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New generalized Jacobi elliptic function rational expansion method

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ABSTRACT

In this work, a new generalized Jacobi elliptic function rational expansion method is based upon twenty-four Jacobi elliptic functions and eight double periodic Weierstrass elliptic functions, which solve the elliptic equation $\phi'^2 = r + p\phi^2 + q\phi^4$, is described. As a consequence abundant new Jacobi–Weierstrass double periodic elliptic functions solutions for (3 + 1)-dimensional Kadmtsev–Petviashvili (KP) equation are obtained by using this method. We show that the new method can be also used to solve other nonlinear partial differential equations (NPDEs) in mathematical physics.

1. Introduction

The investigation of the exact solutions of NPDEs plays an important role in the study of nonlinear physical phenomena. In the past decades, there has been significant progress in the development of methods such as the inverse scattering method [1], Hirota's bilinear method [2], the similarity transformation method [3–6], the non-local symmetries method [7,8], the homogeneous balance method [9], the exp-function method [10–12], the sine–cosine method [13], the tanh function method [14,15], the mapping method [16,17], the F-expansion method [18], the Riccati equation rational expansion method [19], the Jacobi and Weierstrass elliptic function method [20,21] and the new generalized Jacobi elliptic function expansion method [22–27]. In [28–32] Wang and Chen et al. presented a new elliptic function rational expansion method that is more powerful than the existing Jacobi elliptic function method to uniformly construct more new doubly-periodic solutions in terms of a rational formal Jacobi elliptic function of nonlinear evolution equations.

The main objective in this work is to extend the Jacobi elliptic function expansion method by adding rational expansion to the original form. This leads to obtain several new families of exact solutions for the (3 + 1)-dimensional KP equation. The paper is arranged as follows: In Section 2, we briefly describe the generalized Jacobi elliptic function rational expansion method. In Section 3, Several families of solutions to the elliptic equation $\phi'^2 = r + p\phi^2 + q\phi^4$ are obtained. In Section 4, taking to advantage of the solutions developed in Section 3, a great variety of exact solutions for (3 + 1)-dimensional KP equation are obtained. The conclusion is then given in Section 5.

2. Method description

The main idea of our method is to take full advantage of the elliptic equation. This equation is

$$\phi'^2 = r + p\phi^2 + q\phi^4, \tag{2.1}$$

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where r, p, q are constants to be determined and $\phi' = \frac{d\phi}{d\xi}$. For a given NPDE with $u = u(x^1, \ldots, x^s)$ in s independent variables $x^1, \ldots, x^s$

$$H(u, u_{x^i}, u_{x^i x^j}, \ldots) = 0, \quad i, j = 1, \ldots, s, \tag{2.2}$$

where H is polynomial function with respect to the indicated variables or function which can be reduced to polynomial function by using some transformations. We introduce the traveling wave transformations

$$u(x^1, \ldots, x^s) = u(\xi), \quad \xi = k\left(x^1 + \sum_{i=2}^{s} \alpha_i x^i\right) \tag{2.3}$$

where k and α_i are arbitrary constants that can be determined. Under the transformations (2.3), Eq. (2.2) become ordinary differential equations (ODEs) with constant coefficients

$$H(u, u', u'', u''', \ldots) = 0. \tag{2.4}$$

El-Sabbagh and Ali [25] introduced the solution of Eq. (2.4) in the form

$$u(\xi) = A_0 + \sum_{i=1}^{n}(A_i\phi^i(\xi) + B_i\phi^{-i}(\xi)) + \frac{\phi'(\xi)}{\phi^2(\xi)}\left[a_0 + \sum_{i=1}^{n}(a_i\phi^i(\xi) + b_i\phi^{-i}(\xi))\right] \tag{2.5}$$

where $\phi(\xi)$ is the solution of Eq. (2.1) and $\phi'(\xi) = \frac{d\phi(\xi)}{d\xi}$. In this work, we introduced the solution of Eq. (2.4) in the new general form

$$u(\xi) = \sum_{i=0}^{n}[A_i\phi^i(\xi) + a_i\phi'(\xi)\phi^{i-1}(\xi)] + \frac{\Gamma}{\sum_{i=0}^{n}[B_i\phi^i(\xi) + b_i\phi'(\xi)\phi^{i-1}(\xi)]}. \tag{2.6}$$

Balancing the highest derivative term with the nonlinear term in Eq. (2.4) will determine the positive integer number n. Substituting the expansion (2.6) into the ODEs (2.4) with Eq. (2.1) and setting the coefficients of all powers of $\phi(\xi)$ and $\phi'(\xi)$ to zero, we obtain a system of algebraic equations. By solving this system, the coefficients A_i, B_i, a_i, b_i and Γ can be determined.

Remark 1. The new solutions form (2.6) is different from the solution form introduced in [28–32].

3. New solutions of elliptic equation

It is well known that [33], $\text{sn}\xi = \text{sn}(\xi, m)$, $\text{cn}\xi = \text{cn}(\xi, m)$ and $\text{dn}\xi = \text{dn}(\xi, m)$ are called the Jacobian elliptic sine function, the Jacobian elliptic cosine function and the Jacobian elliptic function of third kind respectively, and $0 < m < 1$ is the modulus of the Jacobian elliptic function. The Jacobian elliptic functions which are denoted by Glaisher's symbols can be divided into four groups after carefully studying properties of 12 Jacobi elliptic functions namely,

$$\begin{aligned}
&(1):\ \text{sn}\xi, \quad \text{cn}\xi \quad \text{and} \quad \text{dn}\xi\\
&(2):\ \text{ns}\xi = \frac{1}{\text{sn}\xi}, \quad \text{nc}\xi = \frac{1}{\text{cn}\xi}, \quad \text{and} \quad \text{nd}\xi = \frac{1}{\text{dn}\xi},\\
&(3):\ \text{sc}\xi = \frac{\text{sn}\xi}{\text{cn}\xi}, \quad \text{sd}\xi = \frac{\text{sn}\xi}{\text{dn}\xi}, \quad \text{and} \quad \text{cd}\xi = \frac{\text{cn}\xi}{\text{dn}\xi},\\
&(4):\ \text{cs}\xi = \frac{1}{\text{sc}\xi}, \quad \text{ds}\xi = \frac{1}{\text{sd}\xi}, \quad \text{and} \quad \text{dc}\xi = \frac{1}{\text{cd}\xi}.
\end{aligned}$$

Each group has the following closed relations and any two of four groups have no relation.

