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# Some Clarifications of the Concept of A Garden-of-Eden Configuration

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The relationships between the concepts of Garden-of-Eden configurations and restrictions, mutually erasable configurations, and injective parallel transformations are considered for the tessellation structure which is a model of a uniformly interconnected array of identical finite-state machines.

## 1. INTRODUCTION

There is a significant confusion in the current literature on "cellular" or "tessellation arrays" concerning the concept of a "Garden-of-Eden configuration." A number of current papers on this topic assume that what Moore [1] called "a Garden-of-Eden configuration" is just any array configuration not in the image of the array's global map. This misconception is due, at least in part, to the fact that what Moore called a "configuration" is not what is meant by the term in the current literature. In this note we show that Moore's concept is not the same as a configuration with no preimage, if one is concerned with the class of "finite" configurations. Almost all authors concerned with this topic do limit their attention to this class. Clearing up this point suggests a number of other related questions, all of which are answered and tabulated in Fig. 1.

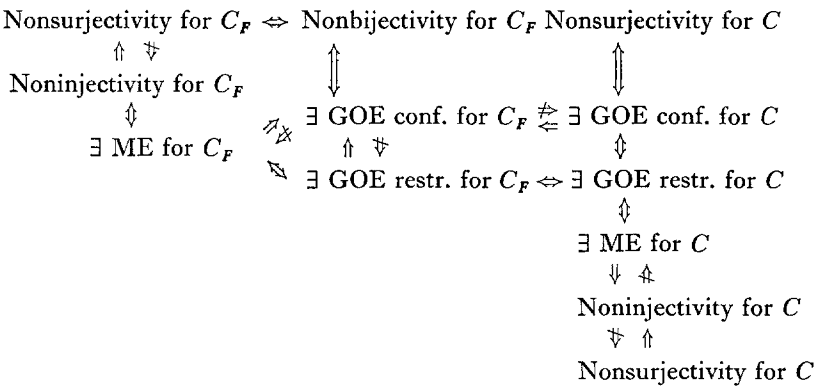


FIG. 1. Summary of the results.

## 2. THE TESSELLATION STRUCTURE

For simplicity, we shall (following Moore [1] and Myhill [2]) limit our attention to arrays of two dimensions. We conjecture that all the concepts and results below carry over to arrays of arbitrary finite dimension.

We use the set $Z^2$ of ordered pairs of integers to name the cells of the tessellation array (we use $Z$ for dimension one). An array configuration, i.e., a symbol from a finite nonempty set $A$ placed in each cell, is formally a mapping $c: Z^2 \to A$. As in [1–3], we can fix the neighbors of any cell to be those cells which have each of their coordinate components differing by at most one from the corresponding coordinate components of the given cell. For one-dimensional arrays, the neighbors of cell $i$ are cells $i - 1$, $i$, and $i + 1$. We designate one symbol in $A$ the quiescent symbol which we denote by 0. An array configuration will be called finite if only finitely many cells contain nonquiescent symbols in the configuration. For any given set $A$, we denote the set of all finite configurations by $C_F$.

Let $\sigma$ be a function which specifies the symbol to be placed in a cell (at time $t$) given the symbols in the neighbors of the cell (at time $t - 1$). We require that $\sigma$ place a 0 (at time $t$) into a cell if all its neighbors contain quiescent symbols (at time $t - 1$). This function $\sigma$, which we call a local transformation, acting on all cells of an array simultaneously, determines a global transformation $\tau$ which maps the set of all array configurations $C$ into $C$. Any global transformation defined in this way, i.e., from a local transformation, will be called a parallel transformation.

## 3. Definitions

The restriction of an array configuration to a subsets $S$ of $Z^2$ will be denoted by $(c)_S$. Let $N(S)$ denote the set of cells containing exactly all neighbors of any cell in $S$. Note $N(S) \supset S$.

