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Vortex sheets with reflection symmetry in exterior domains

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Abstract

In this paper we prove the existence of a weak solution of the incompressible 2D Euler equations in the exterior of a reflection symmetric smooth bluff body with symmetric initial flow corresponding to vortex sheet type data whose vorticity is of distinguished sign on each side of the symmetry axis. This work extends the results proved for full plane flow by the authors in [M.C. Lopes Filho, H.J. Nussenzveig Lopes, Z. Xin, Existence of vortex sheets with reflection symmetry in two space dimensions, Arch. Ration. Mech. Anal. 158 (3) (2001) 235–257].

Keywords: Incompressible ideal flow; Vortex sheets; 2D exterior domain flow; Vorticity

1. Introduction

Let $D \subseteq \mathbb{R}^2$ be a smooth, bounded, simply connected domain with boundary $\partial D = \Gamma$. We assume that D is symmetric with respect to the horizontal coordinate axis. We will be studying the

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initial-boundary value problem for the incompressible 2D Euler equations in the exterior of D, denoted by $\Omega \equiv \mathbb{R}^2 \setminus D$. We will prove the existence of a weak solution of the incompressible 2D Euler equations in Ω with initial flow symmetric with respect to the horizontal axis, with distinguished sign vorticity on each side of the symmetry axis and with vortex sheet initial data. This work extends a similar result proved for full plane flow by the authors in [6].

Let us begin with a bit of terminology. We call measures in the plane which are odd with respect to a straight line and which have a distinguished sign on each side of the symmetry axis, *nonnegative mirror-symmetric* (NMS). In [6] the authors proved the existence of a weak solution to the incompressible 2D Euler equations in the full plane with NMS initial vorticity. This remains the only extension of Delort's existence theorem, see [1], which includes flows with vortex sheets without distinguished sign. In addition, the authors studied, in [6], the validity of the method of images for weak solutions. In this context, the method of images amounts to the equivalence between flow in the half-plane and mirror symmetric flow in the full plane. In order to establish the validity of the method of images for weak solutions, see [6, Theorem 2], the authors introduced a new notion of weak solution for flows in the half-plane, which we called *boundary-coupled weak solutions*. A byproduct of the work in [6] is the existence of such boundary-coupled weak solutions for distinguished sign vortex sheet initial data in the half-plane. The notion of boundary-coupled weak solution is stronger than the standard weak vorticity formulation for flows in domains with boundary (see [1]) because it requires test functions that vanish on the boundary, but are not necessarily compactly supported. Indeed, in the case of one-signed vorticity in special domains, such as in [6] and in this paper, a boundary-coupled weak solution has no vorticity concentration at the boundary. Such concentration may exist for the general classical weak solutions in [1]. One of the motivations of the present work is to study such boundary-coupled weak solutions in more general domains with boundary.

The main ingredients in proving the existence of NMS flows in [6] were the following facts:

(F1) An L^2_{loc} a priori estimate on velocity restricted to the symmetry axis (estimate (4) in [6]);
(F2) An estimate on the mass of vorticity near the symmetry axis in terms of the integral of velocity at the symmetry axis (estimate (5) in [6]);
(F3) Persistence of cancellation in the weak form of the nonlinearity up to the symmetry axis (expressed in identity (10) in [6]).

The proofs of (F1), (F2) and (F3), which ultimately concern half-plane flow, relied heavily on the fact that the boundary of the half-plane is a straight line.

Each one of these facts has a certain independent interest when regarded as information on the behavior of incompressible, ideal 2D flows near a straight rigid boundary. We will show that these facts can be generalized to domains with curved boundaries. The existence result we prove here may be thus regarded as an application of (F1)–(F3).

Beyond the existence of NMS flow outside of a reflection symmetric smooth bluff body we will also establish the validity of the method of images. The method of images provides existence of boundary-coupled weak solutions in certain compactly supported perturbations of the half-plane. Our work also implies the existence of boundary-coupled weak solutions in more general domains with simply connected boundary, such as curved channels. It is our contention that there is additional control over the way weak solutions interact with a material boundary if they are boundary-coupled and we will offer a bit of supporting evidence. The interaction of weak solutions for the incompressible 2D Euler equations with material boundaries is both poorly understood and physically interesting, since this issue is naturally connected with the problem of boundary layers, see the discussion in [5] and references therein.

The remainder of this article is divided into 5 sections. In Section 2 we discuss exterior domain flow and background on NMS flows. In Section 3 we prove nonconcentration of vorticity up to the boundary, extending (F1) and (F2). In Section 4 we prove the extension of (F3) to exterior domain flows. In Section 5 obtain existence of NMS flows and we discuss the method of images. Section 6 contains concluding remarks.

2. Preliminaries

We will consider the 2D Euler equations in vorticity form in an exterior domain. We must contend, however, with the fact that the system coupling velocity to vorticity (namely: $\operatorname{div} u = 0$, $\operatorname{curl} u = \omega$, $u \cdot \hat{n} = 0$ on the boundary, and $|u| \to 0$ at infinity) does not determine u uniquely in terms of ω. This is due to the nonvanishing homology of the exterior domain. This issue was examined in detail in [2], where it was shown that the velocity field is determined by the vorticity up to a harmonic vector field, called the harmonic part. In our problem we will assume that the initial velocity u_0 is mirror symmetric with respect to the horizontal axis. This implies two facts: (1) the vorticity is odd with respect to the variable x_2 and therefore its integral in Ω vanishes, and (2) the circulation of the initial velocity around $\partial\Omega$ vanishes. These facts, together with [2, Lemma 3.1] imply that the harmonic part of the velocity must vanish. Consequently, we can write the Biot–Savart law expressing velocity in terms of vorticity in the following manner. Let $G_\Omega = G_\Omega(x, y)$ be the Greens function for the Laplacian in Ω, and set $K_\Omega \equiv \nabla_x^\perp G_\Omega$. With this notation the Biot–Savart law is given by:

$$u = u(x,t) = K_\Omega[\omega](x,t) \equiv \int_\Omega K_\Omega(x,y)\omega(y,t)\,dy. \tag{1}$$

We now write the vortex sheet initial data problem as:

$$\begin{cases} \omega_t + u \cdot \nabla\omega = 0 & \text{in } \Omega \times (0,\infty), \\ u = K_\Omega[\omega] & \text{in } \Omega \times (0,\infty), \\ \omega(x,0) = \omega_0(x) & \text{on } \Omega \times \{t=0\}, \end{cases} \tag{2}$$

where $\hat{n}$ is the unit exterior normal to the boundary Γ.

Our main result will be the existence of weak solutions to (2) for ω_0 a bounded measure, odd with respect to mirror symmetry, nonnegative in the upper half-plane outside of D. We call $\mu \in \mathcal{BM}(\Omega)$ *nonnegative mirror symmetric* (*NMS*) if it is odd with respect to reflection about the horizontal axis and if it is nonnegative in $\Omega \cap \{x_2 \geqslant 0\}$.