$$\begin{aligned}
&(1):\ \text{cn}^2\xi + \text{sn}^2\xi = 1, \quad \text{dn}^2\xi + m^2\text{sn}^2\xi = 1, \quad m^2(\text{cn}^2\xi - 1) = \text{dn}^2\xi - 1;\\
&(2):\ \text{ns}^2\xi - \text{cs}^2\xi = 1, \quad \text{ns}^2\xi - \text{ds}^2\xi = m^2, \quad \text{ds}^2\xi - \text{cs}^2\xi = 1 - m^2;\\
&(3):\ \text{nc}^2\xi - \text{sc}^2\xi = 1, \quad \text{dc}^2\xi - (1 - m^2)\text{sc}^2\xi = 1, \quad \text{dc}^2\xi - (1 - m^2)\text{nc}^2\xi = m^2;\\
&(4):\ \text{cd}^2\xi + (1 - m^2)\text{sd}^2\xi = 1, \quad \text{nd}^2\xi - m^2\text{sd}^2\xi = 1, \quad m^2\,\text{cd}^2\xi + (1 - m^2)\text{nd}^2\xi = 1;
\end{aligned}$$

In addition, they satisfy

$$\begin{aligned}
&(1):\ (\text{sn}\xi)' = \text{cn}\xi\text{dn}\xi, \quad (\text{cn}\xi)' = -\text{sn}\xi\text{dn}\xi, \quad (\text{dn}\xi)' = -m^2\text{sn}\xi\text{cn}\xi,\\
&(2):\ (\text{ns}\xi)' = -\text{cs}\xi\text{ds}\xi, \quad (\text{cs}\xi)' = -\text{ns}\xi\text{ds}\xi, \quad (\text{ds}\xi)' = -\text{ns}\xi\text{cs}\xi,\\
&(3):\ (\text{sc}\xi)' = \text{nc}\xi\text{dc}\xi, \quad (\text{nc}\xi)' = \text{sc}\xi\text{dc}\xi, \quad (\text{dc}\xi)' = (1 - m^2)\text{nc}\xi\text{sc}\xi,\\
&(4):\ (\text{sd}\xi)' = \text{nd}\xi\text{cd}\xi, \quad (\text{cd}\xi)' = (m^2 - 1)\text{sd}\xi\text{nd}\xi, \quad (\text{nd}\xi)' = m^2\text{cd}\xi\text{sd}\xi.
\end{aligned}$$

From Eq. (2.1) and using symbolic calculations via *Mathematica*, we obtain new general Jacobi elliptic function solutions as the following:

$$\begin{cases} \phi_1 = \dfrac{\omega m}{\sqrt{q}}\mathrm{cn}[\omega\xi], & \omega^2 = \dfrac{p}{2m^2-1}, \quad r = \dfrac{\omega^4 m^2(m^2-1)}{q}, \\ \phi_2 = \dfrac{\imath\omega}{\sqrt{q}}\mathrm{dn}[\omega\xi], & \omega^2 = \dfrac{p}{2-m^2}, \quad r = \dfrac{\omega^4(1-m^2)}{q}, \\ \phi_3 = \dfrac{\omega m}{\sqrt{q}}\mathrm{sn}[\omega\xi], & \omega^2 = \dfrac{-p}{1+m^2}, \quad r = \dfrac{\omega^4 m^2}{q}, \end{cases} \tag{3.1}$$

$$\begin{cases} \phi_4 = \dfrac{\omega}{\sqrt{q}}\mathrm{cs}[\omega\xi], & \omega^2 = \dfrac{p}{2-m^2}, \quad r = \dfrac{\omega^4(1-m^2)}{q}, \\ \phi_5 = \dfrac{\omega}{\sqrt{q}}\mathrm{ds}[\omega\xi], & \omega^2 = \dfrac{p}{2m^2-1}, \quad r = \dfrac{\omega^4 m^2(m^2-1)}{q}, \\ \phi_6 = \dfrac{\omega}{\sqrt{q}}\mathrm{ns}[\omega\xi], & \omega^2 = \dfrac{-p}{1+m^2}, \quad r = \dfrac{\omega^4 m^2}{q}, \end{cases} \tag{3.2}$$

$$\begin{cases} \phi_7 = \dfrac{\omega}{\sqrt{q}}\mathrm{dc}[\omega\xi], & \omega^2 = \dfrac{-p}{1+m^2}, \quad r = \dfrac{\omega^4 m^2}{q}, \\ \phi_8 = \dfrac{\omega\sqrt{1-m^2}}{\sqrt{q}}\mathrm{nc}[\omega\xi], & \omega^2 = \dfrac{p}{2m^2-1}, \quad r = \dfrac{\omega^4 m^2(m^2-1)}{q}, \\ \phi_9 = \dfrac{\omega\sqrt{1-m^2}}{\sqrt{q}}\mathrm{sc}[\omega\xi], & \omega^2 = \dfrac{p}{2-m^2}, \quad r = \dfrac{\omega^4(1-m^2)}{q}, \end{cases} \tag{3.3}$$

$$\begin{cases} \phi_{10} = \dfrac{\omega m}{\sqrt{q}}\mathrm{cd}[\omega\xi], & \omega^2 = \dfrac{-p}{1+m^2}, \quad r = \dfrac{\omega^4 m^2}{q}, \\ \phi_{11} = \dfrac{\omega\sqrt{m^2-1}}{\sqrt{q}}\mathrm{nd}[\omega\xi], & \omega^2 = \dfrac{p}{2-m^2}, \quad r = \dfrac{\omega^4(1-m^2)}{q}, \\ \phi_{12} = \dfrac{\omega m\sqrt{m^2-1}}{\sqrt{q}}\mathrm{sd}[\omega\xi], & \omega^2 = \dfrac{p}{2m^2-1}, \quad r = \dfrac{\omega^4 m^2(m^2-1)}{q}, \end{cases} \tag{3.4}$$

$$\begin{cases} \phi_{13} = \dfrac{m\omega}{2\sqrt{q}}(\mathrm{sn}[\omega\xi] \pm \imath\mathrm{cn}[\omega\xi]), \ \omega^2 = \dfrac{2p}{m^2-2}, \quad r = \dfrac{\omega^4 m^4}{16q}, \\ \phi_{14} = \dfrac{\imath\omega}{2\sqrt{q}}(\mathrm{dn}[\omega\xi] \pm m\,\mathrm{cn}[\omega\xi]), \ \omega^2 = \dfrac{2p}{1+m^2}, \quad r = \dfrac{\omega^4(1-m^2)^2}{16q}, \\ \phi_{15} = \dfrac{\omega}{2\sqrt{q}}(m\mathrm{sn}[\omega\xi] \pm \imath\,\mathrm{dn}[\omega\xi]), \quad \omega^2 = \dfrac{2p}{1-2m^2}, \quad r = \dfrac{\omega^4}{16q}, \end{cases} \tag{3.5}$$

