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Normal toric ideals of low codimension

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ABSTRACT

Every normal toric ideal of codimension two is minimally generated by a Gröbner basis with squarefree initial monomials. A polynomial time algorithm is presented for checking whether a toric ideal of fixed codimension is normal.

1. Introduction

Let $\mathcal{A}$ be a nonnegative integer $d \times n$ matrix of rank d, $C(\mathcal{A})$ the cone in $\mathbb{R}^d$ spanned by the columns of $\mathcal{A}$, and $I_{\mathcal{A}}$ the toric ideal associated to $\mathcal{A}$ as in [1]. The codimension of $I_{\mathcal{A}}$ is $n-d$. If the columns of $\mathcal{A}$ form a Hilbert basis, i.e. $C(\mathcal{A}) \cap \mathbb{Z}\mathcal{A} = \mathbb{N}\mathcal{A}$, then $\mathcal{A}$ and $I_{\mathcal{A}}$ are said to be *normal*. Our main result is:

Theorem 1. *Every normal toric ideal $I_{\mathcal{A}}$ of codimension 2 has a squarefree initial ideal, and the corresponding reduced Gröbner basis minimally generates $I_{\mathcal{A}}$.*

In Section 2 we introduce the necessary tools and prove Theorem 1 in the case of complete intersections. In Section 3 we complete the proof in the case of codimension two normal toric ideals which are not complete intersections. Section 4 deals with detecting normality and we prove the following result.

Theorem 2. *If $\operatorname{codim}(I_{\mathcal{A}}) = n - d$ is fixed, then there is an algorithm to decide whether $\mathcal{A}$ is normal whose running time is polynomial in n and the bit size of $\mathcal{A}$.*

In other words, if $n-d$ is fixed then one can decide in polynomial time whether n vectors in $\mathbb{Z}^d$ form a Hilbert basis of the cone these vectors generate. This answers a question raised by Alexander Barvinok at Snowbird (June 2006). We recently found out that Theorem 2 has been proved independently by F. Eisenbrand, A. Sebö, and G. Shmonin [2]. The authors had announced the result at the 12th Combinatorial Optimization Workshop held in Aussois, France in January 2008. For recent advances on related complexity questions concerning lattice points in polyhedra of fixed dimension or codimension we refer to article [3].

This work is motivated both by questions that are intrinsic to combinatorial commutative algebra and by applications to statistics and optimization. In the former domain, a longstanding conjecture states that every Cohen–Macaulay toric ideal $I_{\mathcal{A}}$

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has a monomial initial ideal that is also Cohen–Macaulay. This holds for toric ideals up to dimension 3 (see [4]), and O'Shea and Thomas proved it for Δ-normal configurations [5]. Theorem 1 offers supporting evidence because normal toric ideals and their squarefree initial ideals are both Cohen–Macaulay.

In algebraic statistics, our result ensures that the sequential importance sampling scheme of Chen, Dinwoodie and Sullivant [6] is applicable to exponential families with few states. In integer programming, it ensures that, for suitably chosen cost functions, every matrix of corank 2 specifies a Gomory family [4]. Finally, in algebraic geometry, where the definition of toric varieties [7] requires them to be normal, Theorem 1 states that every toric variety of codimension 2 admits a Gröbner degeneration to a reduced union of coordinate subspaces.

2. Complete intersections

The aim of this section is to prove Theorem 1 in the case when $I_{\mathcal{A}}$ is a complete intersection. Before we approach this proof we will prove an alternative characterization of normality due to Sebö [8]. We include a proof of this result.

Proposition 3. *An integer matrix $\mathcal{A} = [a_1, \ldots, a_n]$ is normal if and only if for each $x \in \ker(\mathcal{A})$ there exists an integer vector $y \in \ker(\mathcal{A})$ such that $y \leq \lceil x \rceil$.*

Proof. ($\Longrightarrow$) Suppose $\mathcal{A}$ is normal, and $x \in \ker \mathcal{A}$. We have $x_1 a_1 + \cdots + x_n a_n = 0$. The vector $z = \lceil x_1 \rceil \cdot a_1 + \cdots + \lceil x_n \rceil \cdot a_n$ lies in the lattice $\mathbb{Z}\mathcal{A}$, and as $z = (\lceil x_1 \rceil - x_1) \cdot a_1 + \cdots + (\lceil x_n \rceil - x_n) \cdot a_n$, it also lies in the cone $C(\mathcal{A})$. Since $\mathcal{A}$ is normal, we conclude that z is in the semigroup $\mathbb{N}\mathcal{A}$. We can write $z = m_1 a_1 + \cdots + m_n a_n$ with $m_1, \ldots, m_n$ nonnegative integers. The vector y with coordinates $y_i = \lceil x_i \rceil - m_i$ lies in $\ker(\mathcal{A})$ and satisfies $y \leq \lceil x \rceil$.