Let us first define what we mean by weak solution in this context. We introduce $\mathcal{A}$, the set of admissible test functions, defined by:

$$\mathcal{A} \equiv \big\{\varphi \in C_c^\infty\big([0,\infty) \times \bar{\Omega}\big) \mid \varphi \equiv 0 \text{ on } \Gamma\big\}.$$

Definition 1. The function $\omega \in L^\infty([0,\infty); \mathcal{BM}(\Omega))$ is called a boundary-coupled weak solution of the incompressible 2D Euler equations with initial data ω_0 if:

(a) the velocity $u \equiv K_\Omega[\omega]$ belongs to $L^\infty_{\text{loc}}([0,\infty); (L^2(\Omega))^2)$, and
(b) for any test function $\varphi \in \mathcal{A}$, it holds that

$$\mathcal{W}[\omega,\varphi] \equiv \int_0^\infty \int_\Omega \varphi_t \omega(x,t)\,dx\,dt + \int_0^\infty \int_\Omega \int_\Omega H_\varphi^\Omega(x,y,t)\omega(x,t)\omega(y,t)\,dy\,dx\,dt + \int_\Omega \varphi(x,0)\omega_0(x)\,dx = 0, \tag{3}$$

where

$$H_\varphi^\Omega(x,y,t) \equiv \frac{1}{2}\big(\nabla\varphi(x,t)\cdot K_\Omega(x,y) + \nabla\varphi(y,t)\cdot K_\Omega(y,x)\big). \tag{4}$$

Remark. Boundary-coupled weak solutions take on the boundary condition ($u \cdot \hat{n} = 0$) in a stronger manner than classical weak solutions, in which test functions are required to be compactly supported in the interior of Ω while the boundary condition is assumed in the trace sense. The bounded domain version of Delort's theorem guaranteed the existence of a classical weak solution, see [1]. Boundary-coupled weak solutions were introduced by the authors in [6]. Furthermore, it will follow from the analysis in Section 3 that our boundary-coupled weak solution has no vorticity concentration at the boundary, which may occur for the classical weak solutions obtained in [1].

The strategy for obtaining a weak solution is to pass to the weak limit along a suitably constructed approximate solution sequence. The methods of constructing such a sequence of approximations in the context of the initial-value problem for the incompressible 2D Euler equations involve: smoothing out or truncating the initial vorticity, approximation by vanishing viscosity and the use of several numerical methods. Here we will obtain an approximate solution sequence by smoothing out initial data and we will use the available global well-posedness theory which can be found in [3]. Next we will observe that the symmetry of the problem is preserved under smooth flows. We denote the reflection about the horizontal axis by $x = (x_1, x_2) \mapsto \bar{x} = (x_1, -x_2)$.

Proposition 1. *Let $\omega_0 \in C_c^\infty(\Omega)$ be NMS and let $\omega = \omega(x,t)$ be the unique solution of the incompressible 2D Euler equations in Ω. Then ω is NMS for all $t \geqslant 0$.*

Proof. Define $\tilde{\omega}(x,t) = -\omega(\bar{x},t)$. Then $\tilde{\omega}(x,0) = \omega_0(x)$. As the Euler equations are covariant with respect to mirror symmetry it follows that $\tilde{\omega}$ also satisfies the Euler equations in Ω. It follows from the uniqueness that $\tilde{\omega}(x,t) = \omega(x,t)$ for all t. The sign condition is a consequence of the fact that vorticity is transported by the flow, that each half-plane is invariant under symmetric flow and of the hypothesis on the initial data. □

3. Non-concentration of vorticity at the boundary

We will begin with a reasonably straightforward generalization of the argument used in [6] to show non-concentration in mass of vorticity all the way up to the physical boundary or on the interface of two flows with different signs of vorticity. This argument consists of two lemmas which are given below. In fact, it will be shown that versions of (F1) and (F2) hold on certain domains with curved boundaries.

Let $\Omega_+ \equiv \Omega \cap \{x_2 > 0\}$ and $\Gamma_+ = \partial\Omega_+$.

Lemma 1. *Let $\omega = \omega(x,t)$ be the solution of* (2) *with smooth, compactly supported, and NMS initial vorticity. Let $\varphi = \varphi(x)$ be a smooth function on Ω_+ with bounded derivatives up to second order. Then the following identity holds*:

$$\frac{d}{dt}\int_{\Omega_+} \varphi(x)\omega(x,t)\,dx = \frac{1}{2}\int_{\Gamma_+} \left|u\cdot \hat{n}^\perp\right|^2 \nabla\varphi\cdot\hat{n}^\perp\,dS + \int_{\Omega_+} \left[\left((u_1)^2-(u_2)^2\right)\varphi_{x_1x_2} - u_1u_2(\varphi_{x_1x_1}-\varphi_{x_2x_2})\right]dx.$$

Proof. We will prove this identity by direct computations. It holds that

$$\mathcal{I} \equiv \frac{d}{dt}\int_{\Omega_+}\varphi(x)\omega(x,t)\,dx = \int_{\Omega_+}\varphi\omega_t\,dx = -\int_{\Omega_+}\varphi\,\mathrm{div}(u\omega)\,dx = \int_{\Omega_+}(\nabla\varphi\cdot u)\omega\,dx,$$

where the boundary terms have disappeared since u is tangent to Γ_+ (due to symmetry) and bounded everywhere and hence ω has compact support at each fixed time. Re-write $\omega = -\,\mathrm{div}\,u^\perp$ and integrate by parts once more to obtain:

$$\mathcal{I} = \int_{\Omega_+}\nabla(\nabla\varphi\cdot u)\cdot u^\perp\,dx - \int_{\Gamma_+}(\nabla\varphi\cdot u)\left(u^\perp\cdot\hat{n}\right)dS,$$

where again the boundary terms at infinity have vanished, this time because $|u|$ decays sufficiently fast at infinity. Indeed, $|u| = \mathcal{O}(|x|^{-2})$ for large $|x|$, see the discussion in [2, Section 2.2] for a proof. Next, observe that

$$\nabla(\nabla\varphi\cdot u)\cdot u^\perp = \nabla\left(\frac{|u|^2}{2}\right)\cdot\nabla^\perp\varphi + \left((u_1)^2-(u_2)^2\right)\varphi_{x_1x_2} - u_1u_2(\varphi_{x_1x_1}-\varphi_{x_2x_2}).$$

