Robust $H_2$ fuzzy output feedback control for discrete-time nonlinear systems with parametric uncertainties

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Abstract

This paper deals with the robust $H_2$ fuzzy observer-based control problem for discrete-time uncertain nonlinear systems. The Takagi and Sugeno (T-S) fuzzy model is employed to represent a discrete-time nonlinear system with parametric uncertainties. A fuzzy observer is used to estimate the state of the fuzzy system and a non-parallel distributed compensation (non-PDC) scheme is adopted for the control design. A fuzzy Lyapunov function (FLF) is constructed to derive a sufficient condition such that the closed-loop fuzzy system is globally asymptotically stable and an upper bound on the quadratic cost function is provided. A sufficient condition for the existence of a robust $H_2$ fuzzy observer-based controller is presented in terms of linear matrix inequalities (LMIs). Moreover, by using the existing LMI optimization techniques, a suboptimal fuzzy observer-based controller in the sense of minimizing the cost bound is proposed. Finally, an example is given to illustrate the effectiveness of the proposed design method.

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1. Introduction

Recently, fuzzy logic control has attracted increasing attention, because it can provide an effective solution to the control of plants that are complex, uncertain, ill-defined, and have available qualitative knowledge from domain experts for their controllers design. Though the method has been practically successful, it has proved extremely difficult to develop a general analysis and design theory for conventional fuzzy control systems. It is believed that the reason for this is due to the fact that no mathematical model is available for the design of the fuzzy controller.

In recent years, it becomes quite popular to adopt the so-called Takagi–Sugeno (T–S) type fuzzy models [1] to represent or approximate a nonlinear system. This fuzzy model is described by a family of fuzzy IF–THEN rules which represent local linear input–output relations of a nonlinear system. The overall fuzzy model is achieved by smoothly blending these local linear models together through fuzzy membership functions. As a result, the conventional linear system theory can be applied to analysis and synthesis of nonlinear control systems. The stability issue of T–S fuzzy control systems in nonlinear stability frameworks has been studied extensively [2–7]. Recently, many works on T–S fuzzy-model-based control for a nonlinear system were developed to stabilize the T–S fuzzy model with guaranteed $H_2$ or/and $H_\infty$ performance [8–12], or multiobjective control performance [13]. As a common belief, this fuzzy-model-based control technique is conceptually simple and effective for controlling complex nonlinear systems. However, these results are derived by using a single Lyapunov function (SLF) method. The main drawback of this method is that an SLF must work for all linear models, which in general leads to conservative results. To reduce the conservatism of using an SLF, recently, the fuzzy Lyapunov functions (FLFs) have been constructed for the control design of T–S fuzzy systems [14–16].

In many practical systems, there always exist uncertainties which are frequently a source of instability or degraded performance. Thus, recently, the robust control problems have been investigated for uncertain nonlinear systems based on T–S fuzzy model [17–27]. However, most of the existing results are dedicated to the problems of robust stabilization [17–20] and robust $H_\infty$ control [21–25]. Fewer results are available for the robust $H_2$ control of uncertain T–S fuzzy systems with exceptions [26,27]. The objective of robust $H_2$ control is to design a control system that is not only robust stable but also provides an upper bound of quadratic performance for all admissible uncertainties [28,29]. In [26], a robust $H_2$ fuzzy control design is provided for continuous time uncertain nonlinear systems, where both the state-feedback and observer-based control cases are considered. In [27], the robust $H_2$ fuzzy control problem for the discrete-time uncertain nonlinear systems is studied, where the state-feedback case is only considered. Moreover, due to the use of an SLF, the results in [17–24,26,27] are usually conservative. It is worth noting that, more recently, an FLF has been employed to deal with the robust $H_\infty$ fuzzy state-feedback control problem for uncertain T–S fuzzy systems in [25].

In this paper, we study the robust $H_2$ fuzzy observer-based control problem for discrete-time uncertain nonlinear systems. The T–S fuzzy model is employed to represent a discrete-time nonlinear system with parametric uncertainties. Based on a fuzzy observer, which is used to estimate the state of the fuzzy system, a non-parallel distributed compensation (non-PDC) scheme [15] is adopted for the control design of the fuzzy system. An FLF is constructed, which not only includes a fuzzy Lyapunov matrix, but also an
additional fuzzy slack variable. Based on this FLF, a sufficient condition is derived, which can guarantee the stability of the closed-loop fuzzy system and provide an upper bound for the quadratic cost function. From this condition, a sufficient condition for the existence of a robust H₂ fuzzy observer-based controller is presented in the form of a set of linear matrix inequalities (LMIs). The control matrices and observer gain matrices can be obtained by directly solving this set of LMIs. Moreover, by the existing LMI optimization techniques [30,31], a suboptimal robust H₂ fuzzy observer-based controller in the sense of minimizing the cost bound is obtained. Finally, it is also demonstrated, through numerical simulations on a chaotic system, that the proposed design method is effective.