($\Longleftarrow$) Now suppose that for each $x \in \ker(\mathcal{A})$ there is an integral $y \in \ker(\mathcal{A})$ with $y \leq \lceil x \rceil$. Let $z \in C(\mathcal{A}) \cap \mathbb{Z}^d$. This means that $z = r_1 a_1 + \cdots + r_n a_n$ with $r_i \in \mathbb{R}_{\geq 0}$ and $z = m_1 a_1 + \cdots + m_n a_n$ with $m_i \in \mathbb{Z}$. Combining these we obtain:

$$(m_1 - r_1) \cdot a_1 + \cdots + (m_n - r_n) \cdot a_n = 0.$$

By hypothesis we may pick an integral $y \in \ker(\mathcal{A})$ with $y_i \leq \lceil m_i - r_i \rceil$ Then

$$z = (m_1 - y_1) \cdot a_1 + \cdots + (m_n - y_n) \cdot a_n$$

gives a nonnegative integral representation of z in terms of columns of $\mathcal{A}$, since $m_i - y_i \geq 0$ for all i. We conclude that $z \in \mathbb{N}\mathcal{A}$, and hence $\mathcal{A}$ is normal. □

Our main tool in what follows is the Gale diagram of a vector configuration. Let $\mathcal{A}$ be a integer $d \times n$ matrix whose column vectors span $\mathbb{R}^d$. We choose a matrix $\mathcal{B}$ whose rows form a lattice basis of $\ker(\mathcal{A}) \cap \mathbb{Z}^n$. The set of column vectors of $\mathcal{B}$ is said to be a *Gale diagram* [9] of $\mathcal{A}$. Normality of $I_{\mathcal{A}}$ is encoded in both $\mathcal{A}$ and in $\ker(\mathcal{A}) = \mathrm{im}(\mathcal{B})$, by Proposition 3, and hence also in $\mathcal{B}$.

Hochster [10] proved that a normal toric ideal is Cohen–Macaulay. Thus a codimension two normal toric ideal has a minimal free resolution of length two. Peeva and Sturmfels [11] characterized Cohen–Macaulay codimension two lattice ideals. In this paper we consider saturated lattices whose lattice ideal is a toric ideal. For the remainder of this paper we assume that $I_{\mathcal{A}}$ is normal, or equivalently that $\mathcal{A}$ is a Hilbert basis of the cone $C(\mathcal{A})$. We assume that the cone $C(\mathcal{A})$ is pointed and hence $I_{\mathcal{A}}$ is homogeneous in some positive grading.

The following result gives a supply of squarefree monomial terms of the binomial generators of $I_{\mathcal{A}}$. This result has been proven in [12, Proposition 4.1] and [13, Lemma 6.1], and we have also learned it from Winfried Bruns [14].

Proposition 4. *Suppose $\mathcal{A}$ is normal. Then each minimal binomial generator of the toric ideal $I_{\mathcal{A}}$ has at least one squarefree term.*

This implies that the conclusion of Theorem 1 holds when $\mathrm{codim}(I_{\mathcal{A}}) = 1$. In that case, I_A is a principal ideal and the unique binomial generator of $I_{\mathcal{A}}$ is a Gröbner basis with its squarefree term being the leading monomial.

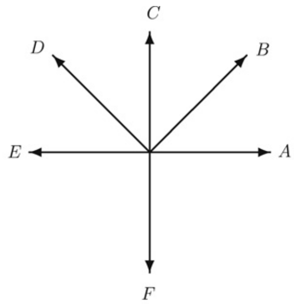
In view of Proposition 4 our approach is to show the existence of a term order selecting the squarefree terms as initial terms. The Gale diagram gives information toward this goal. The following result is [11, Proposition 4.1].