Therefore, this vector calculus identity yields

$$\begin{aligned}
\mathcal{I} &= \int_{\Omega_+}\left[\nabla\left(\frac{|u|^2}{2}\right)\cdot\nabla^\perp\varphi + \left((u_1)^2-(u_2)^2\right)\varphi_{x_1x_2} - u_1u_2(\varphi_{x_1x_1}-\varphi_{x_2x_2})\right]dx \\
&\quad - \int_{\Gamma_+}(\nabla\varphi\cdot u)\left(u^\perp\cdot\hat{n}\right)dS \\
&= \int_{\Omega_+}\left[\left((u_1)^2-(u_2)^2\right)\varphi_{x_1x_2} - u_1u_2(\varphi_{x_1x_1}-\varphi_{x_2x_2})\right]dx \\
&\quad + \int_{\Gamma_+}\left(\frac{|u|^2}{2}\nabla^\perp\varphi - (\nabla\varphi\cdot u)u^\perp\right)\cdot\hat{n}\,dS,
\end{aligned}$$

where, once again, we have used the decay of $|u|$ at infinity. Finally, using the fact that the velocity u is tangent to the boundary, one can compute that there is a simpler expression for the boundary term:

$$\left(\frac{|u|^2}{2}\nabla^{\perp}\varphi - (\nabla\varphi\cdot u)u^{\perp}\right)\cdot\hat{n} = \frac{1}{2}\left|u\cdot\hat{n}^{\perp}\right|^2\nabla\varphi\cdot\hat{n}^{\perp}.$$

This concludes the proof. In fact, since kinetic energy is finite initially, conserved exactly for smooth flows and φ has been assumed to have bounded derivatives up to second order, it follows that the expression on the right-hand side of the identity we have just proved is finite and integrable in time. □

We now use this identity to deduce an a priori estimate for the L^2_{loc}-norm (on $\Gamma_+\times(0,\infty)$) of the tangential component of velocity, namely, a generalization of (F1). For the sake of convenience, we assume that Γ_+ is the graph of a piecewise smooth, compactly supported function $\gamma=\gamma(x_1)$. In this case we will use in Lemma 1 the function $\varphi(x)=\arctan(x_1)$. Note that, for this test function, for each compact subset $\mathcal{K}$ of Γ_+ there exists $\tilde{C}>0$ such that $\nabla\varphi\cdot\hat{n}^{\perp}\geqslant\tilde{C}$ a.e. on $\mathcal{K}$. Indeed, this follows easily from the observations that $\nabla\varphi=((1+x_1^2)^{-1},0)$ and that $\hat{n}^{\perp}=(1,0)$ on the straight portion of Γ_+ and $\hat{n}^{\perp}=(1+(\gamma'(x_1))^2)^{-1/2}(1,\gamma'(x_1))$ on the curved portion of Γ_+. We then obtain that, for every $\mathcal{K}\Subset\Gamma_+$ and for every $T>0$ there exists $C>0$, depending only on $\mathcal{K}$, T, $\|\omega_0\|_{L^1(\Omega)}$ and $\|u_0\|_{L^2(\Omega)}$ such that:

$$\int_0^T\int_{\mathcal{K}}|u|^2\,dS\,dt\leqslant C. \tag{5}$$

This is (the generalization of) (F1). To verify (5) we estimate directly:

$$\begin{aligned}
\int_0^T\int_{\mathcal{K}}|u|^2\,dS\,dt &= \int_0^T\int_{\mathcal{K}}\left|u\cdot\hat{n}^{\perp}\right|^2dS\,dt\leqslant\frac{1}{\tilde{C}}\int_0^T\int_{\mathcal{K}}\left|u\cdot\hat{n}^{\perp}\right|^2\nabla\varphi\cdot\hat{n}^{\perp}\,dS\,dt\\
&=\frac{2}{\tilde{C}}\Bigg(\int_{\Omega_+}\varphi(x)\omega(x,T)\,dx-\int_{\Omega_+}\varphi(x)\omega_0(x)\,dx\\
&\quad+\int_0^T\int_{\Omega_+}\left[\left((u_2)^2-(u_1)^2\right)\varphi_{x_1x_2}+u_1u_2(\varphi_{x_1x_1}-\varphi_{x_2x_2})\right]dx\,dt\Bigg)\\
&\leqslant\frac{2}{\tilde{C}}\left(2T\|\varphi\|_{L^\infty(\Omega_+)}\|\omega_0\|_{L^1(\Omega_+)}+T\left\|D^2\varphi\right\|_{L^\infty(\Omega_+)}\|u_0\|^2_{L^2(\Omega_+)}\right),
\end{aligned}$$

where $D^2\varphi$ stands for a generic second derivative of φ. In the last inequality we have used the fact that smooth incompressible Euler flows preserve the mass of vorticity and kinetic energy. It follows also from the symmetry that $\|\omega_0\|_{L^1(\Omega)}=2\|\omega_0\|_{L^1(\Omega_+)}$ and $\|u_0\|^2_{L^2(\Omega)}=2\|u_0\|^2_{L^2(\Omega_+)}$.

Finally, we obtain the nonconcentration result on the mass of vorticity up to the boundary.

Lemma 2. *Let $\omega_0 \in C_c^\infty(\Omega)$ be NMS and let $\omega = \omega(x,t)$, $u = K_\Omega[\omega]$ be the solution to* (2) *with initial data ω_0. For each $T > 0$ and each compact set $\mathcal{K} \subseteq \bar{\Omega}$ there exists a constant $C = C(\omega_0) > 0$ such that for any $0 < \delta < 1$,*

$$\int_0^T \left(\sup_{x\in\mathcal{K}} \int_{B(x;\delta)\cap\Omega} \left|\omega(y,t)\right| dy \right) dt \leqslant C|\log\delta|^{-1/2}.$$

Proof. Fix $\mathcal{K} \subseteq \bar{\Omega}$ and $0 < \delta < 1$. We make use of the following cut-off function, also used by Schochet in [11]:

$$\eta_\delta(z) = \begin{cases} 1, & \text{if } |z| \leqslant \delta, \\ \frac{\log(|z|)-\log(\sqrt{\delta})}{\log(\sqrt{\delta})}, & \text{if } \delta \leqslant |z| \leqslant \sqrt{\delta}, \\ 0, & \text{if } |z| \geqslant \sqrt{\delta}. \end{cases}$$

Note that for $x \in \mathcal{K}$:

$$\int_{B(x;\delta)\cap\Omega} \left|\omega(y,t)\right| dy = \int_{B(x;\delta)\cap\Omega_+} \omega(y,t)\,dy - \int_{B(x;\delta)\cap(\Omega\setminus\Omega_+)} \omega(y,t)\,dy.$$