$$\begin{cases} \phi_{16} = \dfrac{\omega}{2\sqrt{q}}(\mathrm{ds}[\omega\xi] \pm \mathrm{cs}[\omega\xi]), & \omega^2 = \dfrac{2p}{1+m^2}, \quad r = \dfrac{\omega^4(1-m^2)^2}{16q}, \\ \phi_{17} = \dfrac{\omega}{2\sqrt{q}}(\mathrm{ns}[\omega\xi] \pm \mathrm{cs}[\omega\xi]), & \omega^2 = \dfrac{2p}{1-2m^2}, \quad r = \dfrac{\omega^4}{16q}, \\ \phi_{18} = \dfrac{\omega}{2\sqrt{q}}(\mathrm{ns}[\omega\xi] \pm \mathrm{ds}[\omega\xi]), & \omega^2 = \dfrac{2p}{m^2-2}, \quad r = \dfrac{\omega^4 m^4}{16q}, \end{cases} \tag{3.6}$$

$$\begin{cases} \phi_{19} = \dfrac{\omega}{2\sqrt{q}}(\sqrt{1-m^2}\mathrm{nc}[\omega\xi] \pm \mathrm{dc}[\omega\xi]), & \omega^2 = \dfrac{2p}{m^2-2}, \quad r = \dfrac{\omega^4 m^4}{16q}, \\ \phi_{20} = \dfrac{\omega}{2\sqrt{q}}(\sqrt{m^2-1}\mathrm{sc}[\omega\xi] \pm \mathrm{dc}[\omega\xi]), & \omega^2 = \dfrac{2p}{1-2m^2}, \quad r = \dfrac{\omega^4}{16q}, \\ \phi_{21} = \dfrac{\omega\sqrt{1-m^2}}{2\sqrt{q}}(\mathrm{sc}[\omega\xi] \pm \mathrm{nc}[\omega\xi]), & \omega^2 = \dfrac{2p}{1+m^2}, \quad r = \dfrac{\omega^4(1-m^2)^2}{16q}, \end{cases} \tag{3.7}$$

$$\begin{cases} \phi_{22} = \dfrac{\omega}{2\sqrt{q}}(\sqrt{m^2-1}\mathrm{nd}[\omega\xi] \pm m\mathrm{cd}[\omega\xi]), & \omega^2 = \dfrac{2p}{1-2m^2}, \quad r = \dfrac{\omega^4}{16q}, \\ \phi_{23} = \dfrac{\omega m}{2\sqrt{q}}(\sqrt{m^2-1}\mathrm{sd}[\omega\xi] \pm \mathrm{cd}[\omega\xi]), & \omega^2 = \dfrac{2p}{m^2-2}, \quad r = \dfrac{\omega^4 m^4}{16q} \\ \phi_{24} = \dfrac{\omega\sqrt{m^2-1}}{2\sqrt{q}}(m\,\mathrm{sd}[\omega\xi] \pm \mathrm{nd}[\omega\xi]), & \omega^2 = \dfrac{2p}{1+m^2}, \quad r = \dfrac{\omega^4(1-m^2)^2}{16q}. \end{cases} \tag{3.8}$$

Also, we can obtain Weierstrass elliptic function solutions of Eq. (2.1) as the following:

$$\begin{cases} \phi_{25} = \dfrac{3\wp'(\xi; g_2, g_3)}{\sqrt{q}[6\wp(\xi; g_2, g_3) + p]}, \qquad \phi_{26} = \dfrac{\sqrt{r}[6\wp(\xi; g_2, g_3) + p]}{3\wp'(\xi; g_2, g_3)}, \\ g_2 = rq + \dfrac{p^2}{12}, \qquad g_3 = \dfrac{p(36rq - p^2)}{216}, \\ \phi_{27} = \sqrt{\dfrac{3\wp(\xi; g_2, g_3) - p}{3q}}, \qquad \phi_{28} = \sqrt{\dfrac{3r}{3\wp(\xi; g_2, g_3) - p}}, \\ g_2 = \dfrac{4}{3}(p^2 - 3rq), \qquad g_3 = \dfrac{4}{27}(9rpq - 2p^3), \\ \phi_{29} = \sqrt{\dfrac{-15p^3}{2q}}\left[\dfrac{\wp(\xi; g_2, g_3)}{3\wp(\xi; g_2, g_3) + p}\right], \qquad \phi_{30} = \sqrt{\dfrac{-2r}{15p^3}}\left[\dfrac{3\wp(\xi; g_2, g_3) + p}{\wp(\xi; g_2, g_3)}\right], \\ r = \dfrac{5p^2}{36q}, \qquad g_2 = \dfrac{2p^2}{9}, \qquad g_3 = \dfrac{p^3}{54}, \\ \phi_{31} = \dfrac{12\wp(\xi; g_2, g_3) + C}{\sqrt{2q\,[6\wp(\xi; g_2, g_3) + 2p + C]}}, \qquad \phi_{32} = \dfrac{\sqrt{2r[6\wp(\xi; g_2, g_3) + 2p + C]}}{12\wp(\xi; g_2, g_3) + C}, \\ C = \dfrac{-5p \pm 3\sqrt{p^2 - 4rq}}{2}, \qquad g_2 = \dfrac{p(5C + 4p + 33rq)}{-12}, \\ g_3 = \dfrac{p^2(21C + 20p) - rq(63C - 27p)}{216}, \end{cases} \tag{3.9}$$

where $\wp(\xi; g_2, g_3)$ is called a Weierstrass elliptic function which satisfies

$$\wp' = \frac{d\wp}{d\xi} = \pm\sqrt{4\wp^3 - g_2\wp - g_3}. \tag{3.10}$$

Remark 2. The Jacobi elliptic solutions 1–6 and 13–18 are introduced in our papers [25,34] while the solutions 7–12 and 19–24 are new Jacobi elliptic function solutions for Eq. (2.1).

Remark 3. The Weierstrass elliptic solutions 25, 27, 29 and 31 are introduced in our papers [25,34] while the solutions 26, 28, 30 and 32 are new double periodic Weierstrass elliptic function solutions for Eq. (2.1).

Remark 4. In special cases of p and q the new solutions (3.1)–(3.6) of Eq. (2.1) admit the solutions introduced by Chen and Wang [30]

$$\begin{cases} \phi = \pm \text{sn}\xi, \qquad r = 1, \qquad p = -(1 + m^2), \qquad q = m^2, \\ \phi = \pm \text{cn}\xi, \qquad r = 1 - m^2, \qquad p = 2m^2 - 1, \qquad q = -m^2, \\ \phi = \pm \text{dn}\xi, \qquad r = m^2 - 1, \qquad p = 2 - m^2, \qquad q = -1, \end{cases} \tag{3.11}$$

$$\begin{cases} \phi = \pm \text{sc}\xi, \qquad r = 1, \qquad p = 2 - m^2, \qquad q = 1 - m^2, \\ \phi = \pm \text{sd}\xi, \qquad r = 1, \qquad p = 2m^2 - 1, \qquad q = m^4 - m^2, \\ \phi = \pm \text{dc}\xi, \qquad r = m^2, \qquad p = -(1 + m^2), \qquad q = 1, \end{cases} \tag{3.12}$$