Notations: Throughout the paper, $\mathbb{R}$, $\mathbb{R}^n$, $\mathbb{R}^{n \times m}$ denote the sets of real numbers, $n$ component real vectors, and $n \times m$ real matrices, respectively. For a symmetric matrix $M$, $M > (\geq) 0$ means that it is positive definite (or positive semi-definite). The superscript T represents the transpose. The symbol * denotes the transposed elements in the symmetric positions of a matrix.

2. Problem formulation

Consider a discrete-time uncertain nonlinear system which can be described by the following T–S fuzzy model:

\[
x(k + 1) = \sum_{i=1}^{r} h_i(\theta(k)) \{ (A_i + \Delta A_i(k))x(k) + (B_i + \Delta B_i(k))u(k) \}, \quad x(0) = x_0
\]

\[
z(k) = \sum_{i=1}^{r} h_i(\theta(k)) \{ C_{1i}x(k) + D_{1i}u(k) \}
\]

\[
y(k) = \sum_{i=1}^{r} h_i(\theta(k)) C_{2i}x(k)
\]

where $\theta(k) = [ \theta_1(k) \ \theta_2(k) \ \cdots \ \theta_r(k) ]$ is the premise variable vector, $r$ is the number of model rules. $h_i(\theta(k))$ is the normalized weight for the $i$th rule, $i \in \mathbb{S} \triangleq \{ 1, 2, \ldots, r \}$, that is, $h_i(\theta(k)) \geq 0$, $i \in \mathbb{S}$, and $\sum_{i=1}^{r} h_i(\theta(k)) = 1$ for all $k$. $x(k) \in \mathbb{R}^n$ is the state vector, $u(k) \in \mathbb{R}^m$ is the control input vector, $z(k) \in \mathbb{R}^p$ is the measured output vector, $y(k) \in \mathbb{R}^q$ is the measurement output vector, $x_0$ is the initial state. $A_i$, $B_i$, $C_{1i}$, $C_{2i}$, and $D_{1i}$, $i \in \mathbb{S}$ are known constant matrices that describe the nominal system. $\Delta A_i(k)$, $\Delta B_i(k)$, $i \in \mathbb{S}$ represent the time-varying parametric uncertainties having the following structure:

\[
[ \Delta A_i(k) \ \Delta B_i(k) ] = HF(k) [ E_{1i} \ E_{2i} ], \quad i \in \mathbb{S},
\]

where $F(k) \in \mathbb{R}^{2 \times \beta}$ is an unknown matrix function with Lebesgue-measurable elements and satisfying

\[
F^T(k)F(k) \leq I
\]

and $H$, $E_{1i}$, $E_{2i}$, $i \in \mathbb{S}$ are known constant matrices with appropriate dimensions that specify how the uncertain parameters in $F(k)$ enter the nominal matrices $A_i$ and $B_i$. For the convenience of notations, in the sequel, we will denote
Before concluding this section, we recall a matrix inequality that will be used in next

In this study, it is assumed that all of the premise variables are available for feedback

As in [15], the following non-PDC fuzzy controller is proposed for the system (1)–(3):

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In this paper, we consider the following quadratic cost function:

The objective of this paper is to seek a fuzzy observer-based controller of the form (8)
and determine a cost bound \(J_b < +\infty\) as small as possible such that the closed-loop fuzzy
system (9) and (10) is globally asymptotically stable and the cost function defined by (11)
satisfies \(J \leq J_b\) for all admissible parametric uncertainties.

Before concluding this section, we recall a matrix inequality that will be used in next
section.
Lemma 1 [32]. Let $A$, $H$, $E$, and $F$ be real matrices of appropriate dimensions with $F^TF \preceq I$. For any matrix $P > 0$ and scalar $\varepsilon > 0$ such that $P - \varepsilon H^TH > 0$, then the following inequality holds:

$$(A + HFE)^T P^{-1} (A + HFE) \preceq A^T (P - \varepsilon H^TH)^{-1} A + \varepsilon^{-1} E^TE.$$ 

3. Robust $H_2$ fuzzy observer-based control design

In this section, we first discuss a sufficient condition that not only guarantees the stability of the closed-loop fuzzy system (9) and (10), but also provides an upper bound for the cost function (11). And then, a design method for a robust $H_2$ fuzzy observer-based controller is presented.

