [\,v_1 \quad v_2 \quad \cdots \quad v_p\,], \\ W &= [\,w_1 \quad w_2 \quad \cdots \quad w_p\,], \\ R &= [\,r_1 \quad r_2 \quad \cdots \quad r_p\,]. \end{aligned} \tag{2.4}$$

With these notations, the following result can be easily shown (proof omitted).

LEMMA 2.1. *Let Assumption A1 be valid. Matrix equation (1.2), then, is equivalent to the following group of vector equations:*

$$(A - \sigma E)\, v_i + Bw_i = Ev_{i-1} + r_i, \qquad v_0 = 0, \quad i = 1, 2, \ldots, p. \tag{2.5}$$

The matrix triple $[\,E \quad A \quad B\,]$ is called R-controllable [6], if and only if

$$\text{rank}\,[\,A - sE \quad B\,] = n, \quad \forall\,\text{finite } s \in C. \tag{2.6}$$

For convenience, we introduce the following assumption.

ASSUMPTION A2. *$[\,E \quad A \quad B\,]$ is R-controllable.*

LEMMA 2.2. *Let Assumption A2 be valid. Then, there exists a unimodular matrix $Q(s)$ of order $(n+r)$, such that*

$$[A - sE \quad B]\,Q(s) = [I_n \quad 0]. \tag{2.7}$$

PROOF. Due to Assumption A2, we can find two unimodular matrices $P(s)$ and $U(s)$ of appropriate dimensions, such that

$$P(s)\,[A - sE \quad B]\,U(s) = [I_n \quad 0]. \tag{2.8}$$

Letting

$$Q(s) = U(s)\begin{bmatrix} P(s) & 0 \\ 0 & I_r \end{bmatrix}, \tag{2.9}$$

then, $Q(s)$ is clearly unimodular, and it can be verified that this $Q(s)$ satisfies (2.7). ∎

3. THE MAIN RESULT

The following theorem gives the general complete parametric solution in explicit closed form for matrix equation (1.2). It should be noted that the result does not require the eigenvalue σ of matrix J to be known *a priori*.

THEOREM 3.1. *Let Assumptions A1 and A2 be valid, and $Q(s)$ be a unimodular matrix satisfying (2.7). Then, all the solutions to matrix equation (1.2) are characterized by*

$$\begin{bmatrix} v_k \\ w_k \end{bmatrix} = Q(\sigma)\begin{bmatrix} r_k \\ f_k \end{bmatrix} + Q^{(1)}(\sigma)\begin{bmatrix} r_{k-1} \\ f_{k-1} \end{bmatrix} + \cdots + \frac{1}{(k-1)!}Q^{(k-1)}(\sigma)\begin{bmatrix} r_1 \\ f_1 \end{bmatrix}, \quad k = 1, 2, \ldots, p \tag{3.1}$$

where $f_i \in C^r$, $i = 1, 2, \ldots, p$, are a group of parameter vectors, which represent the degrees of freedom in the solution.

PROOF. Partition $Q(s)$ into

$$Q(s) = \begin{bmatrix} T(s) & N(s) \\ L(s) & D(s) \end{bmatrix}, \tag{3.2}$$

where $N(s)$ and $D(s)$ are a pair of polynomial matrices of dimensions $n \times r$ and $r \times r$, respectively, while $T(s)$ and $L(s)$ are a pair of polynomial matrices of dimensions $n \times n$ and $r \times n$, respectively. It follows from (2.7), that

$$(A - sE)\,N(s) + BD(s) = 0, \tag{3.3}$$

$$(A - sE)\,T(s) + BL(s) = I_n. \tag{3.4}$$

Furthermore, since the matrix $Q(s)$ is unimodular, it is easy to see that the pair of polynomial matrices $N(s)$ and $D(s)$ are right-coprime, and so are the pair of polynomial matrices $T(s)$ and $L(s)$.

Because of (3.2), it is obvious that solution (3.1) has the following equivalent form:

$$\begin{bmatrix} v_k \\ w_k \end{bmatrix} = \begin{bmatrix} T(\sigma) & N(\sigma) \\ L(\sigma) & L(\sigma) \end{bmatrix}\begin{bmatrix} r_k \\ f_k \end{bmatrix} + \cdots + \frac{1}{(k-1)!}\begin{bmatrix} T^{(k-1)}(\sigma) & N^{(k-1)}(\sigma) \\ L^{(k-1)}(\sigma) & L^{(k-1)}(\sigma) \end{bmatrix}\begin{bmatrix} r_1 \\ f_1 \end{bmatrix}, \quad k = 1, 2, \ldots, p. \tag{3.5}$$

To prove the result, let us first show that the vectors v_k and w_k, $k = 1, 2, \ldots, p$, given by (3.5) are solutions to matrix equation (1.2). In view of Lemma 2.1, it suffices only to show that the vectors v_k and w_k, $k = 1, 2, \ldots, p$, given by (3.5) satisfy the equations in (2.5).