$$\begin{cases} \phi = \pm \text{dc}[\omega\xi], \qquad r = m^2, \qquad p = -1 - m^2 \qquad q = 1, \\ \phi = \pm \text{nc}[\omega\xi], \qquad r = -m^2, \qquad p = 2m^2 - 1 \qquad q = 1 - m^2, \\ \phi = \text{sc}[\omega\xi], \qquad r = 1, \qquad p = 2 - m^2 \qquad q = 1 - m^2, \end{cases} \tag{3.13}$$

$$\begin{cases} \phi = \pm \text{cd}[\omega\xi], \qquad r = 1, \qquad p = -1 - m^2 \qquad q = m^2, \\ \phi = \text{nd}[\omega\xi], \qquad r = -1, \qquad p = 2 - m^2 \qquad q = m^2 - 1, \\ \phi = \text{sd}[\omega\xi], \qquad r = 1, \qquad p = 2m^2 - 1 \qquad q = m^4 - m^2, \end{cases} \tag{3.14}$$

$$\begin{cases} \phi = \pm(\text{sn}\xi \pm \imath\text{cn}\xi), \qquad r = q = \dfrac{m^2}{4}, \qquad p = \dfrac{m^2 - 2}{2}, \\ \phi = \pm(m\text{sn}\xi \pm \imath\text{dn}\xi), \qquad r = q = \dfrac{-1}{4}, \qquad p = \dfrac{1 - 2m^2}{2}, \\ \phi = \pm(m\text{cn}\xi \pm \text{dn}\xi), \qquad r = \dfrac{-(1 - m^2)^2}{4}, \qquad p = \dfrac{1 + m^2}{2}, \qquad q = \dfrac{-1}{4}, \end{cases} \tag{3.15}$$

$$\begin{cases} \phi = \dfrac{\pm\mathrm{sn}\xi}{1\pm \mathrm{dn}\xi}, \quad r=\dfrac{1}{4}, \quad p=\dfrac{m^2-2}{2}, \quad q=\dfrac{m^4}{4}, \\ \phi = \dfrac{\pm\mathrm{sn}\xi}{1\pm \mathrm{cn}\xi}, \quad r=q=\dfrac{1}{4}, \quad p=\dfrac{1-2m^2}{2}, \\ \phi = \dfrac{\pm\mathrm{dn}\xi}{1\pm m\mathrm{sn}\xi}, \quad r=q=\dfrac{m^2-1}{4}, \quad p=\dfrac{1+m^2}{2}, \end{cases} \tag{3.16}$$

$$\begin{cases} \phi = \dfrac{\pm\mathrm{dn}\xi}{\sqrt{m^2-1}\pm m\mathrm{cn}\xi}, \quad r=q=\dfrac{1}{4}, \quad p=\dfrac{1-2m^2}{2}, \\ \phi = \dfrac{\pm\mathrm{cn}\xi}{1\pm \mathrm{sn}\xi}, \quad r=q=\dfrac{1-m^2}{4}, \quad p=\dfrac{1+m^2}{2}, \\ \phi = \dfrac{\pm\mathrm{cn}\xi}{\sqrt{1-m^2}\pm \mathrm{dn}\xi}, \quad r=\dfrac{-1}{4}, \quad p=\dfrac{2-m^2}{2}, \quad q=\dfrac{-m^4}{4}, \end{cases} \tag{3.17}$$

$$\begin{cases} \phi = \dfrac{\pm\mathrm{sn}\xi}{\mathrm{dn}\xi\pm \mathrm{cn}\xi}, \quad r=\dfrac{1}{4}, \quad p=\dfrac{1+m^2}{2}, \quad q=\dfrac{(1-m^2)^2}{4}, \\ \phi = \dfrac{\pm\mathrm{cn}\xi}{\sqrt{1-m^2}\mathrm{sn}\xi\pm \mathrm{dn}\xi}, \quad r=q=\dfrac{1}{4}, \quad p=\dfrac{1-2m^2}{2}, \\ \phi = \dfrac{\pm\mathrm{dn}\xi}{\sqrt{m^2-1}\mathrm{sn}\xi\pm \mathrm{cn}\xi}, \quad r=q=\dfrac{m^2}{4}, \quad p=\dfrac{m^2-2}{2}. \end{cases} \tag{3.18}$$

Remark 5. Our proposed method may be called the *new generalized Jacobi elliptic function rational expansion method*. The special solutions of this method are the general results of the Jacobi elliptic function method [28,30,25,34,21,31] and the mapping method [16,17]. Furthermore, the proposed method is computerized in solving nonlinear equations by using symbolic software like *Mathematica* or *Maple*.

4. New exact solution of (3 + 1)-dimensional KP equation

Let us now consider the (3 + 1)-dimensional KP equation [33,35]

$$u_{xt}-6u_x^2-6uu_{xx}-u_{xxxx}-u_{yy}-u_{zz}=0. \tag{4.1}$$

The KP equation is of considerable importance both in physics and Mathematics. The KP equation arises in many physical applications including two-dimensional long waves in shallow [36,37]. Its mathematical significance is related to the fact that it is an integrable soliton equation which was solved in [38,39]. According to Section 2, we first make the following traveling wave transformation:

$$u(x,y,z,t)=u(\xi), \quad \xi=k(x+\alpha y+\beta z-\gamma t), \tag{4.2}$$

where k, α, β and γ are constants to be determined. Substituting (4.2) into (4.1) we gives rise to

$$k^2u''''+(\alpha^2+\beta^2+\gamma)u''+6(uu''+u'^2)=0. \tag{4.3}$$

Balancing the highest derivative term with the nonlinear term in ODE (4.3) we have $n=2$. The solution (2.6) of KP equation becomes

$$u(\xi)=A_0+A_1\phi+A_2\phi^2+\frac{\phi'}{\phi^2}[a_0+a_1\phi+a_2\phi^2]+\frac{\Gamma}{B_0+B_1\phi+B_2\phi^2+\frac{\phi'}{\phi^2}[b_0+b_1\phi+b_2\phi^2]}, \tag{4.4}$$

where ϕ satisfy the Jacobi elliptic equation (2.1). Substituting (4.4) into (4.3) along with (2.1) and using *Mathematica program* yields a system of equations of powers of ϕ and ϕ'. Setting the coefficients of powers of ϕ and ϕ' in the system of equations to zero and solving the system of algebraic equations, with respect to the unknowns $A_0, A_1, A_2, B_0, B_1, B_2, a_0, a_1, a_2, b_0, b_1, b_2$ and Γ. This resulting algebraic system is difficult to solve in all unknowns, therefore we study two cases:

First when $b_0=b_1=b_2=0$. In this case finding solutions becomes less than difficult.

Second when one of the three unknowns b_0, b_1 or b_2 is nonzero, the calculations are very hard, so that to simplify, we choose $b_0\neq 0$ say $b_0=1$, we do not arrive at any solutions. Similarly when $b_1\neq 0$ we have no solutions, but when $b_2\neq 0$ we arrive to two general families of solutions. The details of these two cases and families of solutions are the following:

Case 1. $b_0=b_1=b_2=0$.