Proposition 5. *If $\mathrm{codim}(I_{\mathcal{A}}) = 2$ then the following are equivalent:*

(i) *The toric ideal $I_{\mathcal{A}}$ is not Cohen–Macaulay.*
(ii) *The toric ideal $I_{\mathcal{A}}$ has at least four minimal generators.*
(iii) *The matrix $\mathcal{A}$ has a Gale diagram $\mathcal{B}$ which intersects each of the four open quadrants in $\mathbb{R}^2$. Here the matrix $\mathcal{B}$ is identified with its set of columns.*

We now assume that $I_{\mathcal{A}}$ is normal of codimension 2 and $\mathcal{B}$ is any Gale diagram. Then $\mathcal{B} = \{(\mathcal{B}_{1j}, \mathcal{B}_{2j}) : j = 1, \ldots, n\}$ intersects at most three open quadrants, and that any minimal generating set of $I_{\mathcal{A}}$ has two or three elements. In this section we examine the first case, where $I_{\mathcal{A}} = \langle x^p - x^q, x^r - x^s \rangle$ is a complete intersection, and $\mathcal{B}$ is the $2 \times n$ matrix whose rows are $p - q$ and $r - s$. The Gale diagram $\mathcal{B}$ is said to be *imbalanced* if either $\mathcal{B}_{1j} = 0$ or $\mathcal{B}_{2j} \geq 0$ for all j.

Lemma 6 *([11, Lemma 3.1]). A codimension 2 toric ideal I_A is a complete intersection if and only if there exists an imbalanced Gale diagram $\mathcal{B}$.*



		A	B	C	D	E	F
$p-q$	$=$	$+$	$+$	0	$-$	$-$	0
$r-s$	$=$	0	$+$	$+$	$+$	0	$-$

Fig. 1. The imbalanced Gale diagram of a complete intersection.

In the light of this lemma, we can represent a complete intersection by an imbalanced Gale diagram as depicted in Fig. 1. The arrows represent a *sign class* of columns of $\mathcal{B}$ and not just an individual vector. For instance the class labeled D in Fig. 1 consists of all column vectors with $\mathcal{B}_{1j} < 0$ and $\mathcal{B}_{2j} > 0$.

Proof of Theorem 1 *for complete intersections:* By Proposition 4 both generators $g_1 = x^p - x^q$ and $g_2 = x^r - x^s$ have a squarefree term. The class of vectors F must exist in the Gale diagram since otherwise $C(\mathcal{A})$ would not be pointed.

If the term x^s corresponding to F is squarefree then we can use any term order so that x^s is the initial term of g_2 and the squarefree term of g_1 is its initial term. Then g_1 and g_2 forms the Gröbner basis of $I_{\mathcal{A}}$ because their initial terms are relatively prime. This Gröbner basis is or can be made reduced.

Suppose that x^s is not squarefree. Then $-\mathcal{B}_{2j} = f \geq 2$ for some $j \in F$, and $\mathcal{B}_{2j} = 1$ for $j \in B \cup C \cup D$. Without loss of generality we assume that x^p is the squarefree term of g_1, so that $\mathcal{B}_{1j} = 1$ for $j \in A \cup B$. We choose representatives from the D and E classes, labeling them $-d$ and $-e$ where $d, e \geq 1$:

$$\begin{array}{rcccccccccccc} & & A & & B & & C & & D & & E & & F \\ p-q & = & 1 & \dots & 1 & \dots & 0 & \dots & -d & \dots & -e & \dots & 0 \\ r-s & = & 0 & \dots & 1 & \dots & 1 & \dots & 1 & \dots & 0 & \dots & -f \end{array}$$

Now we consider $u = -\frac{1}{2}(p-q) + \frac{1}{2}(r-s) \in \ker(\mathcal{A})$ and we round it up to get

$$\begin{array}{rcccccccccccc} & & A & & B & & C & & D & & E & & F \\ \lceil u \rceil & = & 0 & \dots & 0 & \dots & 1 & \dots & \left\lceil \dfrac{d+1}{2} \right\rceil & \dots & \left\lceil \dfrac{e}{2} \right\rceil & \dots & \left\lceil -\dfrac{f}{2} \right\rceil \end{array}$$

Note that $-f/2 \leq -1$. By Proposition 3 there exists an integral $v \in \ker(A)$ with $v \leq \lceil u \rceil$. This vector is an integral combination $v = \alpha(p-q) + \beta(r-s)$.