Taking the differential of order l of both sides of (3.3), yields

$$(A - sE)\,N^{(l)}(s) + BD^{(l)}(s) = lEN^{(l-1)}(s), \qquad l = 0, 1, 2, \ldots, k-1. \tag{3.6}$$

Replacing s by σ in (3.6), and then, post-multiplying by the vector $(1/l!)f_{k-1}$ on both sides of (3.6), gives

$$(A-\sigma E)\frac{1}{l!}N^{(l)}(\sigma)\,f_{k-1}+B\frac{1}{l!}D^{(l)}(\sigma)\,f_{k-1}=E\frac{1}{(l-1)!}N^{(l-1)}(\sigma)\,f_{k-1},\qquad l=0,1,2,\dots,k-1. \tag{3.7}$$

Summing up all the equations in (3.7) side-by-side, produces

$$(A-\sigma E)\,v_k'+Bw_k'=Ev_{k-1}',\qquad v_0'=0,\quad k=1,2,\dots,p, \tag{3.8}$$

where

$$\begin{bmatrix} v_k' \\ w_k' \end{bmatrix}=\begin{bmatrix} N(\sigma) \\ D(\sigma) \end{bmatrix}f_k+\begin{bmatrix} N^{(1)}(\sigma) \\ D^{(1)}(\sigma) \end{bmatrix}f_{k-1}+\cdots+\frac{1}{(k-1)!}\begin{bmatrix} N^{(k-1)}(\sigma) \\ D^{(k-1)}(\sigma) \end{bmatrix}f_1. \tag{3.9}$$

Similarly, taking the differential of order l of both sides of (3.4), yields

$$(A-sE)\,T^{(l)}(s)+BL^{(l)}(s)=lET^{(l-1)}(s)+\delta(l)\,I_n,\qquad l=0,1,2,\dots,k-1, \tag{3.10}$$

where $\delta(l)$ is a function, which is 1 at $l=0$ and zero at any other points. Replacing s by σ in (3.10), and post-multiplying by the vector $(1/l!)r_{k-1}$ on both sides of (3.10), gives

$$(A-\sigma E)\frac{1}{l!}T^{(l)}(\sigma)\,r_{k-1}+B\frac{1}{l!}L^{(l)}(\sigma)\,r_{k-1}=E\frac{1}{(l-1)!}T^{(l-1)}(\sigma)\,r_{k-1}+\delta(l)\,\frac{1}{l!}r_{k-1},\qquad l=0,1,2,\dots,k-1. \tag{3.11}$$

Summing up all the equations in (3.11) side-by-side, gives

$$(A-\sigma E)\,v_k''+Bw_k''=Ev_{k-1}''+r_k,\qquad v_0''=0,\quad k=1,2,\dots,p, \tag{3.12}$$

where

$$\begin{bmatrix} v_k'' \\ w_k'' \end{bmatrix}=\begin{bmatrix} T(\sigma) \\ L(\sigma) \end{bmatrix}r_k+\begin{bmatrix} T(\sigma) \\ L(\sigma) \end{bmatrix}r_{k-1}+\cdots+\frac{1}{(k-1)!}\begin{bmatrix} T^{(k-1)}(\sigma) \\ L^{(k-1)}(\sigma) \end{bmatrix}r_1. \tag{3.13}$$

It clearly follows from (3.5), (3.9) and (3.13), that

$$\begin{bmatrix} v_k \\ w_k \end{bmatrix}=\begin{bmatrix} v_k' \\ w_k \end{bmatrix}+\begin{bmatrix} v_k'' \\ w_k'' \end{bmatrix}. \tag{3.14}$$

By summing up the equations (3.8) and (3.12) side-by-side and using relation (3.14), we can obtain the equations in (2.5). Therefore, we have proved that vectors v_k and w_k, $k=1,2,\dots,p$, given by (3.5) satisfy the equations in (2.5).