Family 1.1. $A_1 = B_0 = B_1 = a_0 = a_1 = a_2 = 0,\ B_2 = 1.$

$$\begin{cases} 1.\ A_0 = \dfrac{-1}{6}(4pk^2+\alpha^2+\beta^2+\gamma), & A_2 = -2qk^2, \quad \Gamma = 0, \\ 2.\ A_0 = \dfrac{-1}{6}(4pk^2+\alpha^2+\beta^2+\gamma), & \Gamma = -2rk^2, \quad A_2 = 0, \\ 3.\ A_0 = \dfrac{-1}{6}(4pk^2+\alpha^2+\beta^2+\gamma), & A_2 = -2qk^2, \quad \Gamma = -2rk. \end{cases} \tag{4.5}$$

The solutions of the KP equation (4.1) in the general form can be taken as:

$$u_{1.1}(x,y,z,t) = -\frac{4pk^2+\alpha^2+\beta^2+\gamma}{6} - 2k^2[\varepsilon q\phi^2(\xi) + \eta r\phi^{-2}(\xi)], \tag{4.6}$$

where $p, q, k, \alpha, \beta, \gamma$ are arbitrary constants, ε, η are arbitrary elements of $\{0, 1\}$ and $\xi = k(x + \alpha y + \beta z - \gamma t)$. Then the set of solutions (3.1)–(3.9) of Eq. (2.1) yield 2-parameter families of Jacobi and Weierstrass double periodic elliptic functions solutions of Eq. (4.1) which are introduced by El-Sabbagh and Ali [14]. Thus we shall not write them here.

Family 1.2. $A_1 = a_1 = B_0 = B_1 = 0, B_2 = 1.$

$$\begin{cases} 4.\ A_0 = \dfrac{-1}{6}(pk^2+\alpha^2+\beta^2+\gamma), & A_2 = -qk^2, \quad a_2 = \pm k^2\sqrt{q}, \quad \Gamma = a_0 = 0, \\ 5.\ A_0 = \dfrac{-1}{6}(pk^2+\alpha^2+\beta^2+\gamma), & \Gamma = -rk^2, \quad a_0 = \pm k^2\sqrt{q}, \quad A_2 = a_2 = 0, \\ 6.\ A_0 = \dfrac{a_0a_2}{k^2} - \dfrac{1}{6}(pk^2+\alpha^2+\beta^2+\gamma), & \begin{cases} A_2 = -qk^2, & \Gamma = -rk^2, \\ a_2 = \pm k^2\sqrt{q}, & a_0 = \pm k^2\sqrt{r}. \end{cases} \end{cases} \tag{4.7}$$

The exact solution of KP equation (4.1) in the general form leads to the following solutions

$$\begin{aligned} u_{1.2}(x,y,z,t) &= -\frac{pk^2+\alpha^2+\beta^2+\gamma}{6} + \varepsilon\eta\varepsilon_1\eta_1k^2\sqrt{qr} \\ &\quad - \varepsilon k^2[q\phi^2(\xi) - \sqrt{q}\varepsilon_1\phi'(\xi)] - \eta k^2\phi^{-2}(\xi)[r - \eta_1\sqrt{r}\phi'(\xi)], \end{aligned} \tag{4.8}$$

where $p, q, k, \alpha, \beta, \gamma$ are arbitrary constants, ε, η are arbitrary elements of $\{0, 1\}$, $\varepsilon_1 = \pm 1$, $\eta_1 = \pm 1$, and $\xi = k(x + \alpha y + \beta z - \gamma t)$. Then the set of solutions (3.1)–(3.9) of Eq. (2.1) yield 4-parameters families of Jacobi and Weierstrass double periodic elliptic functions solutions of Eq. (4.1) which are introduced by El-Sabbagh and Ali [33]. Thus we shall not write them here also.

Family 1.3. $A_1 = A_2 = a_0 = a_1 = a_2 = B_1 = 0,\ B_2 = 1.$

$$\begin{cases} \text{(a)}\ A_0 = \mu_0 + k^2\theta, & \Gamma = -\dfrac{k^2\theta(p+\theta)}{q}, \quad B_0 = \dfrac{p+\theta}{2q}, \\ \text{(b)}\ A_0 = \mu_0 - k^2\theta, & \Gamma = \dfrac{k^2\theta(p-\theta)}{q}, \quad B_0 = \dfrac{p-\theta}{2q}, \end{cases} \tag{4.9}$$

where $\mu_0 = \frac{2pk^2-\alpha^2-\beta^2-\gamma}{6}$ and $\theta = \sqrt{p^2-4qr}$. The exact solution (4.4) can take the general form

$$u_{1.3}(x,y,z,t) = \frac{2pk^2-\alpha^2-\beta^2-\gamma}{6} + \eta_1k^2\sqrt{p^2-4qr}\left[\frac{2q\phi^2(\xi) - \eta_1\sqrt{p^2-4qr} - p}{2q\phi^2(\xi) + \eta_1\sqrt{p^2-4qr} + p}\right], \tag{4.10}$$

where $a, b, c, d, k, \alpha, \gamma$ are arbitrary constants, $\varepsilon_1 = \pm 1$, $\eta_1 = \pm 1$ and $\xi = k(x + \alpha y + \beta z - \gamma t)$. Then the set of solutions (3.1)–(3.9) yield new Jacobi and Weierstrass double periodic elliptic function solutions for Eq. (4.1) which are the completely new solutions. The exact solutions of the KP equation corresponding to ϕ_1, ϕ_{16} and ϕ_{25}, for example, are:

$$\begin{aligned} u_{1.3.1} &= \frac{2pk^2-\alpha^2-\beta^2-\gamma}{6} + k^2\eta_1\sqrt{p^2-4m^2\omega^4(m^2-1)} \\ &\quad \times \left[\frac{2\omega^2 m^2\,\mathrm{cn}^2[\omega\xi] - \eta_1\sqrt{p^2-4m^2\omega^4(m^2-1)} - p}{2\omega^2 m^2\,\mathrm{cn}^2[\omega\xi] + \eta_1\sqrt{p^2-4m^2\omega^4(m^2-1)} + p}\right], \end{aligned} \tag{4.11}$$

$$\begin{aligned} u_{1.3.16} &= \frac{2pk^2-\alpha^2-\beta^2-\gamma}{6} - \frac{k^2\eta_1\sqrt{4p^2-(1-m^2)^2\omega^4}}{4} \\ &\quad \times \left[\frac{4p + \eta_1\sqrt{4p^2-(1-m^2)^2\omega^4} - 2\omega^2(\mathrm{ds}[\omega\xi] \pm \mathrm{cs}[\omega\xi])^2}{4p + \eta_1\sqrt{4p^2-(1-m^2)^2\omega^4} + 2\omega^2(\mathrm{ds}[\omega\xi] \pm \mathrm{cs}[\omega\xi])^2}\right], \end{aligned} \tag{4.12}$$

$$u_{1.3.25} = \frac{1}{6}(2pk^2-\alpha^2-\beta^2-\gamma) - \frac{3dk^2\eta_1\sqrt{p^2-4qr}}{c} \times \left[\frac{9(p+\eta_1\sqrt{p^2-4qr})\wp'(\xi;g_2,g_3) - 2qr(p+6\wp(\xi;g_2,g_3))}{9(p+\eta_1\sqrt{p^2-4qr})\wp'(\xi;g_2,g_3) + 2qr(p+6\wp(\xi;g_2,g_3))}\right], \tag{4.13}$$

where $g_2 = rq + \frac{p^2}{12}$ and $g_3 = \frac{p(36rq-p^2)}{216}$.