Each integral above can be estimated by using the fact that ω has a distinguished sign in each of Ω_+ and $\Omega \setminus \Omega_+$. Indeed, for the first integral, one has

$$\begin{aligned} \int_{B(x;\delta)\cap\Omega_+} \omega(y,t)\,dy &\leqslant \int_{B(x;\sqrt{\delta})\cap\Omega_+} \eta_\delta(x-y)\omega(y,t)\,dy \\ &= \int_{B(x;\sqrt{\delta})\cap\Omega_+} \nabla_y\eta_\delta(x-y)\cdot u^\perp(y,t)\,dy \\ &\quad + \int_{B(x;\sqrt{\delta})\cap\Gamma_+} \eta_\delta(x-y)u(y,t)\cdot\hat{n}^\perp(y)\,dS, \end{aligned}$$

by integrating by parts. Note that the other boundary terms vanish since $\eta_\delta(x-y) = 0$ for $y \in \partial B(x;\sqrt{\delta})$. Therefore,

$$\begin{aligned} \int_{B(x;\delta)\cap\Omega_+} \omega(y,t)\,dy &\leqslant C|\log\delta|^{-1/2}\|u_0\|^2_{L^2(\Omega_+)} \\ &\quad + \left(\int_{B(x;\sqrt{\delta})\cap\Gamma_+} \left|u\cdot\hat{n}^\perp\right|^2 dS \right)^{1/2} \left|B(x;\sqrt{\delta})\cap\Gamma_+\right|^{1/2}. \end{aligned}$$

Finally, since the boundary Γ_+ was assumed to be piecewise smooth it follows easily that $|B(x;\sqrt{\delta}) \cap \Gamma_+| \leqslant C\sqrt{\delta}$, which, together with (5), yields the desired estimate of the first integral.

The estimate of the second integral follows in an analogous way. □

It should be noted that Γ_+ was assumed to be a graph of a piecewise smooth, compactly supported function. This simplified the derivation of (5) by allowing us to explicitly produce an appropriate test function φ. However, this hypothesis is not needed and the derivation of (5) can be obtained, for example, through deformation of Γ_+ into a graph. Lemma 2 is the curved domain generalization of (F2).

4. Desingularization of the nonlinearity

Let $\omega_0 \in \mathcal{BM}_c(\Omega)$ be NMS and assume that $u_0 \equiv K_\Omega[\omega_0] \in L^2(\Omega)$. To produce a weak solution to (2) with initial data ω_0, it is a key step to study the concentration–cancellation effects of the nonlinearity in the Euler equations. To this end, we will show the persistence of cancellation in the weak form of the nonlinearity up to the symmetry axis, (F3). We begin by considering a smooth approximation of the initial data. Let ω_0^n be a sequence in $C_c^\infty(\Omega)$ such that

1. $\omega_0^n \rightharpoonup \omega_0$ weak-$*$ in $\mathcal{BM}(\Omega)$,
2. $\|\omega_0^n\|_{L^1(\Omega)}$ and $\|u_0^n \equiv K_\Omega[\omega_0^n]\|_{L^2(\Omega)}$ are uniformly bounded with respect to n,
3. ω_0^n is NMS.

One way of building such a sequence of approximations is to solve the heat equation in Ω with ω_0 as initial data for time $1/n$ and then smoothly truncate near infinity.

Let $\omega^n = \omega^n(x,t)$ be the smooth solution of (2) with $u^n = K_\Omega[\omega^n]$ and initial vorticity ω_0^n, given by Kikuchi's theorem, see [3]. Of course, since ω_0^n is NMS we have that ω^n is NMS as well for all n and t. Therefore the conclusion of Lemma 2 can be re-formulated as a uniform a priori estimate on the mass of vorticity in small balls. For any $T > 0$ and any compact set $\mathcal{K} \Subset \bar{\Omega}$ there exists a constant $C > 0$ such that, for all n,

$$\int_0^T \left(\sup_{x\in\mathcal{K}} \int_{B(x;\delta)\cap\Omega} \left|\omega^n(y,t)\right| dy \right) dt \leqslant C|\log\delta|^{-1/2}. \tag{6}$$

We wish to pass to the limit in the weak formulation of (2) given in Definition 1 for this approximate solution sequence. The crucial step is to pass the limit in the nonlinearity. To do so we will need to establish the boundedness of the auxiliary function H_φ^Ω, for $\varphi \in \mathcal{A}$, where H_φ^Ω was defined in (15). This is the content of the theorem below.

Theorem 1. *Let $\varphi \in \mathcal{A}$. Then there exists $C > 0$ such that*

$$\left|H_\varphi^\Omega(x,y,t)\right| \leqslant C,$$

for all $x, y \in \bar{\Omega}$ and $t \in [0,\infty)$.

Proof. Note that, as φ has compact support in $\bar{\Omega} \times [0, \infty)$, it is enough to prove the boundedness of H_φ^Ω in a compact set $\mathcal{K} \Subset \bar{\Omega}$ and on a finite interval $[0, T]$. Re-write H_φ^Ω as:

$$H_\varphi^\Omega(x, y, t) = \frac{1}{2}\big(\nabla\varphi(x, t) - \nabla\varphi(y, t)\big) \cdot K_\Omega(x, y) + \frac{1}{2}\nabla\varphi(y, t) \cdot \big(K_\Omega(x, y) + K_\Omega(y, x)\big) \equiv \mathcal{I} + \mathcal{J}.$$

We will use the basic framework developed in [2, Sections 2.1 and 2.2]. Let $U = \{|x| > 1\}$ and let $T : \Omega \to U$ be the biholomorphic mapping given in [2, Lemma 2.1]. The mapping T induces a diffeomorphism between Γ and $\{|x| = 1\}$. Recall that the Biot–Savart kernel K_Ω can be explicitly expressed using this mapping in the following manner:

$$K_\Omega(x, y) = \frac{((T(x) - T(y))DT(x))^\perp}{2\pi|T(x) - T(y)|^2} - \frac{((T(x) - (T(y))^*)DT(x))^\perp}{2\pi|T(x) - (T(y))^*|^2}, \tag{7}$$

where $z \mapsto z^* = z/|z|^2$ is the inversion with respect to the unit circle. Note that $K_\Omega(x, y) = (DT(x))^t K_U(T(x), T(y))$. Next we recall an estimate obtained in [2, Section 2.2], namely,

$$\big|K_\Omega(x, y)\big| \leqslant C\frac{|T(y) - (T(y))^*|}{|T(x) - T(y)||T(x) - (T(y))^*|},$$

for some constant $C > 0$. It is easy to see that, for each $z \in U$ fixed, we have, for any $w \in U$,

$$\frac{|z - z^*|}{|w - z^*|} \leqslant \frac{|z - z^*|}{|(z/|z|) - z^*|} = |z| + 1.$$

Hence,

$$\big|K_\Omega(x, y)\big| \leqslant \frac{C}{|T(x) - T(y)|} \leqslant \frac{C}{|x - y|},$$

for all $(x, y) \in \mathcal{K} \times \mathcal{K}$, since DT and its inverse are bounded. This implies that $\mathcal{I}$ is bounded.

