Geometric programming with fuzzy parameters in engineering optimization

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Received 20 August 2006; received in revised form 29 December 2006; accepted 4 January 2007
Available online 30 January 2007

Abstract

Geometric programming provides a powerful tool for solving a variety of engineering optimization problems. Many applications of geometric programming are engineering design problems in which some of the problem parameters are estimates of actual values. When the parameters in the problem are imprecise, the calculated objective value should be imprecise as well. This paper develops a procedure to derive the fuzzy objective value of the fuzzy posynomial geometric programming problem when the exponents of decision variables in the objective function, the cost and the constraint coefficients, and the right-hand sides are fuzzy numbers. The idea is based on Zadeh’s extension principle to transform the fuzzy geometric programming problem into a pair of two-level of mathematical programs. Based on duality algorithm and a simple algorithm, the pair of two-level mathematical programs is transformed into a pair of conventional geometric programs. The upper bound and lower bound of the objective value are obtained by solving the pair of geometric programs. From different values of $\alpha$, the membership function of the objective value is constructed. Two examples are used to illustrate that the whole idea proposed in this paper.
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Keywords: Geometric programming; Fuzzy set; Optimization

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doi:10.1016/j.ijar.2007.01.004
1. Introduction

Geometric programming, a technique developed for solving algebraic nonlinear programming problems subject to linear or nonlinear constraints, is useful in the study of a variety of optimization problems. Its great impact has been in the area of integrated circuit design [4, 8, 11], manufacturing system design [3, 7, 13], and project management [16]. The familiar posynomial geometric programming is

\[ Z = \min_x s_0 \sum_{i=1}^{s_0} c_{oi} \prod_{j=1}^{n} x_j^{a_{ij}} \]

\[ \text{s.t. } \sum_{i=1}^{s_i} c_{ii} \prod_{j=1}^{n} x_j^{\gamma_{ij}} \leq 1, \quad i = 1, \ldots, m, \]

\[ x_j > 0, \quad j = 1, \ldots, n. \]  

(1)

The objective function contains \( s_0 \) terms, while inequality constraints contain \( s_i \) terms for \( i = 1, 2, \ldots, m \). Exponents \( a_{ij} \) and \( \gamma_{ij} \) are arbitrary constants and coefficients \( c_{oi} \) and \( c_{ii} \) are positive. Since there is a strong duality theorem for geometric programming problems, the geometric program with highly nonlinear constraints can be stated equivalently as one with only linear constraints. If the primal problem is in posynomial form, then a global minimizing solution to that problem can be obtained by solving the dual maximization. The dual problem has the desirable features of being linearly constrained and having an objective function with attractive structural properties. This allows for the development of powerful solution techniques for geometric programs.

Efficient algorithms have been developed for solving the geometric programming problems when the cost and constraint coefficients are known exactly. However, many applications of geometric programming are engineering design problems in which some of the problem parameters are estimates of actual values [2]. There are also cases that these coefficients may not be presented in a precise manner. For example, the tool life in machining economics model may fluctuate due to different machining operations and conditions. To deal quantitatively with imprecise information in making decisions, Bellman and Zadeh [1] and Zadeh [18] introduce the notion of fuzziness.


Intuitively, when the exponents of decision variables in the objective function, the cost and the constraint coefficients, and the right-hand sides are fuzzy numbers, the derived objective value is fuzzy as well. In this paper, we develop a solution procedure that is able to calculate the fuzzy objective value, where at least one of the parameters in the geometric program is a fuzzy number. The idea is to apply Zadeh’s extension principle [18, 19]. A pair
of two-level mathematical programs is formulated to calculate the upper and lower bounds of the objective value at possibility level \( \alpha \). The membership function of the fuzzy objective value is derived numerically by enumerating different values of \( \alpha \).

The remainder of this paper is organized as follows. The fuzzy geometric programming problem is first introduced. Next, a pair of geometric programs for calculating the \( \alpha \)-cuts of the objective value is formulated based on the extension principle. We use two engineering optimization examples to illustrate the method proposed in this paper. Finally, a summary of the research is presented.

2. Mathematical formulation

Suppose we modify the right-hand sides of the constraints in the geometric program (1) as follow:

\[
Z = \min_{x} \sum_{i=1}^{n} c_{0i} \prod_{j=1}^{m} x_{ij}^{a_{0ij}} \\
\text{s.t.} \quad \sum_{i=1}^{n} c_{it} \prod_{j=1}^{m} x_{ij}^{e_{itj}} \leq b_{i}, \quad i = 1, \ldots, m, \\
x_{j} > 0, \quad j = 1, \ldots, n,
\]

where all \( b_{i} \) are positive numbers. If \( b_{i} = 1, \forall i \), then this modified geometric program is the original geometric program.