Family 1.4. $A_1 = A_2 = a_0 = a_1 = a_2 = B_2 = 0,\ B_1 = 1.$

$$\begin{cases} \text{(a) } A_0 = \mu_0 + \dfrac{k^2\theta}{2}, & \Gamma = k^2\theta\sqrt{\dfrac{-\theta-p}{2q}}, \quad B_0 = -\sqrt{\dfrac{-\theta-p}{2q}}, \\ \text{(b) } A_0 = \mu_0 + \dfrac{k^2\theta}{2}, & \Gamma = -k^2\theta\sqrt{\dfrac{-\theta-p}{2q}}, \quad B_0 = \sqrt{\dfrac{-\theta-p}{2q}}, \\ \text{(c) } A_0 = \mu_0 - \dfrac{k^2\theta}{2}, & \Gamma = k^2\theta\sqrt{\dfrac{\theta-p}{2q}}, \quad B_0 = \sqrt{\dfrac{\theta-p}{2q}}, \\ \text{(d) } A_0 = \mu_0 - \dfrac{k^2\theta}{2}, & \Gamma = -k^2\theta\sqrt{\dfrac{\theta-p}{2q}}, \quad B_0 = -\sqrt{\dfrac{\theta-p}{2q}}, \end{cases} \tag{4.14}$$

where $\mu_0 = \frac{2pk^2-\alpha^2-\beta^2-\gamma}{6}$ and $\theta = \sqrt{p^2-4qr}$. The exact solution (4.4) takes the general form

$$u_{1.4}(x,y,z,t) = \frac{2pk^2-\alpha^2-\beta^2-\gamma}{6} + \frac{\eta_1 k^2\sqrt{p^2-4qr}}{2}\left[\frac{\sqrt{\eta_1\sqrt{p^2-4qr}-p} + \varepsilon_1\sqrt{2q}\,\phi(\xi)}{\sqrt{\eta_1\sqrt{p^2-4qr}-p} - \varepsilon_1\sqrt{2q}\,\phi(\xi)}\right], \tag{4.15}$$

where $a, b, c, d, k, \alpha, \gamma$ are arbitrary constants, $\varepsilon_1 = \pm 1$, $\eta_1 = \pm 1$ and $\xi = k(x+\alpha y+\beta z-\gamma t)$. Then the set of solutions (3.1)–(3.9) yield new Jacobi and Weierstrass double periodic elliptic function solutions for Eq. (4.1) which are new solutions. The exact solutions of the KP equation corresponding to ϕ_1, ϕ_{16} and ϕ_{25}, for example, are:

$$u_{1.4.1} = \frac{2pk^2-\alpha^2-\beta^2-\gamma}{6} + \frac{k^2\eta_1\sqrt{p^2-4m^2\omega^4(m^2-1)}}{\sqrt{2}} \times \left[\frac{\sqrt{\eta_1\sqrt{p^2-4m^2\omega^4(m^2-1)}-p} - \sqrt{2}\varepsilon_1\omega\, m\,\mathrm{cn}[\omega\xi]}{\sqrt{\eta_1\sqrt{p^2-4m^2\omega^4(m^2-1)}-p} + \sqrt{2}\varepsilon_1\omega\, m\,\mathrm{cn}[\omega\xi]}\right], \tag{4.16}$$

$$u_{1.4.16} = \frac{1}{6}(2pk^2-\alpha^2-\beta^2-\gamma) + \frac{k^2\eta_1\sqrt{4p^2-(1-m^2)^2\omega^4}}{2\sqrt{2}} \times \left[\frac{\sqrt{\eta_1\sqrt{4p^2-(1-m^2)^2\omega^4}-2p} + \varepsilon_1\omega\,(\mathrm{ds}[\omega\xi] \pm \mathrm{cs}[\omega\xi])}{\sqrt{\eta_1\sqrt{4p^2-(1-m^2)^2\omega^4}-2p} - \varepsilon_1\omega\,(\mathrm{ds}[\omega\xi] \pm \mathrm{cs}[\omega\xi])}\right], \tag{4.17}$$

$$u_{1.4.25} = \frac{2pk^2-\alpha^2-\beta^2-\gamma}{6} + \frac{k^2\eta_1\sqrt{p^2-4qr}}{\sqrt{2}} \times \left[\frac{\sqrt{\eta_1\sqrt{p^2-4qr}-p}(6\wp(\xi;g_2,g_3)+p) - 3\sqrt{2}\varepsilon_1\wp'(\xi;g_2,g_3)}{\sqrt{\eta_1\sqrt{p^2-4qr}-p}(6\wp(\xi;g_2,g_3)+p) + 3\sqrt{2}\varepsilon_1\wp'(\xi;g_2,g_3)}\right], \tag{4.18}$$

where $g_2 = rq + \frac{p^2}{12}$ and $g_3 = \frac{p(36rq-p^2)}{216}$.

Case 2. $b_0 = b_1 = 0$ and $b_2 = 1$.

Family 2.1. $A_1 = A_2 = B_1 = a_0 = a_1 = a_2 = 0.$

$$\begin{cases} \text{(a) } A_0 = \mu_0, & \Gamma = -\dfrac{k^2(p^2-4qr)}{4\sqrt{q}}, \quad B_0 = \dfrac{p}{2\sqrt{q}}, \quad B_2 = \sqrt{q}, \\ \text{(b) } A_0 = \mu_0, & \Gamma = \dfrac{k^2(p^2-4qr)}{4\sqrt{q}}, \quad B_0 = -\dfrac{p}{2\sqrt{q}}, \quad B_2 = -\sqrt{q}, \end{cases} \tag{4.19}$$

where $\mu_0 = \frac{2pk^2-\alpha^2-\beta^2-\gamma}{6}$. The exact solution (4.4) takes the general form

$$u_{2.1}(x,y,z,t) = \frac{2pk^2-\alpha^2-\beta^2-\gamma}{6} - \frac{k^2(p^2-4qr)}{2[p+2q\phi^2(\xi)+2\eta_1\sqrt{q}\phi'(\xi)]}, \quad (4.20)$$