Let $S_1 = N(N(S))$ for some finite set $S \subset Z^2$. Restrictions $(c_1)_{S_1}$ and $(c_2)_{S_1}$ of array configurations $c_1, c_2 \in C$ (or $C_F$) will be called (after Moore) mutually erasable with respect to $\tau$, if conditions (a)–(c) below hold.

(a) $(c_1)_{S_1-S} = (c_2)_{S_1-S}$,

(b) $(c_1)_S \neq (c_2)_S$,

(c) $(\tau(c_1))_{N(S)} = (\tau(c_2))_{N(S)}$.

The concept of a Garden-of-Eden (GOE) configuration with respect to a parallel transformation $\tau$ and $C$ (or $C_F$) was defined in [3] as an array configuration $c \in C(C_F)$ for which no $c' \in C(C_F)$ exists such that $\tau(c') = c$. This is also what we shall mean when we speak of a GOE configuration here. It was implied in [3] that this was what Moore in [1] and Myhill in [2] meant when they spoke of a Garden-of-Eden configuration. Actually, the concept that they had in mind for this phrase was the following.

A restriction $(c)_S$ of an array configuration $c \in C$ (or $C_F$) to a finite set $S$ will be called a Garden of Eden (GOE) restriction with respect to $\tau$ and $C$ (or $C_F$) if for any extension $c' \in C$ (or $C_F$) there is no $c''$ such that $\tau(c'') = c'$.

Note that what Moore called a "Garden-of-Eden configuration" was not an array configuration, but a restriction of an array configuration to a finite set $S$ of cells. The following section should help to clear up some of the confusion which may exist after [3].

Proposition 1. (Moore and Myhill). *The existence of* GOE *restrictions is necessary and sufficient for the existence of mutually erasable restrictions for universe $C$ or $C_F$.*

The proof of this result is found in [1] and [2]. The reader can verify that their arguments are valid independent of whether the universe is chosen to be $C$ or $C_F$, although this is not explicitly stated there.

Proposition 2. *For universe $C$, the existence of* GOE *configurations is necessary and sufficient for the existence of* GOE *restrictions.*

*Proof.* The proof of necessity is immediate. We first give the sufficiency proof for a one-dimensional array. Assume $c$ is a GOE configuration for $\tau$. For some fixed $i \in Z$, let $K_i$ be the unit set containing a triple $(a_1a_2a_3)$ mapped by $\sigma$ to $c(i)$, where $\sigma$ is the local transformation defining $\tau$. Suppose $K_{i+k}$, $k \geqslant 0$ is defined, then let $K_{i+k+1}$

be the set of all triples $(b_1 b_2 b_3)$ such that for some $d$, $(d\, b_1 b_2)$ is in $K_{i+k}$ and $\sigma(b_1 b_2 b_3) = c(i+k+1)$. Note that these sets have been defined "moving to the right from cell $i$." We would likewise define sets $K_{i-1}$, $K_{i-2}$,..., by "moving left." Since $c$ is a GOE configuration, there is some $j$, which may be positive or negative, such that $K_{i+j} = \varnothing$. This means that for any $c'$ such that $c'(i-1) = a_1$, $c'(i) = a_2$, and $c'(i+1) = a_3$, then $\tau(c') \neq c''$ where $c''$ is any configuration agreeing with $c$ on all cells between and including $i$ and $i+j$. Repeating the above argument for each triple mapped by $\sigma$ to $c(i)$ leads to the existence of integers $m, n$ such that $c$ restricted to $\{i: m \leqslant i \leqslant n\}$ is a GOE restriction.

The extension of this proof to the two-dimensional case should be clear. $K_i$ in this case would initially be some state of the neighborhood of cell $i$ mapped by $\sigma$ to $c(i)$. $K_{i+1}$ would be a set of states of a square "frame" of cells of width 2 surrounding the neighborhood of cell $i$. These states would be consistent with the contents of $K_i$ analogous to the one-dimensional case above. $K_{i+2}$ would be a set of states of a still larger frame of cells again of width 2 surrounding the last square frame. The reader should be able to fill in the necessary details to complete the proof.