Next we re-write $\mathcal{J}$ in the following manner:

$$\begin{aligned}\mathcal{J} &= \frac{1}{2}\nabla\varphi(y, t)\big[\big(DT(x)\big)^t - \big(DT(y)\big)^t\big]K_U\big(T(x), T(y)\big) \\ &\quad + \frac{1}{2}\nabla\varphi(y, t)\big(DT(y)\big)^t\big[K_U\big(T(x), T(y)\big) + K_U\big(T(y), T(x)\big)\big] \\ &\equiv \mathcal{J}_1(x, y, t) + \mathcal{J}_2(x, y, t).\end{aligned}$$

As before we find that

$$\big|K_U\big(T(x), T(y)\big)\big| \leqslant \frac{C}{|x - y|},$$

for x, y in $\mathcal{K}$, so that, since D^2T is also bounded, we conclude that $\mathcal{J}_1$ is bounded in $\mathcal{K} \times \mathcal{K}$. We are left with the estimate of $\mathcal{J}_2$, which is the heart of the matter. We will need the following claim.

Claim. *If $y \in \Gamma$ then $\mathcal{J}_2(x, y, t) \equiv 0$.*

Proof. Let $y \in \Gamma$. For each $\theta \in [0, 2\pi)$ let $\mathcal{C}_\theta \equiv T^{-1}(\{re^{i\theta} \mid r \in (1, \infty)\})$. Of course, $\mathcal{C}_\theta$ is a smooth curve in Ω, naturally parametrized by $r \in (1, \infty)$. Let $A(z) \equiv \arg(T(z))$. Note that $A(z) = \theta$ for $z \in \mathcal{C}_\theta$. Therefore, ∇A is orthogonal to the family of curves $\mathcal{C}_\theta$ and we have that

$$\nabla A(z) = \big(DT(z)\big)^t \frac{(T(z))^\perp}{|T(z)|^2}. \tag{8}$$

As φ is admissible, the boundary Γ is a level curve of φ and hence $\nabla\varphi(y,t)$ is orthogonal to Γ. On the other hand, the curves $\mathcal{C}_\theta$ are also orthogonal to Γ because T is conformal and $T(\mathcal{C}_\theta)$ is a straight ray perpendicular to $T(\Gamma) = \{|z| = 1\}$. Therefore,

$$\nabla\varphi(y,t) \cdot \nabla A(y) = 0. \tag{9}$$

Let z and w be points in the plane such that $|z| \geqslant 1$ and $|w| \geqslant 1$. We use (7), with T being the identity, and a straightforward calculation to obtain

$$K_U(z,w) + K_U(w,z) = -\frac{1}{2\pi}\left\{\frac{(|w|^2-1)z^\perp}{|w|^2|z-w^*|^2} + \frac{(|z|^2-1)w^\perp}{|z|^2|w-z^*|^2}\right\}. \tag{10}$$

Since $y \in \Gamma$, we have that $|T(y)| = 1$, and therefore,

$$K_U\big(T(x),T(y)\big) + K_U\big(T(y),T(x)\big) = -\frac{1}{2\pi}\frac{(|T(x)|^2-1)(T(y))^\perp}{|T(x)|^2|T(y)-(T(x))^*|^2}. \tag{11}$$

Putting together (9), (11) and (8) it follows that $\mathcal{J}_2(x,y,t) \equiv 0$ for $y \in \Gamma$, which concludes the proof of the claim. □

For $x \neq 0$ in the plane, we write $\hat{x} = x/|x|$. Let $(x,y) \in \mathcal{K} \times \mathcal{K}$. First we observe that, for z and w with $|z| \geqslant 1$ and $|w| \geqslant 1$ we have the following elementary fact

$$|w|\big|z-w^*\big| = |z|\big|w-z^*\big|. \tag{12}$$

Next, using (10) and (12) we write

$$\begin{aligned}
-4\pi\mathcal{J}_2(x,y,t) &= \nabla\varphi(y,t)\big(DT(y)\big)^t \frac{(|T(y)|^2-1)(T(x))^\perp}{|T(x)|^2|T(y)-(T(x))^*|^2} \\
&\quad + \nabla\varphi(y,t)\big(DT(y)\big)^t \frac{(|T(x)|^2-1)(T(y))^\perp}{|T(y)|^2|T(x)-(T(y))^*|^2} \equiv \mathcal{M}_1 + \mathcal{M}_2.
\end{aligned}$$

Let us first estimate $\mathcal{M}_1$. Using the claim above we find:

$$\begin{aligned}
\mathcal{M}_1 &= \nabla\varphi(y,t)\big(DT(y)\big)^t \frac{(|T(y)|^2-1)(T(x))^\perp}{|T(x)|^2|T(y)-(T(x))^*|^2} \\
&= \big[\nabla\varphi(y,t) - \nabla\varphi\big(T^{-1}\big(\widehat{T(x)}\big),t\big)\big]\big(DT(y)\big)^t \frac{(|T(y)|^2-1)(T(x))^\perp}{|T(x)|^2|T(y)-(T(x))^*|^2} \\
&\quad + \nabla\varphi\big(T^{-1}\big(\widehat{T(x)}\big),t\big)\big[\big(DT(y)\big)^t - \big(DT\big(T^{-1}\big(\widehat{T(x)}\big)\big)\big)^t\big]\frac{(|T(y)|^2-1)(T(x))^\perp}{|T(x)|^2|T(y)-(T(x))^*|^2},
\end{aligned}$$

as $T^{-1}(\widehat{T(x)}) \in \Gamma$. Therefore, there exists $C > 0$, depending only on φ and T and their derivatives up to second order, such that

$$|\mathcal{M}_1| \leqslant C\big|y - T^{-1}\big(\widehat{T(x)}\big)\big|\left|\frac{(|T(y)|^2 - 1)(T(x))^\perp}{|T(x)|^2|T(y) - (T(x))^*|^2}\right|.$$

Next we note that $|T(y)|^2 - 1 = (T(y) - \widehat{T(x)})(T(y) + \widehat{T(x)})$, so that, since $|T(x)|$ and $|T(y)|$ are bounded for x and y in $\mathcal{K}$, it follows that

$$|\mathcal{M}_1| \leqslant C\frac{|y - T^{-1}(\widehat{T(x)})||T(y) - \widehat{T(x)}|}{|T(y) - (T(x))^*|^2} \leqslant C\frac{|T(y) - \widehat{T(x)}|^2}{|T(y) - (T(x))^*|^2},$$

as T^{-1} is a diffeomorphism with bounded derivative in $\mathcal{K}$. We conclude by observing that

$$\big|T(y) - \big(T(x)\big)^*\big| \geqslant \big|\widehat{T(x)} - \big(T(x)\big)^*\big|, \tag{13}$$

as $\widehat{T(x)}$ is the point in $\bar{U}$ closest to $(T(x))^*$, and

$$\big|T(y) - \widehat{T(x)}\big|^2 \leqslant 2\big(\big|T(y) - \big(T(x)\big)^*\big|^2 + \big|\big(T(x)\big)^* - \widehat{T(x)}\big|^2\big).$$

These inequalities allow us to conclude that $|\mathcal{M}_1| \leqslant 4C$.