For convenience, we denote

$$\tilde{\mathcal{P}}(k) = \mathcal{F}^{-1}(k) \mathcal{P}(k) \mathcal{F}^{-1}(k),$$

where $\mathcal{F}(k) = \sum_{i=1}^{m} h_i(\theta(k)) \mathcal{G}_i$ is non-singular, $\mathcal{P}(k) = \sum_{i=1}^{m} h_i(\theta(k)) \mathcal{P}_i > 0$, and $\mathcal{G}_i, \mathcal{P}_i \in \mathbb{R}^{2n \times 2n}$. Let us consider the following FLF candidate [15]:

$$V(\xi(k)) = \xi^T(k) \tilde{\mathcal{P}}(k) \xi(k).$$

Remark 1. Observe that the FLF candidate (13) includes not only a fuzzy Lyapunov matrix $\mathcal{P}(k)$, but also an additional fuzzy slack variable $\mathcal{G}(k)$. Moreover, it has been shown in [15] that (13) is indeed a candidate Lyapunov function.

Along the trajectory of system (9), we have

$$\Delta V \triangleq V(\xi(k+1)) - V(\xi(k)) = \xi^T(k+1) \tilde{\mathcal{P}}(k+1) \xi(k+1) - \xi^T(k) \tilde{\mathcal{P}}(k) \xi(k)$$

$$= \xi^T(k) \Upsilon \xi(k)$$

(14)

where

$$\Upsilon \triangleq [\mathcal{A}_a(k) + \mathcal{H}_aF(k) \mathcal{E}_a(k)]^T \tilde{\mathcal{P}}(k+1) [\mathcal{A}_a(k) + \mathcal{H}_aF(k) \mathcal{E}_a(k)] - \tilde{\mathcal{P}}(k).$$

From (10) and (14), we get

$$\Delta V + \xi^T(k) z(k) = \xi^T(k) [\Upsilon + \mathcal{G}_a^T(k) \mathcal{E}_a(k)] \xi(k).$$

(15)

Obviously, if the following inequality holds:

$$\Upsilon + \mathcal{G}_a^T(k) \mathcal{E}_a(k) < 0,$$

(16)

then we have

$$\Delta V + \xi^T(k) z(k) \leq 0.$$ 

(17)

Therefore, we have the following theorem which presents a sufficient condition such that the closed-loop fuzzy system (9) and (10) is globally asymptotically stable and an upper bound is provided for the cost function (11).

Theorem 1. Consider the closed-loop fuzzy system (9) and (10). If there exist a scalar $\varepsilon > 0$, a time-varying matrix $\mathcal{P}(k) > 0$ and a non-singular time-varying matrix $\mathcal{G}(k)$ such that
then the system is globally asymptotically stable and the cost function (11) satisfies
\[ J \leq \xi_0^T \overline{\mathcal{F}}^{-1}(0) \overline{\mathcal{F}}(0) \xi_0 \] for all admissible parametric uncertainties.