If $v_1 = 0$ then $\alpha = 0$ and v must be a *positive* multiple of $(r-s)$ to ensure that $v_F = -\beta f \leq \lceil -f/2 \rceil \leq -1$. This implies the contradiction $v_B = \beta > 0$.

Next suppose that $v_1 \leq -1$. Then $\alpha \leq -1$ (considering the A component) and $\beta \geq 1$ (considering the F component). The D representative requires that $d + 1 \leq -\alpha d + \beta \leq \left\lceil \frac{d+1}{2} \right\rceil$. But this implies that $d = 0$, a contradiction. We conclude that the D class is not present in the Gale diagram. By rotating the diagram by 90 degrees counterclockwise we can assume that we have an imbalanced Gale diagram where the B class is missing. For the two new minimal generators $x^p - x^q$ and $x^r - x^s$ we are either in the first case analyzed above (i.e. x^s and x^p are squarefree and relatively prime) or in the second case where x^p and x^r are squarefree and relatively prime, as these do not contain any B variable. In both cases the two generators form a squarefree Gröbner basis. $\square$

3. Normal but not complete intersection

We now assume that the toric ideal $I_{\mathcal{A}}$ is not a complete intersection, but it is normal and hence Cohen–Macaulay. This time we can assume that the Gale diagram is of the form as in Fig. 2. As in Section 2, the vectors in the diagram represent a sign class of vectors. The class B_1 represents vectors (x, y) where $x \geq y > 0$ and B_2 represents vectors (x, y) where $y > x > 0$. Similarly, F_1 represents those with $x \leq y < 0$, and F_2 with $y < x < 0$. The minimal free resolution of I_A has the form

$$0 \to R^2 \to R^3 \to R \to R/I_{\mathcal{A}} \to 0$$

where $R = \mathbb{K}[x_1, \dots, x_n]$. A matrix representing the map $R^2 \to R^3$ in the resolution can be determined using Construction 5.1 and Remark 5.8 in [11]. More precisely, using the two *syzygy triangles* in Figs. 3 and 4 we find that this matrix equals

$$\begin{bmatrix} AB_1 & D^*E^*F_1^* \\ B_2CD & F_2^*G^* \\ F_1F_2 & B_1^*B_2^* \end{bmatrix}.$$

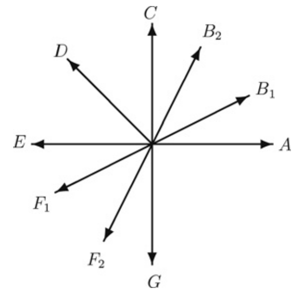


Fig. 2. Gale diagram of a CM but not complete intersection configuration.

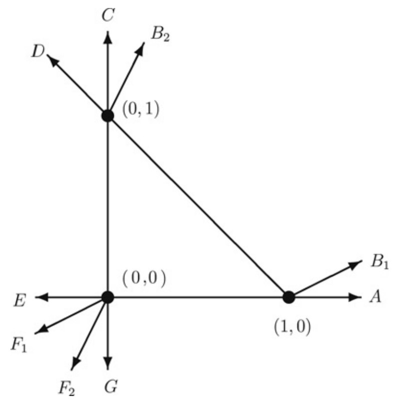


Fig. 3. A syzygy triangle as in [11].

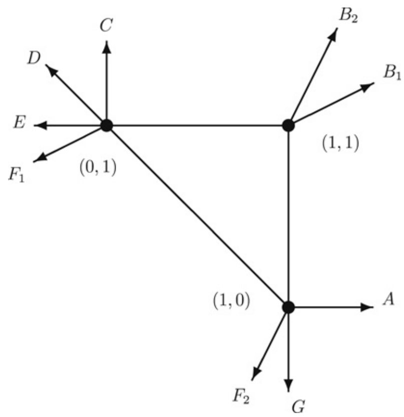


Fig. 4. Another syzygy triangle as in [11].