Secondly, we show the completeness of solution (3.5) in the sense that it contains the maximum degree of freedom. It follows from the main result in [3], that the maximum degree of freedom existing in the solution to matrix equation (1.2) is $r\times p$, while solution (3.5) has exactly $r\times p$ parameters represented by the elements of the vectors f_i, $i=1,2,\dots,p$. Since $N(s)$ and $D(s)$ are right-coprime, rank $[N^{\mathsf T}(s)\ D^{\mathsf T}(s)]=n$ holds, for all $s\in\mathbb{C}$. Therefore, it follows from the format of solution (3.5), that all the elements of the vectors f_i, $i=1,2,\dots,p$, contribute to the solution. Therefore, all the $p\times r$ parameters in solution (3.5) represented by the elements of the vectors f_i, $i=1,2,\dots,p$, are indeed an effective degree of freedom in the solution. This shows the completeness of solution (3.5), or equivalently, solution (3.1). ∎

To finish this section, we finally give some remarks about Theorem 3.1.

REMARK 3.1. When $R=0$, both solutions (3.1) and (3.5) reduce to the main result in [10], which discusses the solution to matrix equation (1.1).

REMARK 3.2. It follows from the proof of Lemma 2.2, that the unimodular matrix $Q(s)$ can be easily obtained based on the Smith form reduction (2.8), while the Smith form reduction (2.8) can be easily realized by manually using some simple elementary matrix transformations for relatively lower-order cases. For higher-order cases, the Maple function `smith` can be readily used.

REMARK 3.3. Instead of solving the unimodular matrix $Q(s)$, we can also find the right-coprime polynomial matrices $N(s)$ and $D(s)$, and $T(s)$ and $L(s)$ satisfying (3.3) and (3.4), respectively, and derive the general solution to the equation based on (3.5). Regarding methods for solving these polynomial matrices, please refer to the comments in [1,2,10].

REMARK 3.4. Besides simplicity and neatness, solution (3.1) has the advantage that it can be applied to matrix equation (1.2) with J only structurally known. In other words, it only requires the order of the Jordan block J to be known, while the eigenvalue σ is allowed to be unknown. This offers a great advantage in certain control applications, where the closed-loop eigenvalues can be regarded undetermined and used as a part of the design parameters (see, e.g., [12–14]).

4. AN ILLUSTRATIVE EXAMPLE

Consider an equation in the form of (1.2), with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$
$$R = \begin{bmatrix} 2 & 3 \\ 3 & 4 \\ 4 & 5 \end{bmatrix}, \qquad J = \begin{bmatrix} \sigma & 1 \\ 0 & \sigma \end{bmatrix}, \qquad \sigma \in C.$$

It can be easily verified that Assumption A2 holds. By applying matrix elementary transformations to the matrix $[\,A - sE \quad B\,]$, we obtain the unimodular matrices $P(s) = I_3$ and

$$Q(s) = U(s) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & s & 0 \\ 0 & 0 & 0 & 0 & 1 \\ s & 1 & 0 & s^2 & -1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix},$$

which satisfy equations (2.7) and (2.8).

According to Theorem 3.1, the general solution to the equation is given by

$$\begin{bmatrix} v_1 \\ w_1 \end{bmatrix} = Q(\sigma) \begin{bmatrix} r_1 \\ f_1 \end{bmatrix},$$
$$\begin{bmatrix} v_2 \\ w_2 \end{bmatrix} = Q(\sigma) \begin{bmatrix} r_2 \\ f_2 \end{bmatrix} + \frac{d}{ds} Q(\sigma) \begin{bmatrix} r_1 \\ f_1 \end{bmatrix},$$

where r_j represents the j^{th} column of matrix R. Writing

$$f_i = [x_{1i} \; x_{2i}]^\top, \qquad x_{ij} \in C, \quad i, j = 1, 2,$$

we can obtain the general solution to the equation as

$$\begin{aligned} V &= \begin{bmatrix} x_{11} & x_{12} \\ 2 + \sigma x_{11} & 3 + x_{11} + \sigma x_{12} \\ x_{21} & x_{22} \end{bmatrix}, \\ W &= \begin{bmatrix} 2\sigma + 3 + \sigma^2 x_{11} - x_{21} & 6 + 3\sigma + 2\sigma x_{11} + \sigma^2 x_{12} - x_{22} \\ 4 + x_{21} & 5 + x_{22} \end{bmatrix}, \end{aligned} \tag{4.3}$$

where x_{ij}, $i, j = 1, 2$, are arbitrary complex numbers.

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