**Proof.** From the fact \( [\overline{\mathcal{F}}(k + 1) - \overline{\mathcal{F}}(k + 1)] \overline{\mathcal{F}}^{-1}(k + 1) [\overline{\mathcal{F}}(k + 1) - \overline{\mathcal{F}}(k + 1)]^T \geq 0 \), we have
\[ -\overline{\mathcal{P}}^{-1}(k + 1) = -\overline{\mathcal{F}}(k + 1) \overline{\mathcal{F}}^{-1}(k + 1) \overline{\mathcal{F}}(k + 1) \leq -\overline{\mathcal{F}}(k + 1) - \overline{\mathcal{F}}^T(k + 1) + \overline{\mathcal{F}}(k + 1). \] Thus, it follows from (18) and (20) that
\[ \begin{bmatrix} -\mathcal{P}(k) & * & * & * & * \\ \mathcal{A}(k) \overline{\mathcal{F}}(k) & -\overline{\mathcal{P}}^{-1}(k + 1) & * & * & * \\ 0 & \varepsilon \mathcal{H}^T_a & -\varepsilon I & * & * \\ \mathcal{E}(k) \overline{\mathcal{F}}(k) & 0 & 0 & -\varepsilon I & * \\ \mathcal{E}(k) \overline{\mathcal{F}}(k) & 0 & 0 & 0 & -I \end{bmatrix} < 0. \] Define matrix \( \mathcal{F} \triangleq \text{diag} \{ \overline{\mathcal{F}}^{-1}(k), I, I, I, I \} \). Pre- and post-multiplying (21) by \( \mathcal{F}^T \) and \( \mathcal{F} \), respectively, and considering (12), yield
\[ \begin{bmatrix} -\mathcal{P}(k) & * & * & * & * \\ \mathcal{A}(k) & -\overline{\mathcal{P}}^{-1}(k + 1) & * & * & * \\ 0 & \varepsilon \mathcal{H}^T_a & -\varepsilon I & * & * \\ \mathcal{E}(k) & 0 & 0 & -\varepsilon I & * \\ \mathcal{E}(k) & 0 & 0 & 0 & -I \end{bmatrix} < 0. \] By Schur complement, it follows that (22) is equivalent to
\[ \begin{bmatrix} -\mathcal{P}(k) + \varepsilon^{-1} \mathcal{E}_a^T(k) \mathcal{E}_a(k) + \mathcal{E}_a^T(k) \mathcal{E}_a(k) & * \\ \mathcal{A}(k) & -\overline{\mathcal{P}}^{-1}(k + 1) + \varepsilon \mathcal{H}_a \mathcal{H}_a^T \end{bmatrix} < 0 \] which is equivalent to
\[ \overline{\mathcal{P}}^{-1}(k + 1) - \varepsilon \mathcal{H}_a \mathcal{H}_a^T > 0 \] (24)
and
\[ \mathcal{A}_a^T(k) [\overline{\mathcal{P}}^{-1}(k + 1) - \varepsilon \mathcal{H}_a \mathcal{H}_a^T]^{-1} \mathcal{A}_a(k) - \overline{\mathcal{P}}(k) + \varepsilon^{-1} \mathcal{E}_a^T(k) \mathcal{E}_a(k) + \mathcal{E}_a^T(k) \mathcal{E}_a(k) < 0. \] (25)
By Lemma 1, we have
\[
[\mathcal{A}_a(k) + H_a F(k) \mathcal{E}_a(k)]^T \mathcal{P}(k+1)[\mathcal{A}_a(k) + H_a F(k) \mathcal{E}_a(k)] \\
\leq \mathcal{A}_a(k) \left[ \mathcal{P}^{-1}(k+1) - \varepsilon H_a H_a^T \right]^{-1} \mathcal{A}_a(k) + \mathcal{E}_a(k) \mathcal{E}_a(k).
\]
It follows from (25) and (26) that (16) holds, and thus (17) is fulfilled. It is clear from (14) and (16) that \( \Delta V < 0 \) for any \( \xi(k) \neq 0 \). Hence, the system (9) and (10) is globally asymptotically stable. In this case, \( V(\xi(\infty)) = 0 \). Summing the inequality (17) from \( k = 0 \) to \( k = \infty \), we have
\[
V(\xi(\infty)) - V(\xi(0)) = -\xi_0^T \mathcal{P}(0) \xi_0 \leq -\sum_{k=0}^{\infty} z^T(k) z(k).
\]
Thus, from (27), we have (19). \( \square \)

Let \( \mathcal{P}_i \) be partitioned as
\[
\mathcal{P}_i = \begin{bmatrix} P_{1i} & * \\ P_{2i} & P_{3i} \end{bmatrix} \quad \text{where} \quad P_{1i} > 0, \ P_{2i}, \ \text{and} \ P_{3i} > 0 \ \text{are} \ n \times n.
\]
Moreover, to obtain a LMI formulation, we let \( \mathcal{G}_i \) be the following form:
\[
\mathcal{G}_i = \begin{bmatrix} G_{1i} & G_{2i}^{-1} \\ 0 & G_{2i}^{-1} \end{bmatrix} \quad \text{where} \quad G_{1i} \ \text{and} \ G_{2i} \ \text{are} \ n \times n \ \text{and non-singular}.
\]
Then, based on Theorem 1, the following result can be obtained:

**Theorem 2.** Consider the fuzzy system (1)–(3). For a given scalar \( \varepsilon > 0 \), if there exist matrices \( G_2, G_{1i}, P_{1i} > 0, Q_{2i} \in \mathbb{R}^{n \times n}, Q_{3i} > 0 \in \mathbb{R}^{n \times n}, K_i, \) and \( S_i \in \mathbb{R}^{n \times q}, i \in \mathbb{S} \) satisfying the following LMIs:
\[
\Phi_{ii} < 0, \quad i \in \mathbb{S},
\]
\[
\frac{1}{r-1} \Phi_{ii} + \frac{1}{2} (\Phi_{ij} + \Phi_{ji}) < 0, \quad i,j \in \mathbb{S}, \quad i \neq j,
\]
where
\[
\Phi_{ij} \triangleq \begin{bmatrix} -P_{1i} & * & * & * & * & * & * \\ -Q_{2i} & -Q_{1i} & * & * & * & * & * \\ A_i G_{1j} + B_i K_j & A_i & -G_{1i} - G_{1j}^T + P_{1i} & * & * & * & * \\ 0 & G_{2j} A_i - S_i C_{2j} & -I + Q_{2j} & -G_{2j}^T - G_{2i} + Q_{3i} & * & * & * \\ 0 & 0 & -eH^T & -eH^T G_{2i} & -eI & * & * \\ E_{1j} G_{1j} + E_{2i} K_j & E_{1i} & 0 & 0 & 0 & -eI & * \\ C_{1j} G_{1j} + D_{1i} K_j & C_{1i} & 0 & 0 & 0 & 0 & -I \end{bmatrix},
\]
then there exists a fuzzy observer-based controller of the form (8) with the observer gain matrices
\[
L_i = G_2^T S_i, \quad i \in \mathbb{S}
\]
such that the closed-loop fuzzy system (9) and (10) is globally asymptotically stable and the cost function (11) satisfies (19) with
It is clear from (30) that \( G_{1l} + G_{1l}^T > P_{1l} > 0 \) and \( G_2^T + G_2 > Q_{3l} > 0, l \in \mathbb{S} \). Thus, we have \( \overline{G}_1(k) + \overline{G}_1^T(k) > 0 \) and \( G_2^T + G_2 > 0 \) which ensure the non-singularity of \( \overline{G}_1(k) \) and \( G_2 \). From (28) and (29), we have