Intuitively, if any of the parameters \( a_{0ij}, b_{i}, c_{0i}, \) or \( c_{it} \) is fuzzy, the objective value should be fuzzy as well. The conventional geometric programming problem defined in (2) then turns into fuzzy geometric programming problem. We suppose that the exponents \( a_{0ij} \), the right-hand side \( b_{i} \), the cost coefficient \( c_{0i} \), and the constraint coefficient \( c_{it} \) are approximately known and can be represented by the convex fuzzy sets \( \tilde{A}_{0ij}, \tilde{B}_{i}, \tilde{C}_{0i}, \) and \( \tilde{C}_{it} \), respectively. Let \( \mu_{\tilde{A}_{0ij}}, \mu_{\tilde{B}_{i}}, \mu_{\tilde{C}_{0i}}, \) and \( \mu_{\tilde{C}_{it}} \) denote their membership functions, respectively. We have

\[
\tilde{A}_{0ij} = \{ (a_{0ij}, \mu_{\tilde{A}_{0ij}}(a_{0ij})) | a_{0ij} \in S(\tilde{A}_{0ij}) \}, \quad (3a) \\
\tilde{B}_{i} = \{ (b_{i}, \mu_{\tilde{B}_{i}}(b_{i})) | b_{i} \in S(\tilde{B}_{i}) \}, \quad (3b) \\
\tilde{C}_{0i} = \{ (c_{0i}, \mu_{\tilde{C}_{0i}}(c_{0i})) | c_{0i} \in S(\tilde{C}_{0i}) \}, \quad (3c) \\
\tilde{C}_{it} = \{ (c_{it}, \mu_{\tilde{C}_{it}}(c_{it})) | c_{it} \in S(\tilde{C}_{it}) \}, \quad (3d)
\]

where \( S(\tilde{A}_{0ij}), S(\tilde{B}_{i}), S(\tilde{C}_{0i}), \) and \( S(\tilde{C}_{it}) \) are the supports of \( \tilde{A}_{0ij}, \tilde{B}_{i}, \tilde{C}_{0i}, \) and \( \tilde{C}_{it} \), which denote the universe sets of the decision variables of the objective function, the right-hand side, the cost coefficient, and the constraint coefficient, respectively. The fuzzy objective function \( \bar{Z} = \sum_{i=1}^{n} \tilde{C}_{it} \prod_{j=1}^{m} x_{ij}^{e_{itj}} \), which is to be minimized, together with the following constraints, constitute the fuzzy geometric programming problem:

\[
\text{s.t.} \quad \sum_{i=1}^{n} \tilde{C}_{it} \prod_{j=1}^{m} x_{ij}^{e_{itj}} \leq \tilde{B}_{i}, \quad i = 1, \ldots, m, \\
x_{j} > 0, \quad j = 1, \ldots, n.
\]
Without loss of generality, all $\tilde{A}_{0ij}$, $\tilde{B}_i$, $\tilde{C}_0$, and $\tilde{C}_d$ in Model (4) are assumed to be convex fuzzy numbers, as crisp values can be represented by degenerated membership functions which only have one value in their domains.

Denote the $x$-cuts of $A_{0ij}$, $B_i$, $C_0$, and $C_d$ as

\[
(A_{0ij})_x = [(A_{0ij})_x^L, (A_{0ij})_x^U] = \{a_{0ij} \in S(\tilde{A}_{0ij}) | \mu_{\tilde{A}_{0ij}}(a_{0ij}) \geq x\}, \max_{a_{0ij}} \{a_{0ij} \in S(\tilde{A}_{0ij}) | \mu_{\tilde{A}_{0ij}}(a_{0ij}) \geq x\},
\]

(5a)

\[
(B_i)_x = [(B_i)_x^L, (B_i)_x^U] = \{b_i \in S(\tilde{B}_i) | \mu_{\tilde{B}_i}(b_i) \geq x\}, \max_{b_i} \{b_i \in S(\tilde{B}_i) | \mu_{\tilde{B}_i}(b_i) \geq x\},
\]

(5b)

\[
(C_0)_x = [(C_0)_x^L, (C_0)_x^U] = \{c_0 \in S(\tilde{C}_0) | \mu_{\tilde{C}_0}(c_0) \geq x\}, \max_{c_0} \{c_0 \in S(\tilde{C}_0) | \mu_{\tilde{C}_0}(c_0) \geq x\},
\]

(5c)

\[
(C_d)_x = [(C_d)_x^L, (C_d)_x^U] = \{c_d \in S(\tilde{C}_d) | \mu_{\tilde{C}_d}(c_d) \geq x\}, \max_{c_d} \{c_d \in S(\tilde{C}_d) | \mu_{\tilde{C}_d}(c_d) \geq x\}.
\]

(5d)

These intervals indicate where the exponents of the decision variables in the objective function, the right-hand side, the cost coefficients, and the constraint coefficients lie at possibility level $x$. We are interested in deriving the membership function of the objective value $\tilde{Z}$. Since $\tilde{Z}$ is a fuzzy number rather than a crisp number, we apply Zadeh’s extension principle [18,19] to transform the problem into a family of conventional geometric programs to be solved.