From the above results we have the analog of the Moore–Myhill Theorem for GOE configurations limited to universe $C$.

PROPOSITION 3. *For universe $C$, the existence of* GOE *configurations is necessary and sufficient for the existence of mutually erasable restriction.*

PROPOSITION 4. *For universe $C_F$, parallel transformation $\tau$ is not injective is necessary and sufficient for the existence of mutually erasable restrictions.*

*Proof.* Let $\tau(c_1) = \tau(c_2)$ where $c_1 \neq c_2$, and let $S \subset Z^2$ include all cells containing nonquiescent symbols in either $c_1$ or $c_2$. Then $(c_1)_S$ and $(c_2)_S$ are mutually erasable.

PROPOSITION 5. *For universe $C_F$, the existence of* GOE *restrictions is sufficient but not necessary for the existence of* GOE *configurations.*

*Proof.* In [3] the local transformation specified by

| | |
|---|---|
| 000 | 0 |
| 001 | 1 |
| 010 | 1 |
| 011 | 0 |
| 100 | 1 |
| 101 | 0 |
| 110 | 1 |
| 111 | 0 |

defined a parallel transformation which was injective but not surjective with respect to $C_F$. That injectivity precludes the possibilities of GOE restrictions is established by Propositions 1 and 4.

PROPOSITION 6. *For universe $C_F$, the existence of* GOE *configurations is necessary but not sufficient for the existence of mutually erasable restrictions.*

*Proof.* That the condition is not sufficient was shown in [3]. That the condition is necessary follows from Propositions 1 and 5.

PROPOSITION 7. *For universe $C$, parallel transformation $\tau$ is not injective is necessary but not sufficient for the existence of mutually erasable restrictions.*

*Proof.* To prove the nonsufficiency, one must show the existence of a noninjective $\tau$ with no mutually erasable restrictions, or from Proposition 3, that is surjective. Such parallel transformations are easy to find.

PROPOSITION 8. (Amoroso and Cooper [3]). *For universe $C_F$, parallel transformation $\tau$ is surjective is necessary and sufficient for $\tau$ to be bijective.*

COROLLARY 8.1. (Amoroso and Cooper [3]). *For universe $C_F$, parallel transformation $\tau$ is bijective is necessary and sufficient for* GOE *configurations to exist.*

PROPOSITION 9. *For universe $C_F$, parallel transformation $\tau$ is injective is necessary but not sufficient for $\tau$ to be surjective.*

*Proof.* The nonsufficiency is established by the example from [3] cited above in the proof of Proposition 5. The necessity follows from Proposition 8.

PROPOSITION 10. (Richardson [4]). *For universe $C$, parallel transformation $\tau$ is injective is sufficient but not necessary that $\tau$ be surjective.*

The proof of this is found in [4].

PROPOSITION 11. *The existence of a* GOE *configuration for $\tau$ and universe $C_F$ is necessary but not sufficient for the existence of a* GOE *configuration for $\tau$ and $C$.*

*Proof.* The necessity follows easily from Proposition 2. The example from [3] used in the proof of Proposition 5 has GOE configuration for $C_F$ but not for $C$.

PROPOSITION 12. *With respect to some $\tau$, $(c)_S$ is a* GOE *restriction for universe $C_F$ is necessary and sufficient for $(c)_S$ to be a* GOE *restriction for $C$.*

*Proof.* The necessity is trivial. Let $(c)_S$ be a GOE restriction for $C_F$ but not for $C$, i.e., some extension $c'$ of $(c)_S$ has an infinite configuration as preimage under $\tau$. But some restriction of this preimage extended to a finite configuration would have as image under $\tau$, a finite configuration which is an extension of $(c)_S$ contradicting our premise.

PROPOSITION 13 (Richardson). *Let $\tau$ be defined on $C$ and let $\tau'$ be its restriction to $C_F$. Then $\tau'$ is injective is necessary and sufficient for to be surjective.*

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