\[
\overline{P}(k) = \begin{bmatrix}
-\overline{P}_1(k) & * & * & * & * & * & * \\
-\overline{P}_2(k) & -\overline{P}_3(k) & * & * & * & * & * \\
0 & 0 & \varepsilon H^T & \varepsilon H^T & -\varepsilon I & * & * \\
0 & 0 & 0 & 0 & 0 & -\varepsilon I & * \\
0 & 0 & 0 & 0 & 0 & 0 & -I
\end{bmatrix}
\]

(34)

where

\[
\Xi_{31}(k) \triangleq \overline{A}(k) \overline{G}_1(k) + \overline{B}(k) \overline{K}(k), \quad \Xi_{61}(k) \triangleq \overline{E}_1(k) \overline{G}_1(k) + \overline{E}_2(k) \overline{K}(k), \]

\[
\Xi_{71}(k) \triangleq \overline{C}_1(k) \overline{G}_1(k) + \overline{D}_1(k) \overline{K}(k), \quad \overline{A}_L(k) \triangleq \overline{A}(k) - \overline{L}(k) \overline{C}_2(k),
\]

\[
\Xi_{33}(k+1) \triangleq -\overline{G}_1(k+1) - \overline{G}_1^T(k+1) + \overline{P}_1(k+1),
\]

\[
\Xi_{44}(k+1) \triangleq -G_2^{-1} - G_2^T + \overline{P}_3(k+1).
\]

Define matrix \( T_1 \triangleq \text{diag}\{I, G_2, I, G_2^T, I, I, I\} \). Pre- and post-multiplying (34) by \( T_1^T \) and \( T_1 \), respectively, and letting

\[
Q_{2l} = G_2^T P_{2l}, \quad Q_{3l} = G_2^T P_{3l} G_2, \quad i \in \mathbb{S},
\]

we get

\[
\Xi_{31}(k) \triangleq \overline{A}(k) \overline{G}_1(k) + \overline{B}(k) \overline{K}(k), \quad \Xi_{61}(k) \triangleq \overline{E}_1(k) \overline{G}_1(k) + \overline{E}_2(k) \overline{K}(k),
\]

\[
\Xi_{71}(k) \triangleq \overline{C}_1(k) \overline{G}_1(k) + \overline{D}_1(k) \overline{K}(k), \quad \overline{A}_L(k) \triangleq \overline{A}(k) - \overline{L}(k) \overline{C}_2(k),
\]

\[
\Xi_{33}(k+1) \triangleq -\overline{G}_1(k+1) - \overline{G}_1^T(k+1) + \overline{P}_1(k+1),
\]

\[
\Xi_{44}(k+1) \triangleq -G_2^{-1} - G_2^T + \overline{P}_3(k+1).
\]

Define matrix \( T_1 \triangleq \text{diag}\{I, G_2, I, G_2^T, I, I, I\} \). Pre- and post-multiplying (34) by \( T_1^T \) and \( T_1 \), respectively, and letting

\[
Q_{2l} = G_2^T P_{2l}, \quad Q_{3l} = G_2^T P_{3l} G_2, \quad i \in \mathbb{S},
\]

we get
where $\bar{Q}_2(k) = \sum_{i=1}^{r} h_i(\theta(k)) Q_{2i}$, $\bar{Q}_3(k) = \sum_{i=1}^{r} h_i(\theta(k)) Q_{3i}$, and

$$\Gamma_{44}(k+1) = -G_2^T G_2 + \bar{Q}_3(k+1).$$

Letting

$$S_i = G_2^T L_i, \quad i \in \mathcal{S},$$

then (36) can be written as

$$\sum_{i=1}^{r} h_i(\theta(k+1)) \sum_{j=1}^{r} h_i(\theta(k)) h_j(\theta(k)) \Phi_{ij} < 0.$$  \hspace{1cm} (38)

By using Theorem 2.2 in [5], if the conditions (30) and (31) hold, then (38) is fulfilled. From (35), we have

$$P_{2i} = G_2^{-T} Q_{2i}, \quad P_{3i} = G_2^{-T} Q_{3i} G_2^{-1}, \quad i \in \mathcal{S}.\hspace{1cm} (39)$$