where $a, b, c, d, k, \alpha, \gamma$ are arbitrary constants, $\eta_1 = \pm 1$ and $\xi = k(x+\alpha y+\beta z-\gamma t)$. Then the set of solutions (3.1)–(3.9) yield new Jacobi and Weierstrass double periodic elliptic function solutions for Eq. (4.1) which are also new solutions. The exact solutions of the KP equation corresponding to ϕ_1, ϕ_{16} and ϕ_{25}, for examples, are:

$$u_{2.1.1} = \frac{2pk^2-\alpha^2-\beta^2-\gamma}{6} + \frac{k^2}{2}\left[\frac{p^2-4m^2\omega^4(m^2-1)}{p+2\omega^2 m^2\,\mathrm{cn}^2[\omega\xi]-2\eta_1 m\omega^2\mathrm{sn}[\omega\xi]\mathrm{dn}[\omega\xi]}\right], \quad (4.21)$$

$$u_{2.1.16} = \frac{2pk^2-\alpha^2-\beta^2-\gamma}{6} - \frac{k^2}{4}\left[\frac{4p^2-(1-m^2)^2\omega^4}{2p+\omega^2(\mathrm{ds}[\omega\xi]\pm\mathrm{cs}[\omega\xi])^2-2\eta_1\omega^2\mathrm{ns}[\omega\xi](\mathrm{cs}[\omega\xi]\pm\mathrm{ds}[\omega\xi])}\right], \quad (4.22)$$

$$\begin{aligned} u_{2.1.25} = {} & \frac{2pk^2-\alpha^2-\beta^2-\gamma}{6} - \frac{3dk^2(p^2-4qr)(p+6\wp(\xi;g_2,g_3))}{2c} \\ & \times\Big[p^3+12p^2\wp(\xi;g_2,g_3)-36p(\eta_1-1)\wp^2(\xi;g_2,g_3)-216\eta_1\wp^3(\xi;g_2,g_3) \\ & +3\eta_1 g_2(p+6\wp(\xi;g_2,g_3))+18(1+2\eta_1)\wp'^2(\xi;g_2,g_3)\Big]^{-1}, \end{aligned} \quad (4.23)$$

where $g_2 = rq + \frac{p^2}{12}$ and $g_3 = \frac{p(36rq-p^2)}{216}$.

Family 2.2. $A_1 = A_2 = B_1 = a_0 = a_1 = a_2 = 0$.

$$\begin{cases} \text{(a) } A_0 = \mu_0 - \frac{k^2(p+2\sqrt{qr})}{2}, & \Gamma = -k^2\sqrt{r}(p+2\sqrt{qr}), \quad B_0 = -\sqrt{r}, \quad B_2 = \sqrt{q}, \\ \text{(b) } A_0 = \mu_0 - \frac{k^2(p-2\sqrt{qr})}{2}, & \Gamma = k^2\sqrt{r}(p-2\sqrt{qr}), \quad B_0 = \sqrt{r}, \quad B_2 = \sqrt{q}, \\ \text{(c) } A_0 = \mu_0 - \frac{k^2(p+2\sqrt{qr})}{2}, & \Gamma = k^2\sqrt{r}(p+2\sqrt{qr}), \quad B_0 = \sqrt{r}, \quad B_2 = -\sqrt{q}, \\ \text{(d) } A_0 = \mu_0 - \frac{k^2(p-2\sqrt{qr})}{2}, & \Gamma = -k^2\sqrt{r}(p-2\sqrt{qr}), \quad B_0 = -\sqrt{r}, \quad B_2 = -\sqrt{q}, \end{cases} \quad (4.24)$$

where $\mu_0 = \frac{2pk^2-\alpha^2-\beta^2-\gamma}{6}$. The exact solution of Eq. (4.1) in general form takes the following form

$$u_{2.2}(x,y,z,t) = \frac{2pk^2-\alpha^2-\beta^2-\gamma}{6} - \frac{k^2(p+2\varepsilon_1\sqrt{qr})}{2} + \frac{k^2\sqrt{r}(p+\varepsilon_1\sqrt{qr})}{\left[\sqrt{r}-\varepsilon_1\sqrt{q}\phi^2(\xi)+\eta_1\phi'(\xi)\right]}, \quad (4.25)$$

where $a, b, c, d, k, \alpha, \gamma$ are arbitrary constants, $\varepsilon_1 = \pm 1$, $\eta_1 = \pm 1$ and $\xi = k(x+\alpha y+\beta z-\gamma t)$. Then the set of solutions (3.1)–(3.9) yield new Jacobi and Weierstrass double periodic elliptic function solutions for Eq. (4.1). The exact solutions of the KP equation corresponding to ϕ_1, ϕ_{16} and ϕ_{25}, for example, are:

$$\begin{aligned} u_{2.2.1} = {} & \frac{2pk^2-\alpha^2-\beta^2-\gamma}{6} - \frac{k^2}{2}(p+2\varepsilon_1\omega^2 m\sqrt{m^2-1}) \\ & + \frac{k^2\omega^2 m\sqrt{m^2-1}(p+\varepsilon_1 m\,\omega^2\sqrt{m^2-1})}{m\omega^2\sqrt{m^2-1}-\varepsilon_1 m^2\omega^2\,\mathrm{cn}^2[\omega\xi]-\eta_1 m\omega^2\mathrm{sn}[\omega\xi]\mathrm{dn}[\omega\xi]}, \end{aligned} \quad (4.26)$$

$$\begin{aligned} u_{2.2.16} = {} & \frac{2pk^2-\alpha^2-\beta^2-\gamma}{6} - \frac{k^2}{4}\Bigg[2p-\varepsilon_1\omega^2(1-m^2) \\ & + \frac{6(m^2-1)(2p+\varepsilon_1\omega^2(m^2-1))}{1-m^2+\varepsilon_1(\mathrm{cs}[\omega\xi]+\mathrm{ds}[\omega\xi])^2+2\eta_1\,\mathrm{ns}[\omega\xi](\mathrm{cs}[\omega\xi]+\mathrm{ds}[\omega\xi])}\Bigg], \end{aligned} \quad (4.27)$$

$$\begin{aligned} & u_{2.2.25} = \frac{1}{6}(2pk^2-\alpha^2-\beta^2-\gamma) - \frac{k^2}{2}\left[\frac{H}{G}\right], \\ & H = p+2\varepsilon_1\sqrt{qr}+24\sqrt{qr}(p+2\varepsilon_1\sqrt{qr})(p+6\wp(\xi;g_2,g_3))^2 \\ & G = 4\sqrt{qr}(p+6\wp(\xi;g_2,g_3))^2-9\varepsilon_1(g_2-12\wp(\xi;g_2,g_3))^2 \\ & \quad +36\eta_1(g_2+4\wp(\xi;g_2,g_3)[p+3\wp(\xi;g_2,g_3)])\wp'(\xi;g_2,g_3), \end{aligned} \quad (4.28)$$

where $g_2 = rq + \frac{p^2}{12}$ and $g_3 = \frac{p(36rq-p^2)}{216}$.