Based on the extension principle, the membership function $\mu_{\tilde{Z}}$ can be defined as

\[
\mu_{\tilde{Z}}(z) = \sup \min_{a,b,c} \{\mu_{\tilde{A}_{0ij}}(a_{0ij}), \mu_{\tilde{B}_i}(b_i), \mu_{\tilde{C}_0}(c_0), \mu_{\tilde{C}_d}(c_d), \forall i,j,t | z = Z(a,b,c)\},
\]

(6)

where $Z(a,b,c)$ is the function of the conventional geometric program that is defined in Model (2). In Eq. (6), several membership functions are involved. To derive $\mu_{\tilde{Z}}$ in closed form is hardly possible. According to (6), $\mu_{\tilde{Z}}$ is the minimum of $\mu_{\tilde{A}_{0ij}}$, $\mu_{\tilde{B}_i}$, $\mu_{\tilde{C}_0}$, and $\mu_{\tilde{C}_d}$, $\forall i,j,t$. We need $\mu_{\tilde{A}_{0ij}}(a_{0ij}) \geq x$, $\mu_{\tilde{B}_i}(b_i) \geq x$, $\mu_{\tilde{C}_0}(c_0) \geq x$, and $\mu_{\tilde{C}_d}(c_d) \geq x$, and at least one $\mu_{\tilde{A}_{0ij}}(a_{0ij})$, $\mu_{\tilde{B}_i}(b_i)$, $\mu_{\tilde{C}_0}(c_0)$, or $\mu_{\tilde{C}_d}(c_d)$, $\forall i,j,t$, equal to $x$ such that $z = Z(a,b,c)$ to satisfy $\mu_{\tilde{Z}}(z) = x$. To find the membership function $\mu_{\tilde{Z}}$, it suffices to find the right shape function and left shape function of $\mu_{\tilde{Z}}$, which is equivalent to finding the upper bound of the objective value $Z_{x}^U$ and lower bound of the objective $Z_{x}^L$ at specific $x$ level. Since $Z_{x}^U$ is the maximum of $Z(a,b,c)$ and $Z_{x}^L$ is the minimum of $Z(a,b,c)$, they can be expressed as

\[
Z_{x}^U = \max \left\{Z(a,b,c) | (A_{0ij})_x^L \leq a_{0ij} \leq (A_{0ij})_x^U, (B_i)_x^L \leq b_i \leq (B_i)_x^U,\right.\]

(7a)

\[
(C_0)_x^L \leq c_0 \leq (C_0)_x^U, (C_d)_x^L \leq c_d \leq (C_d)_x^U, \forall i,j,t\right\},
\]

(7b)

\[
Z_{x}^L = \min \left\{Z(a,b,c) | (A_{0ij})_x^L \leq a_{0ij} \leq (A_{0ij})_x^U, (B_i)_x^L \leq b_i \leq (B_i)_x^U,\right.\]

(7c)

\[
(C_0)_x^L \leq c_0 \leq (C_0)_x^U, (C_d)_x^L \leq c_d \leq (C_d)_x^U, \forall i,j,t\right\}.
\]

(7d)
which can be reformulated as the following pair of two-level mathematical programs:

\[
Z_x^U = \max_{\substack{(A_{0ij})_x^L \leq a_{0ij} \leq (A_{0ij})_x^U \quad \forall i, j, t \\ (B_i)_x^L \leq b_i \leq (B_i)_x^U \quad \forall i, j, t \\ (C_0)_x^L \leq c_0 \leq (C_0)_x^U \quad \forall i, j, t \\ (C_{it})_x^L \leq c_{it} \leq (C_{it})_x^U \quad \forall i, j, t}} \min_x \sum_{i=1}^{s_i} c_{0i} \prod_{j=1}^{n} x_{ij}^{a_{0ij}} \\
\text{s.t.} \quad \sum_{i=1}^{s_i} c_{it} \prod_{j=1}^{n} x_{ij}^{a_{0ij}} \leq b_i, \quad i = 1, \ldots, m, \\
x_j > 0, \quad j = 1, \ldots, n,
\]

(8a)

\[
Z_x^L = \min_{\substack{(A_{0ij})_x^L \leq a_{0ij} \leq (A_{0ij})_x^U \quad \forall i, j, t \\ (B_i)_x^L \leq b_i \leq (B_i)_x^U \quad \forall i, j, t \\ (C_0)_x^L \leq c_0 \leq (C_0)_x^U \quad \forall i, j, t \\ (C_{it})_x^L \leq c_{it} \leq (C_{it})_x^U \quad \forall i, j, t}} \min_x \sum_{i=1}^{s_i} c_{0i} \prod_{j=1}^{n} x_{ij}^{a_{0ij}} \\
\text{s.t.} \quad \sum_{i=1}^{s_i} c_{it} \prod_{j=1}^{n} x_{ij}^{a_{0ij}} \leq b_i, \quad i = 1, \ldots, m, \\
x_j > 0, \quad j = 1, \ldots, n.
\]