Moreover, it follows from (30) that

$$\begin{bmatrix} P_{1i} & * \\ Q_{2i} & Q_{3i} \end{bmatrix} > 0, \quad i \in \mathcal{S}.\hspace{1cm} (40)$$

Pre- and post-multiplying (40) by $\text{diag} \{ I, G_2^{-T} \}$ and $\text{diag} \{ I, G_2^{-1} \}$, respectively, and considering (39), we have

$$\begin{bmatrix} P_{1i} & * \\ P_{2i} & P_{3i} \end{bmatrix} = \begin{bmatrix} P_{1i} & * \\ G_2^{-T} Q_{2i} & G_2^{-T} Q_{3i} G_2^{-1} \end{bmatrix} > 0, \quad i \in \mathcal{S}.\hspace{1cm} (41)$$

This implies that there exist a matrix $\overline{\Phi}(k) = \begin{bmatrix} \overline{P}_{1i}(k) & * \\ G_2^{-T} \overline{Q}_{2i}(k) & G_2^{-T} \overline{Q}_{3i} G_2^{-1} \end{bmatrix} > 0$ and a non-singular matrix $\overline{\mathcal{G}}(k) = \begin{bmatrix} \mathcal{G}_1(k) & G_2^{-1} \\ 0 & G_2 \end{bmatrix}$ satisfying (18). Therefore, it follows from Theorem 1 that the closed-loop fuzzy system (9) and (10) is globally asymptotically stable and the cost function (11) satisfies (19) with (33) for all admissible parametric uncertainties. Furthermore, by (37), we have (32). \hfill $\Box$

Theorem 2 provides an LMI-based condition for the existence of a robust $H_2$ fuzzy observer-based controller for the fuzzy system (1)–(3). Different from the two-stage procedure of [12,20,24,26], where the resolution needs to separate the inequalities into two sets, one for the observer gain matrices, the second for the control gain matrices, the method proposed in this paper can give the control matrices and observer gain matrices of the fuzzy controller (8) by directly solving the LMIs of (30) and (31) for a fixed $\varepsilon > 0$. Moreover, the resulting fuzzy controller can guarantee that the cost function (11) satisfies (19) with (33) for all admissible uncertainties.

**Remark 2.** Notice that in the proof of Theorem 2, we have to face the following problem: Find the less conservative sufficient conditions so that the inequality (38) is fulfilled. A trivial solution would be for a fixed $l : \Phi_{lij} < 0$, $\Phi_{lij} + \Phi_{lijk} < 0$, $l, i, j \in \mathcal{S}$, $j > i$ [3]. However, it has been shown that this result is too conservative [4–7,11]. To outperform this result, several relaxation techniques have been proposed, for example [4–7,11]. These
techniques can be applied to this study straightforwardly. However, the methods in [4,6,7,11] must introduce a huge number of additional variables, which may be not compatible with the actual LMI solvers. Thus, we prefer adopting the relaxation technique without additional variables in [5] to give the less conservative sufficient conditions for (38) in this study.

To minimize the upper bound on the cost function (11), we seek to minimize the following upper bound of $J_b$,

$$J_b \leq \xi_0^T V \xi_0$$

(42)

where $V \triangleq \begin{bmatrix} V_1 & * \\ V_2 & V_3 \end{bmatrix}$, $V_i \in \mathbb{R}^{n \times n}$, $i = 1, 2, 3$ satisfying

$$\mathcal{F}^{-T}(0) \mathcal{F}(0) \mathcal{F}^{-1}(0) < V.$$

(43)

**Remark 3.** Note that the upper bound of the cost function (14) given by (42) depends on the initial condition $x_0$. To avoid this dependence, we will assume that $x_0$ is a zero mean random variable satisfying $\mathbb{E}\{x_0 x_0^T\} = I$ where $\mathbb{E}\{\cdot\}$ denotes the expectation operator. Then, from (19) and (42), we have the following cost bound:

$$J \triangleq \mathbb{E}\{J\} \leq \mathbb{E}\{J_b\} \leq \text{trace} \left\{ \begin{bmatrix} V_1 & * \\ V_2 & V_3 \end{bmatrix} \begin{bmatrix} \mathbb{E}\{x_0 x_0^T\} & * \\ \mathbb{E}\{x_0 x_0^T\} & \mathbb{E}\{x_0 x_0^T\} \end{bmatrix} \right\}$$

$$= \text{trace} \left\{ V_1 + V_2 + V_2^T + V_3 \right\}.$$  

(44)

By Schur complement, it follows that (43) is equivalent to

$$\begin{bmatrix} -V & * \\ I & -\mathcal{F}(0) \mathcal{F}^{-1}(0) \mathcal{F}^T(0) \end{bmatrix} < 0.$$  