Next we estimate $\mathcal{M}_2$. We have, once again using the claim proved above,

$$\begin{aligned}
\mathcal{M}_2 &= \nabla\varphi(y,t)\big(DT(y)\big)^t \frac{(|T(x)|^2 - 1)(T(y))^\perp}{|T(y)|^2|T(x) - (T(y))^*|^2} \\
&= \big[\nabla\varphi(y,t) - \nabla\varphi\big(T^{-1}\big(\widehat{T(y)}\big), t\big)\big]\big(DT(y)\big)^t \frac{(|T(x)|^2 - 1)(T(y))^\perp}{|T(y)|^2|T(x) - (T(y))^*|^2} \\
&\quad + \nabla\varphi\big(T^{-1}\big(\widehat{T(y)}\big), t\big)\big[\big(DT(y)\big)^t - \big(DT\big(T^{-1}\big(\widehat{T(y)}\big)\big)\big)^t\big]\frac{(|T(x)|^2 - 1)(T(y))^\perp}{|T(y)|^2|T(x) - (T(y))^*|^2},
\end{aligned}$$

as $T^{-1}(\widehat{T(y)}) \in \Gamma$. Therefore, as before, we find

$$|\mathcal{M}_2| \leqslant C\frac{|T(y) - \widehat{T(y)}||T(x) - \widehat{T(y)}|}{|T(x) - (T(y))^*|^2},$$

for some constant $C > 0$ depending on φ, T, T^{-1}, their derivatives up to second order, and the diameter of $T(\mathcal{K})$. Using again (13) we obtain

$$|\mathcal{M}_2| \leqslant C\frac{|T(y) - \widehat{T(y)}||T(x) - \widehat{T(y)}|}{|\widehat{T(y)} - (T(y))^*||T(x) - (T(y))^*|} \leqslant C\frac{|T(x) - (T(y))^*| + |(T(y))^* - \widehat{T(y)}|}{|T(x) - (T(y))^*|} \leqslant 2C.$$

This concludes the proof. □

Remark 1. The proof of Theorem 1 can be easily adapted to a general simply connected bounded domain with smooth boundary.

Remark 2. Note that, if φ is supported away from the boundary Γ, i.e., compactly supported in $\Omega \times [0,\infty)$, then the proof of Theorem 1 can be substantially simplified. To see that, just note that the delicate part of the proof involves estimating $\mathcal{J}_2$, which has terms of the form $|T(y) - T(x)^*|$ in the denominator; these are trivially bounded if both x and y are bounded away from the boundary of Ω. Furthermore, in this case, Theorem 1 could be regarded as an easy adaptation of what was proved by Delort in [1].

Next we discuss the relation between this result and (F3). Let us begin by considering the case $\Omega = \mathbb{R}^2$. The nonlinearity in the vorticity Eq. (2) has the form $u\omega = K_{\mathbb{R}^2}[\omega]\omega$, where

$$K_{\mathbb{R}^2}[\omega] = K * \omega \equiv \frac{1}{2\pi} \int\limits_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2} \omega(y)\, dy.$$

If $\omega \in L^p$, $1 < p < 2$, then, by the Hardy–Littlewood–Sobolev inequality, u belongs to L^{p^*}, with $p^* = 2p/(p-2)$. The naïve condition needed to make sense of $u\omega$ is therefore $p \geqslant 4/3$. However, due to the antisymmetry of the kernel $K_{\mathbb{R}^2}$, we easily deduce that, for any test function $\varphi \in C_c^\infty((0,\infty) \times \mathbb{R}^2)$, the auxiliary function $H_\varphi^{\mathbb{R}^2}$ is smooth away from $x = y$ and globally bounded, see [1], Proposition 1.2.3 and see [11] for an alternative proof. The weak form of the nonlinearity, which is

$$\int\limits_{\mathbb{R}^2} \nabla\varphi(x) u(x)\omega(x)\, dx = \int\limits_{\mathbb{R}^2}\int\limits_{\mathbb{R}^2} H_\varphi^{\mathbb{R}^2}(x,y)\omega(x)\omega(y)\, dx\, dy,$$

clearly makes sense for any $\omega \in L^1$, and even for ω a continuous measure in $\mathcal{BM}$. It is the boundedness of $H_\varphi^{\mathbb{R}^2}$ which we refer to as cancellation in the weak form of the nonlinearity.

For domains with boundary, the kernel K_Ω is no longer antisymmetric. Nevertheless, in the case of bounded domains, Delort observed that H_φ^Ω is still bounded if φ is compactly supported in the interior of Ω, see the comment following identity (2.3.12) in [1]. In [6] we proved that, if Ω is the half-plane $\mathbb{H}$, then $H_\varphi^{\mathbb{H}}$ is bounded for all $\varphi \in C_c^\infty(\bar{\mathbb{H}})$, with $\varphi = 0$ on $\partial\mathbb{H}$. This is what we are calling *persistence* of cancellation in the weak form of the nonlinearity up to the boundary, i.e. (F3). Theorem 1 is thus a generalization of (F3) to domains with curved boundaries.

The boundedness of H_φ^Ω is a key ingredient in the proof of existence of weak solution of (2) with vortex sheet initial data, see [1,4,7,11,13], and the stronger version where φ is not required to be compactly supported was used both to prove existence for NMS data and to conclude existence of a boundary-coupled weak solution in [6]. The boundedness of H_φ^Ω can be also be useful in other problems regarding (2), see, for example, the proof of [2, Theorem 2.1] and the discussion following Definition 1.1 in [9]. Moreover, a very similar idea applies to the Vlasov–Poisson system and was used in [8,10] (it should be noted that these applications were made for flows in domains without boundary, where there is no distinction between classical and boundary-coupled weak solutions). In conclusion, Theorem 1 is an unexpected fact with potential applicability beyond the results which we will present in the next section.

5. Existence of weak solutions and the method of images

We are now ready to state and prove our main result, extending the half-plane existence result in [6] to exterior domains.

Theorem 2. *Let D be a closed bounded region of the plane with smooth boundary and symmetric with respect to reflection about the horizontal coordinate axis and let $\Omega = \mathbb{R}^2 - D$. Let $\omega_0 \in \mathcal{BM}_c(\Omega)$, be NMS and assume that $u_0 = K_\Omega[\omega_0] \in (L^2(\Omega))^2$. Then there exists a boundary-coupled weak solution of the 2D incompressible Euler equations in Ω with initial data ω_0.*

Proof. Let $\omega_0^n \in C_c^\infty(\Omega)$ be such that $\omega_0^n \rightharpoonup \omega_0$ weak-$*$ $\mathcal{BM}(\Omega)$. Let ω^n, $u^n = K_\Omega[\omega^n]$, be the unique global smooth solutions of (2) with initial data ω_0^n. The existence of such solutions follow from the well-posedness of 2D Euler with smooth initial data in exterior domains, due to Kikuchi in [3]. It is an easy calculation to verify that ω^n satisfies Definition 1.