Let us briefly explain how this matrix is constructed. Each syzygy triangle is the convex hull of three lattice points. These lattice points correspond to three monomials which have the same multidegree as the syzygy they collectively represent. For each syzygy triangle these monomials are obtained as follows: recall that $\mathcal{B}$ is a Gale diagram and let $P = \{y \in \mathbb{R}^2 : y\mathcal{B} \leq u\}$ be the polytope that minimally contains the three lattice points. For each lattice point $z \in P$ the exponent vector of the corresponding monomial is $u - z\mathcal{B}$. To illustrate this let $I_{\mathcal{A}}$ be the defining ideal of the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^2$. This toric ideal of codimension two is normal but not a complete intersection. One can take

$$\mathcal{B} = \begin{bmatrix} 1 & -1 & 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & -1 & 0 & 1 \end{bmatrix}$$

as a Gale diagram. We denote the columns of $\mathcal{B}$ as well as the corresponding indeterminates by B_1, E, G, F_1, A and C, respectively. The three monomials corresponding to $(0, 0)$, $(1, 0)$ and $(0, 1)$ in the first syzygy triangle (Fig. 3) are AB_1C, CEF_1 and AF_1G. The ones corresponding to $(1, 0)$, $(0, 1)$ and $(1, 1)$ in the second syzygy triangle (Fig. 4) are B_1CE, AB_1G and EF_1G. Now let $M = \{m_1, m_2, m_3\}$ be the monomials coming from one of these syzygies. Then the column of the above matrix corresponding to this syzygy has entries $\gcd(M \setminus m_i)$ for $i = 1, 2, 3$. In our example, this matrix is

$$\begin{bmatrix} A & E \\ C & G \\ F_1 & B_1 \end{bmatrix}.$$

In the earlier more general matrix the letters represent classes of variables corresponding to the classes of vectors in the Gale diagram. Products of letters correspond to monomials in these classes of variables. The letters with an asterisk correspond to the same class as those without an asterisk, but they might have different exponent vectors (since they come from different syzygy triangles). The first column of the matrix corresponds to the first triangle and the second column to the second triangle. Moreover, by the Hilbert–Burch Theorem, the three 2×2-minors of this matrix are precisely the minimal generators of $I_{\mathcal{A}}$:

$$AB_1F_2^*G^* - B_2CDD^*E^*F_1^* \tag{1}$$

$$AB_1B_1^*B_2^* - D^*E^*F_1F_1^*F_2 \tag{2}$$

$$B_1^*B_2B_2^*CD - F_1F_2F_2^*G^*. \tag{3}$$

The next result completes the proof of Theorem 1.

Lemma 7. *There exists a term order such that the binomial generators (1), (2), (3) of the toric ideal $I_{\mathcal{A}}$ form a Gröbner basis with squarefree initial monomials.*

Proof. We have a few cases to consider. First, either the D class exists or it does not. If it exists then the monomial DD^* is not squarefree and hence the squarefree term of (1) is the first term. Note that the first terms of (2) and (3) cannot be squarefree simultaneously: if they were, the B_1 and B_2 vectors cannot be present in the Gale diagram, and this would be an imbalanced Gale diagram. Similarly, the second terms of these binomials cannot be squarefree simultaneously. This gives two cases to consider. In the first case we have

$$\underline{AG^*} - B_2CDD^*E^*F_1^*, \qquad \underline{AB_2^*} - D^*E^*F_1F_1^*, \qquad B_2B_2^*CD - \underline{F_1G^*}. \tag{4}$$

Here B_1 and F_2 are absent because otherwise $B_1B_1^*$ and $F_2F_2^*$ are not squarefree. If we choose a lexicographic term order where $A > G > \{B_2, C, D, E, F_1\}$ then the underlined terms are the leading terms in (4). The S-pair $S(1, 2) = D^*E^*F_1F_1^*G^* - B_2B_2^*CDD^*E^*F_1^*$ is reduced to zero by the third binomial, and the S-pair $S(1, 3) = AB_2B_2^*CD - B_2CDD^*E^*F_1F_1^*$ is reduced to zero by the second binomial. The S-pair $S(2, 3)$ reduces to zero since the leading terms AB_2^* and F_1G^* are relatively prime, and hence (4) is a squarefree Gröbner basis.