(45)

From the fact $[\mathcal{F}(0) - \mathcal{F}(0)] \mathcal{F}^{-1}(0) [\mathcal{F}(0) - \mathcal{F}(0)]^T \geq 0$, we have

$$-\mathcal{F}(0) \mathcal{F}^{-1}(0) \mathcal{F}^T(0) \leq -\mathcal{F}(0) - \mathcal{F}^T(0) + \mathcal{F}(0).$$

(46)

Obviously, if the following inequality holds:

$$\begin{bmatrix} -V & * \\ I & -\mathcal{F}(0) - \mathcal{F}^T(0) + \mathcal{F}(0) \end{bmatrix} < 0$$

(47)

which can be written as

$$\sum_{i=1}^{r} h_i(\theta(0)) \begin{bmatrix} -V_1 & * & * & * \\ -V_2 & -V_3 & * & * \\ I & 0 & -G_{1i} - G_{1i}^T + P_{1i} & * \\ 0 & I & -G_{2i}^T + P_{2i} & -G_{2i}^{-1} - G_{2i}^{-T} + P_{3i} \end{bmatrix} < 0,$$  

(48)
then (45) or (43) holds. By considering (39), it is clear that if
\[
\begin{bmatrix}
-V_1 & * & * & * \\
-V_2 & -V_3 & * & * \\
I & 0 & -G_{1i}^T - G_{1i}^T + P_{1i} & * \\
0 & I & -G_2^T + G_2^T Q_{2i} & -G_2^{-1} - G_2^{-1} + G_2^T Q_{2i} G_2^{-1} \\
\end{bmatrix} < 0, \quad i \in \mathbb{S}.
\]

then (48) holds, and thus (43) holds. Define matrix \( T_2 \triangleq \text{diag}\{I, I, I, G_2\}. \) Pre- and post-multiplying (49) by \( T_2^T \) and \( T_2 \), respectively, yield the following LMIs:
\[
\begin{bmatrix}
-V_1 & * & * & * \\
-V_2 & -V_3 & * & * \\
I & 0 & -G_{1i}^T - G_{1i}^T + P_{1i} & * \\
0 & G_2^T & -I + Q_{2i} & -G_2 - G_2^T + Q_{2i} \\
\end{bmatrix} < 0, \quad i \in \mathbb{S}.
\]

(50)

Therefore, for some given \( \varepsilon > 0 \), a suboptimal robust H_2 fuzzy observer-based control problem can be addressed in the sense of minimizing the bound in (44) as the following LMI optimization problem:
\[
\min_{\Omega} \text{trace} \left\{ V_1 + V_2 + V_2^T + V_3 \right\} \quad \text{subject to the LMIs of (30), (31), and (50)}
\]

(51)

where \( \Omega \triangleq \{ V_1, V_2, V_3, G_2, G_{1i}, P_{1i}, Q_{2i}, Q_{3i} > 0, K_i, S_i, i \in \mathbb{S} \}. \) By using the existing LMI optimization techniques [30,31], the problem (51) can be efficiently solved.

Remark 4. Note that the LMIs of (30) and (31) in Theorem 2 are not strictly linear due to the fact that it involves a tuning parameter \( \varepsilon > 0 \). Obviously, different values of \( \varepsilon \) can give rise to different optimized bounds \( \text{trace} \left\{ V_1 + V_2 + V_2^T + V_3 \right\} \) by solving the problem (51). Thus, one can still optimize this bound by an enumeration method or a one-dimensional search for \( \varepsilon > 0 \) to obtain a suboptimal H_2 fuzzy observer-based controller.

4. Numerical example

In this section, a numerical example is presented to demonstrate the effectiveness of the proposed design method. Consider the following second-order discrete-time chaotic system [33]:
\[
\begin{align*}
x_1(k+1) &= ax_1(k) - x_1^3(k) + x_2(k) + u(k) \\
x_2(k+1) &= bx_1(k) \\
z(k) &= [0.5x_1(k) \ 0.5u(k)]^T \\
y(k) &= x_1(k)
\end{align*}
\]

(52)-(55)

where the nonlinear term is \( x_1^3(k) \). It is assumed that the parameters \( a \) and \( b \) are uncertain and satisfy \( a_0 - \delta_a \leq a \leq a_0 + \delta_a \) and \( b_0 - \delta_b \leq b \leq b_0 + \delta_b \), where \( a_0 \) and \( b_0 \) are the nominal values of \( a \) and \( b \), respectively, and \( \delta_a \geq 0 \) and \( \delta_b \geq 0 \).