The sequence ω^n satisfies the following a priori estimates:

1. $\|\omega^n\|_{L^\infty((0,\infty);L^1(\Omega))} \leqslant C < \infty$;
2. $\|u^n\|_{L^\infty((0,\infty);(L^2(\Omega))^2)} \leqslant C < \infty$;
3. $\{\omega^n\}$ is *equicontinuous* from $(0, T)$, for any $T > 0$, to $H^{-M}(\Omega)$ for some $M > 0$.

Indeed, the first estimate follows from conservation in time of L^1 norm of vorticity for smooth solutions and the second follows from the standard energy estimate. The third estimate is a bit more complicated. To prove it we consider a test function $\varphi \in C_c^\infty((0,\infty) \times \Omega)$. We use this test function in identity (14), noting that the initial data term disappears because $\varphi(x, 0) \equiv 0$, to get:

$$\int_0^\infty \int_\Omega \varphi_t \omega^n(x,t)\,dx\,dt + \int_0^\infty \int_\Omega \int_\Omega H_\varphi^\Omega(x,y,t)\omega^n(x,t)\omega^n(y,t)\,dy\,dx\,dt = 0.$$

Hence,

$$\left| \int_0^\infty \int_\Omega \varphi_t \omega^n(x,t)\,dx\,dt \right| \leqslant \big\|H_\varphi^\Omega\big\|_{L^1((0,\infty);L^\infty(\Omega))} \big\|\omega^n\big\|^2_{L^\infty((0,\infty);L^1(\Omega))}.$$

It follows from the first a priori estimate above and Theorem 1 that

$$\left| \int_0^\infty \int_\Omega \varphi_t \omega^n(x,t)\,dx\,dt \right| \leqslant C_\varphi.$$

The dependence of C_φ on φ comes from Theorem 1. Examining the proof of Theorem 1 it is possible to infer that $C_\varphi = C\|\varphi\|_{W^{2,\infty}(\Omega)}$. This, together with the Sobolev imbedding theorem, gives, by duality, an estimate of ω_t in $L^\infty((0,T); H^{-M}(\Omega))$, for any $T > 0$ and some $M > 3$. This clearly implies the third a priori estimate.

It follows, by the Aubin–Lions lemma and the Banach–Alaoglu theorem, that there exists $\omega \in L^\infty((0,\infty); \mathcal{BM}(\Omega)) \cap C((0,T); H^{-L}(\Omega))$, for any $T > 0$ and some $L < M$, such that, passing to a subsequence if necessary, we have $\omega^n \rightharpoonup \omega$ weak-$*$ in $L^\infty((0,\infty); \mathcal{BM}(\Omega))$ and $\omega^n \to \omega$ strongly in $C((0,T); H^{-L}(\Omega))$. We will observe that ω is a weak solution with initial data ω_0. Let $\varphi \in \mathcal{A}$. As usual, the only difficulty in passing to the limit in each of the terms in (14) is the nonlinear term,

$$\mathcal{W}_{NL}\big[\omega^n,\varphi\big] \equiv \int_0^\infty \int_\Omega \int_\Omega H_\varphi^\Omega(x,y,t)\omega^n(x,t)\omega^n(y,t)\,dy\,dx\,dt.$$

By Lemma 2 we have that there are no time-averaged concentrations, i.e., $|\omega^n|$ does not form Diracs when $n \to \infty$. This fact, together with the boundedness of H_φ^Ω derived in Theorem 1, and the fact that H_φ^Ω is continuous off of the diagonal $x = y$, allows us to deduce that $\mathcal{W}_{NL}[\omega^n,\varphi] \to \mathcal{W}_{NL}[\omega,\varphi]$ as $n \to \infty$. The proof of this last convergence follows precisely the same argument of the proof of Theorem 1 in [6] so we choose not to repeat it. □

Remark. We can simplify the argument leading to Theorem 2 if we restrict ourselves to proving existence of a classical weak solution, i.e., with test functions supported away from the boundary and with the boundary condition attained in the trace sense. Indeed, to prove existence of classical weak solutions one only needs to have H_φ^Ω bounded for test functions supported away from the boundary, which, as already noted in Remark 2 following Theorem 1, is a relatively easy result to obtain.

One issue that was discussed at length in [6] was the method of images. The relevant formulation states that smooth flow on a half-plane is a solution of the incompressible 2D Euler equations if and only if its symmetric extension is a solution in the full plane. This is not true for weak solutions if one uses the classical definition of weak solution in domains with boundary, as in [1], but if one uses a stronger notion of weak solution then an analogous equivalence may be obtained. Such an equivalence was proved in [6] for the half-plane. The extension of the method of images to weak solutions also works in the present case, mirror symmetric flow in the exterior of a bluff body.

First, we recall the definition of boundary-coupled weak solution in a general domain with boundary. Let U be a simply connected domain in the plane with rectifiable boundary. Let $\mathcal{A}$ be the set of admissible test functions, defined by:

$$\mathcal{A} \equiv \big\{\varphi \in C_c^\infty\big([0,\infty)\times \bar{U}\big) \mid \varphi \equiv 0 \text{ on } \partial U\big\}.$$

Definition 2. The function $\omega \in L^\infty([0,\infty);\mathcal{BM}(U))$ is called a boundary-coupled weak solution of the incompressible 2D Euler equations in U with initial data ω_0 if:

(a) the velocity $u \equiv K_U[\omega]$ belongs to $L^\infty_{\text{loc}}([0,\infty);(L^2(U))^2)$, and
(b) for any test function $\varphi \in \mathcal{A}$, it holds that

$$\mathcal{W}[\omega,\varphi] \equiv \int_0^\infty \int_U \varphi_t\omega(x,t)\,dx\,dt + \int_0^\infty \int_U \int_U H_\varphi^U(x,y,t)\omega(x,t)\omega(y,t)\,dy\,dx\,dt + \int_U \varphi(x,0)\omega_0(x)\,dx = 0, \tag{14}$$

where

$$H_\varphi^U(x,y,t) \equiv \frac{1}{2}\big(\nabla\varphi(x,t)\cdot K_U(x,y) + \nabla\varphi(y,t)\cdot K_U(y,x)\big). \tag{15}$$

We can now give the extension of the method of images to mirror symmetric flows outside a symmetric body.

Theorem 3. *The function* $\omega = \omega(x,t) \in L^\infty([0,\infty); \mathcal{BM}(\Omega_+)$ *is a boundary-coupled weak solution of the incompressible* 2D *Euler equations in* Ω_+ *if and only if its odd extension is a boundary-coupled weak solution in* Ω.

The proof of Theorem 3 is somewhat involved, but a faithful adaptation of the argument used in [6, Theorem 2] works in this case as well. For the sake of brevity, we omit the proof. One immediate consequence of Theorems 2 and 3 is the following.

Corollary 1. *Let* $\omega_0 \in \mathcal{BM}_c^+(\Omega_+) \cap H^{-1}(\Omega_+)$. *There exists a boundary-coupled weak solution* ω *of the incompressible* 2D *Euler equations in* Ω_+ *with initial data* ω_0.