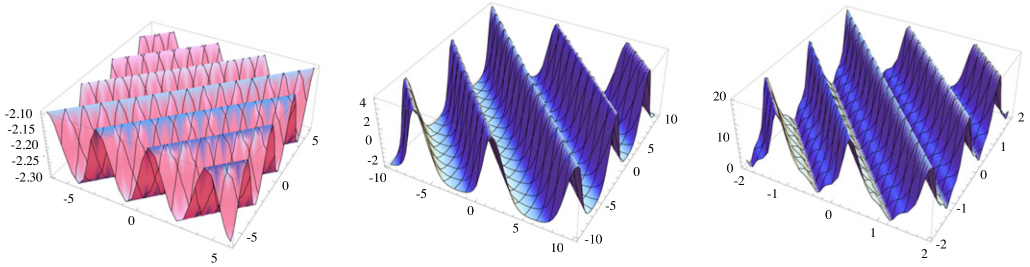

Fig. 1. Solutions corresponding to $u_{1.3.1}$, $u_{1.3.16}$ and $u_{1.3.25}$ for $\alpha = \beta = 0$.

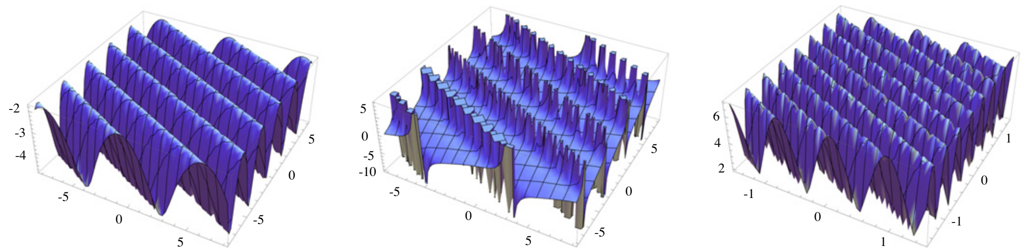

Fig. 2. Solutions corresponding to $u_{1.4.1}$, $u_{1.4.16}$ and $u_{1.4.25}$ for $\alpha = \beta = 0$.

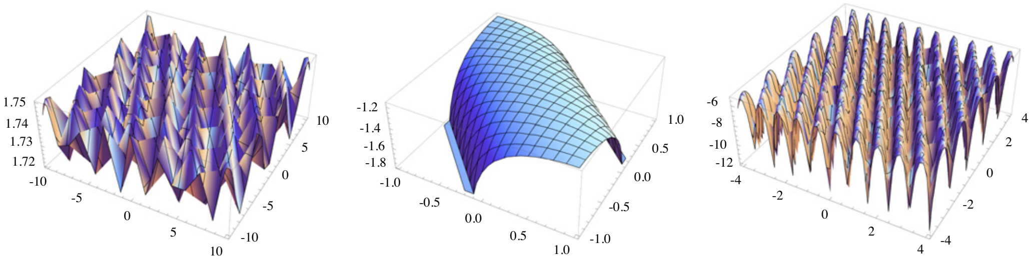

Fig. 3. Solutions corresponding to $u_{2.1.1}$, $u_{2.1.16}$ and $u_{2.1.25}$ for $\alpha = \beta = 0$.

Remark 6. When $m \to 0$, then $\text{sn}\xi \to \sin\xi$, $\text{cn}\xi \to \cos\xi$ and $\text{dn}\xi \to 1$, we obtain new triangular periodic wave solutions of the KP equation.

Remark 7. When $m \to 1$, then $\text{sn}\xi \to \tanh\xi$, $\text{cn}\xi \to \text{sech}\,\xi$ and $\text{dn}\xi \to \text{sech}\,\xi$, we can obtain new hyperbolic soliton wave solutions of the KP equation.

Remark 8. The $(3+1)$-dimensional KP equation (4.1) can be written in the simple form:

$$(u_t - 6uu_x - u_{xxx})_x - u_{yy} - u_{zz} = 0. \tag{4.29}$$

One can see immediately that every solution of the KdV equation

$$u_t - 6uu_x - u_{xxx} = 0, \tag{4.30}$$

gives the solution to the KP equation by giving u trivial y and z-dependence. So that, when fixing the variables y and z by putting $\alpha = \beta = 0$ in the new solutions of the KP equations (4.10), (4.15), (4.20) and (4.25), we obtain a new solution of the once-differentiated KdV equation $(u_t - 6uu_x - u_{xxx})_x = 0$. On other hand, by using Mathematica, the KdV equation (4.30) is verified by the solutions (4.10), (4.15), (4.20) and (4.25) for $\alpha = \beta = 0$.

Remark 9. It would be very nice if we had true figures which illustrate graphically some of the obtained new solutions of the KP equation (4.1) as well as the KdV equation (4.30) corresponding to a Family 1.3 (Fig. 1), Family 1.4 (Fig. 2), Family 2.1 (Fig. 3) and Family 2.2 (Fig. 4) in a special cases of the constants when fixing the variables y and z.

5. Conclusion

We introduced a new generalized Jacobi elliptic function rational expansion method and used it for constructing many new exact traveling wave solutions for nonlinear PDEs in a unified way. Also, we obtained many new Jacobi and Weierstrass

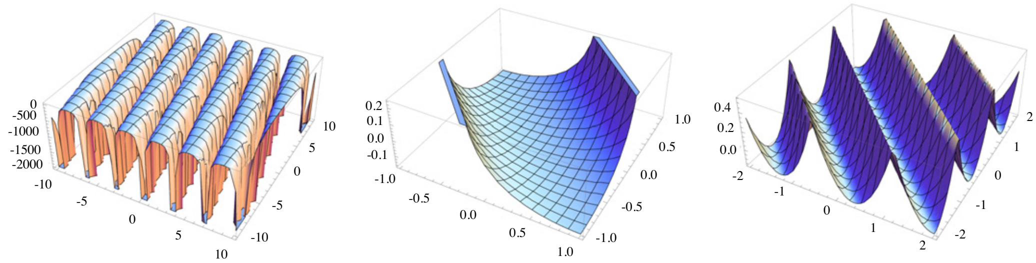

Fig. 4. Solutions corresponding to $u_{2.2.1}$, $u_{2.2.16}$ and $u_{2.2.25}$ for $\alpha = \beta = 0$.

double periodic elliptic function solutions for the (3 + 1)-dimensional KP equation as an application of this method. This generalized method can be applied to many other equations such as: the generalized Klein–Gordan equation [40], the (2+1)-dimensional Burger's equations [41], the Broer–Kaup–Kupershmidt equations [42], the foam drainage equation [43] and etc..., in a similar systematic way.

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