Suppose that \( x_1(k) \in [-d,d] \) and \( d > 0 \). Using the sector nonlinearity approach [3], we can exactly represent the system (52)-(55) by the T–S fuzzy model (1)–(3) with \( r = 2 \) and
\[ x(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}, \quad A_1 = \begin{bmatrix} a_0 & 1 \\ b_0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} a_0 - d^2 & 1 \\ b_0 & 0 \end{bmatrix}, \quad B_1 = B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \]
\[
C_{11} = C_{12} = \begin{bmatrix} 0.5 & 0 \\ 0 & 0 \end{bmatrix}, \quad D_{11} = D_{12} = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}, \quad C_{21} = C_{22} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} \delta_a \\ \delta_b \end{bmatrix}^T, \quad E_{11} = E_{12} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad E_{21} = E_{22} = 0, \]
\[ h_1(x_1(k)) = 1 - x_1^2(k)/d^2, \quad h_2(x_1(k)) = x_1^2(k)/d^2. \]

In this study, the model parameters are given as \( a_0 = 1.9, \ b_0 = 0.5, \ \delta_a = 0.25, \ \delta_b = 0.2, \) and \( d = 2. \)

By using the Matlab LMI toolbox, it can be found that the problem (51) has a solution for \( \varepsilon \in [0.00001, 0.4] \). The optimized bound in (51) versus the scaling parameter \( \varepsilon \) is tabulated in Table 1.

By performing a simple one-parameter-search minimization of the optimized bound, it can be found that the optimal scaling parameter is given by \( \varepsilon = 0.1805 \). The corresponding optimized bound is 245.9656 and the corresponding control and observer gain matrices are obtained as

\[
K_1 = \begin{bmatrix} 3.8787 \times 10^{-8} & -4.0000 \end{bmatrix}, \quad K_2 = \begin{bmatrix} -1.7881 \times 10^{-9} & -4.0000 \end{bmatrix},
\]
\[
G_{11} = \begin{bmatrix} 0.1274 & 0.0137 \\ -0.2542 & 3.9757 \end{bmatrix}, \quad G_{12} = \begin{bmatrix} 0.1281 & 0.0183 \\ 0.2708 & 4.0346 \end{bmatrix},
\]
\[
L_1 = \begin{bmatrix} 2.5010 \\ 0.5912 \end{bmatrix}, \quad L_2 = \begin{bmatrix} -1.7771 \\ 0.4317 \end{bmatrix}. \]

Now, we apply the fuzzy observer-based controller (8) with the above matrices to the system (52)–(55) under the initial condition \( x(0) = [1.5 \\ -0.8]^T \). Assume that \( a = a_0 + \Delta a(k) \) and \( b = b_0 + \Delta b(k) \) where \( \Delta a(k) \) and \( \Delta b(k) \) are random variables

<table>
<thead>
<tr>
<th>( \varepsilon )</th>
<th>0.00001</th>
<th>0.001</th>
<th>0.1</th>
<th>0.15</th>
<th>0.1805</th>
<th>0.2</th>
<th>0.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Optimized bound</td>
<td>( 2.0144 \times 10^6 )</td>
<td>( 2.0201 \times 10^4 )</td>
<td>298.6727</td>
<td>252.1023</td>
<td>245.9656</td>
<td>248.2712</td>
<td>3.4377 \times 10^3</td>
</tr>
</tbody>
</table>

![Fig. 1. State responses \( x_1(k) \) and \( \hat{x}_1(k) \).](image-url)
uniformly distributed in the intervals \([-0.25, 0.25]\) and \([-0.2, 0.2]\), respectively. The simulation results are shown in Figs. 1–3. The solid lines in Figs. 1 and 2 show the actual trajectories of states \(x_1(k)\) and \(x_2(k)\) of the system, respectively, and the dotted lines show those of the estimated states \(\hat{x}_1(k)\) and \(\hat{x}_2(k)\) of the fuzzy observer, respectively. The control action \(u(k)\) is shown in Fig. 3. The simulation results show that the resulting fuzzy controller stabilizes the uncertain nonlinear system (52)–(55) and provides an optimized quadratic cost bound for the expected value of cost function (11).

5. Conclusions

In this paper, based on the T–S fuzzy model, we have studied the robust H\(_2\) fuzzy observer-based control problem for discrete-time nonlinear systems with parametric uncertainties. The sufficient condition for the existence of a robust H\(_2\) fuzzy observer-based controller is presented in the form of a set of LMIs. The control matrices and observer gain matrices can be obtained by directly solving this set of LMIs. Moreover, by the existing LMI optimization techniques, a suboptimal robust H\(_2\) fuzzy observer-based controller can be obtained, which can not only guarantee that the closed-loop fuzzy system is globally asymptotically stable, but also provide an optimized upper bound on the quadratic cost function for all admissible parametric uncertainties. Finally, simulation results indicate that the proposed method is effective. One of the future research topics is to extend the result developed in this paper to the robust H\(_\infty\) fuzzy observer-based control design for discrete-time uncertain T–S fuzzy systems.
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