6. Concluding remarks

Several remarks are in order. First, all the conclusions discussed above remain true if the initial vorticity is perturbed by an integrable function with reflection symmetry as we did in [6]. Second, it would be extremely interesting to study the limits of approximate solutions generated by either Navier–Stokes approximations as in [7] or vortex methods as in [4]. The weak solution obtained in Theorem 2 is a limit of the approximate solutions obtained by regularizing the initial data and exactly solving the Euler equations.

It is natural to investigate the problem of existence of weak solutions in the sense of Definition 1 in a general domain. We already understand the special cases of the half-plane (see [6]) and certain compactly supported perturbations of the half plane, as noted above. The argument we presented here can be used to prove such a result for domains in the plane with simply connected boundary (plus technical assumptions on the behavior of said boundary at infinity), such as curved boundary half planes and channels. Indeed, let $U \subset \mathbb{R}^2$ be a domain with smooth, simply connected boundary and assume well-posedness of the Euler equations with smooth data in U. Take ω^n an approximate solution sequence obtained by regularizing vortex sheet initial data $\omega_0 \in \mathcal{B}M_+(U)$ and exactly solving the equations. It is easy to adapt the proofs of Lemmas 1 and 2 and of Theorem 1 to U. With this the argument used to prove Theorem 2 follows. We note that there is no well-posedness result explicitly available in the literature which applies to such general unbounded domains. This is the reason why we have not formulated a result corresponding to Theorem 2 in this context. We refer the reader to [12] for a broad discussion in this direction.

In the case of a bounded domain, a version of Corollary 1 would constitute a slight improvement of Delort's theorem for bounded domains. However, our argument cannot be adapted to prove existence for bounded domain flow. This is somewhat surprising, and the difficulty stems from the derivation of the a priori L^2_{loc} bound on the tangential velocity at the boundary, given by (5). To derive (5), we need to exhibit a test function with derivatives bounded up to second order which is monotonic when restricted to each connected component of the boundary. Otherwise, the identity obtained in Lemma 1 does not lead to an actual a priori estimate. Such a test function cannot exist on a domain with compact boundary components. It may be that this is just a technical difficulty, and that a boundary-coupled weak solution does exist for distinguished signed vortex sheet initial data in a bounded domain, but we would like to argue that this might not be the case. In fact we observe that this restriction might be the result of a meaningful physical distinction between compact and noncompact boundary components with respect to concentrations of vorticity. For a noncompact boundary component, vorticity that concentrates, forming a Dirac, and at the same time approaching the boundary, tends to move with large velocity and leave the compact parts of the flow domain. Since the test functions involved in the definition of weak solutions are compactly supported this kind of concentration ends up being irrelevant. However, with a compact boundary component, concentration of vorticity near the boundary leads to this vorticity moving faster and faster around this boundary component, without disappearing. Such concentration behavior would be entirely consistent with Lemma 1 and would require a substantially different approach to handle existence.

Finally, it is natural to ask what is gained by obtaining a boundary-coupled weak solution. As we have already noted, boundary-coupled weak solutions assume the boundary condition in a stronger fashion than classical weak solutions. As a consequence, we can prove the method of images for boundary-coupled weak solutions, for problems with reflection symmetry, which was the reason we introduced the concept. In what follows, we will prove an additional property of boundary-coupled weak solutions which we do not know how to prove for classical weak solutions.

For simplicity, we will focus on flow in the unit disk, a situation where existence of a boundary-coupled weak solution is an open problem. We recall that ideal flow should actually be regarded as small viscosity flow, and that the limiting behavior of solutions of the incompressible Navier–Stokes equations with small viscosity in the disk is poorly understood. A key issue in this problem is how vorticity is generated through the interaction of nearly ideal flow with material boundaries. If such "nearly ideal" flows actually become ideal in the limit (an important open question) then the resulting weak solution may well contain vorticity being produced or destroyed at the boundary. So, one interesting question is: are there weak solutions of the incompressible 2D Euler equations for which the total mass of vorticity is not conserved? This question makes sense for any domain in the plane. For smooth flows in a disk, another exactly conserved quantity is the moment of inertia, $\int |x|^2\omega$. Again, weak solutions may generate moment of inertia at the boundary. At the boundary of the unit disk, vorticity and density of moment of inertia are the same and hence it is reasonable that the net flux of these two quantities across the boundary be exactly the same. In fact, we can prove this property for boundary-coupled weak solutions.

Proposition 2. *Let ω be a boundary-coupled weak solution for the incompressible* 2D *Euler equations in the unit disk $\mathcal{D}$. Then the quantity*

$$\int \left(1-|x|^2\right)\omega(x,t)\,dx$$

is constant in time.

Proof. Let $\eta \in C_c^\infty((0,\infty))$. The test function $\varphi = \varphi(x,t) \equiv \eta(t)(1-|x|^2)$ is an admissible test function which can be used in the definition of boundary-coupled weak solution. Moreover, a direct calculation shows that $H_\varphi^{\mathcal{D}}$ vanishes identically (this is connected with the fact that the moment of inertia is exactly conserved). Let

$$F = F(t) \equiv \int_{\mathcal{D}} \left(1-|x|^2\right)\omega(x,t)\,dx.$$

Since $1-|x|^2 \in C_0(\mathcal{D})$ and $\omega \in L^\infty((0,\infty);\mathcal{BM}(\mathcal{D}))$, it follows that $F \in L^\infty(0,\infty)$.

Therefore, Definition 2 implies that

$$\int_0^\infty \eta'(t)F(t)\,dt = 0,$$

which in turn implies that F is constant almost everywhere. □

Let us conclude by explaining the difficulty in obtaining the result in Proposition 2 for *classical* weak solutions. Suppose ω is a classical weak solution in (the open set) $\mathcal{D}$, i.e., ω satisfies the estimates and weak formulation in Definition 2 for test functions $\varphi \in C_c^\infty([0,\infty) \times \mathcal{D})$, and let $\eta \in C_c^\infty((0,\infty))$ be as in the proof of Proposition 2. Clearly, one must approximate $\varphi \equiv \eta(t)(1-|x|^2)$ by a sequence $\{\varphi^n\} \in C_c^\infty((0,\infty) \times \mathcal{D})$ in such a way as to guarantee that $H^{\mathcal{D}}_{\varphi^n} \to H^{\mathcal{D}}_{\varphi} \equiv 0$ uniformly. It is easy to approximate φ by functions in C_c^∞, however, it is not possible to approximate φ by such functions with respect to the $W^{2,\infty}$-norm because the trace of $\nabla\varphi$ does not vanish on the boundary of $\mathcal{D}$. Note that the L^∞-estimate on $H^{\mathcal{D}}_{\varphi^n}$ is given in terms of (uniform estimates on) derivatives of φ^n up to second order, as can be verified by following carefully the proof of Theorem 1. One interesting open problem is to find, if possible, an example of a classical weak solution which is not boundary-coupled.

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