In the second case, the minimal generators and their squarefree terms are

$$\underline{AB_1F_2^*G^*} - CDD^*E^*, \qquad AB_1B_1^* - \underline{D^*E^*F_2}, \qquad \underline{B_1^*CD} - F_2F_2^*G^*. \tag{5}$$

The product of the three underlined terms is equal to the product of the three non-underlined terms. Hence no term order selects the underlined terms as leading terms. However, the squarefreeness of these three monomials implies

$$A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad D = \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$
$$E = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad F_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

From the diagonal edge in the two syzygy triangles we see that $B_1 = F_2^* = 1$. This means that the non-underlined terms of the second and third binomials in (5) are actually squarefree, and we are back in the previous case (4).

Now suppose that the D vectors are not present in the Gale diagram depicted in Fig. 2. Then the binomial generators (1)–(3) have the form

$$AB_1F_2^*G^* - B_2CE^*F_1^*, \qquad AB_1B_1^*B_2^* - E^*F_1F_1^*F_2, \qquad B_1^*B_2B_2^*C - F_1F_2F_2^*G^*.$$

If in the first binomial the first term is squarefree we are back to (4) or (5). If the second term is squarefree, then we rotate the Gale diagram 180 degrees. This leads to the same binomials but now with the first term of the first binomial squarefree. Once again we are back to (4) or (5). This concludes the proof. □

4. Checking normality

In this section we assume that the codimension $m = n - d$ of $I_{\mathcal{A}}$ is fixed. First we reformulate Proposition 3. Let z be an integral vector in $\ker(\mathcal{A})$. We define

$$P_z = \left\{x \in \ker(\mathcal{A}) : \lceil x_i \rceil \geq z_i \text{ for } i = 1, \ldots, n\right\}.$$

Since $P_z = \{x \in \ker(\mathcal{A}) : x_i > z_i - 1 \; i = 1, \ldots, n\}$ and since we assume that the cone $C(\mathcal{A})$ is pointed, P_z is a relatively open polytope in $\ker(\mathcal{A}) \simeq \mathbb{R}^m$.

Remark 8. If u and z are lattice vectors in $\ker(\mathcal{A})$ then $P_{u+z} = z + P_u$.

Now let $\mathcal{B}$ be an $n \times m$ matrix whose columns form a lattice basis of $\ker(\mathcal{A})$, and let b_i be the rows of $\mathcal{B}$. Then P_z is affinely isomorphic to $Q_v = \{y \in \mathbb{R}^m : b_i \cdot y > b_i \cdot v - 1,\ i = 1, \ldots, n\}$ where v is the unique lattice point in $\mathbb{Z}^m$ such that $\mathcal{B}v = z$. Remark 8 implies that for two lattice points v and w in $\mathbb{Z}^m$ we have $Q_{v+w} = w + Q_v$. Note that $Q_0 = \{y \in \mathbb{R}^m : b_i \cdot y > -1,\ i = 1, \ldots, n\}$. We now see that the following is equivalent to Proposition 3.

Theorem 9. *The toric ideal $I_{\mathcal{A}}$ is normal if and only if $Q_0 + \mathbb{Z}^m = \mathbb{R}^m$.*

Given any polytope $Q = \{y \in \mathbb{R}^m : Cy \geq d\}$ of dimension m the smallest positive real number t such that $tQ + \mathbb{Z}^m = \mathbb{R}^m$ is called the *covering radius* of Q. If Q is a rational polytope it is known that the covering radius of Q is a rational number with a bit-size that is a polynomial in the bit-size of C and d.

Corollary 10. *The toric ideal $I_{\mathcal{A}}$ is normal if and only if the covering radius of $\bar{Q}_0$, the closure of the polytope Q_0, is less than* 1.

Proof of Theorem 2. Ravi Kannan [15, Section 5] has shown that, for fixed m, and given a rational m-dimensional polytope $Q = \{y \in \mathbb{R}^m : Cy \geq d\}$ where $C \in \mathbb{Z}^{n \times m}$ and $d \in \mathbb{Z}^n$, there exists an algorithm to find the covering radius of Q with runtime a polynomial in n and the bit-size of C and the vector d. Since one can compute a $\mathcal{B}$ whose bit-size is a polynomial in the bit-size of $\mathcal{A}$ in polynomial time, the above corollary implies the result. $\square$

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