(8b)

In the inner program of Model (8), one can divide the constraint coefficients \(c_{it}\) by the right-hand side value \(b_i\), \(\forall i\), to be the following standard geometric program form:

\[
Z_x^U = \max_{\substack{(A_{0ij})_x^L \leq a_{0ij} \leq (A_{0ij})_x^U \quad \forall i, j, t \\ (B_i)_x^L \leq b_i \leq (B_i)_x^U \quad \forall i, j, t \\ (C_0)_x^L \leq c_0 \leq (C_0)_x^U \quad \forall i, j, t \\ (C_{it})_x^L \leq c_{it} \leq (C_{it})_x^U \quad \forall i, j, t}} \min_x \sum_{i=1}^{s_i} c_{0i} \prod_{j=1}^{n} x_{ij}^{a_{0ij}} \\
\text{s.t.} \quad \sum_{i=1}^{s_i} c_{it} \prod_{j=1}^{n} x_{ij}^{a_{0ij}} \leq 1, \quad i = 1, \ldots, m, \\
x_j > 0, \quad j = 1, \ldots, n,
\]

(9a)

\[
Z_x^L = \min_{\substack{(A_{0ij})_x^L \leq a_{0ij} \leq (A_{0ij})_x^U \quad \forall i, j, t \\ (B_i)_x^L \leq b_i \leq (B_i)_x^U \quad \forall i, j, t \\ (C_0)_x^L \leq c_0 \leq (C_0)_x^U \quad \forall i, j, t \\ (C_{it})_x^L \leq c_{it} \leq (C_{it})_x^U \quad \forall i, j, t}} \min_x \sum_{i=1}^{s_i} c_{0i} \prod_{j=1}^{n} x_{ij}^{a_{0ij}} \\
\text{s.t.} \quad \sum_{i=1}^{s_i} c_{it} \prod_{j=1}^{n} x_{ij}^{a_{0ij}} \leq 1, \quad i = 1, \ldots, m, \\
x_j > 0, \quad j = 1, \ldots, n.
\]

(9b)

The inner program in Model (9a) calculates the objective value for each set of \((a_{0ij}, b_i, c_0, c_{it})\) defined by the outer program, while the outer program determines the set of \((a_{0ij}, b_i, c_0, c_{it})\) that derives the largest objective value. Likewise, in Model (9b) the inner program calculates the objective value for each given set of \((a_{0ij}, b_i, c_0, c_{it})\), while the outer program determines the set of \((a_{0ij}, b_i, c_0, c_{it})\) that produces the smallest objective value. In the next section we shall develop a solution method to transform Models (9a) and (9b) into one-level conventional geometric programs.
3. Solution approach

3.1. Upper bound

In Model (9a) we want to find a set of \((a_{0ij}, b_i, c_{0r}, c_{1t})\) that derive the maximal objective value. The outer program and inner program of (9a) have different directions for optimization, one for maximization and one for minimization. A transformation is required to make a solution obtainable. To solve Model (9a), the dual of the inner program is formulated to become a maximization problem to be consistent with the maximization operation of outer program. It is well-known from the duality theorem of geometric programming that the primal model and the dual model have the same objective value. According to Beightler and Phillips [2] and Duffin et al. [5], one can transform inner program to its corresponding dual geometric program. Thus Model (9a) becomes

\[
Z^U_z = \max_{(A_{0ij})^U_z \leq a_{0ij} \leq (A_{0ij})^L_z} \left\{ \max_{w} \prod_{t=1}^{s_0} \left( \frac{c_{0t}}{w_{0t}} \right)^{w_{0t}} \prod_{i=1}^{m} \prod_{t=1}^{s_i} \left( \frac{c_{1t} w_{1t}}{b_{iw_{it}}} \right)^{w_{it}} \right\}
\]

\[
s.t. \quad \sum_{t=1}^{s_0} w_{0t} = 1, \quad i = 1, \ldots, m,
\]

\[
\sum_{t=1}^{s_0} a_{0ij} w_{0t} + \sum_{i=1}^{m} \sum_{t=1}^{s_i} \gamma_{ij} w_{it} = 0, \quad j = 1, \ldots, n,
\]

\[
w_{it} \geq 0, \quad \forall i, t.
\]

(10)

Since both inner program and outer program perform the same maximization operation and the variables \(b_i, c_{0r}, c_{1t}\) are all in the objective function, one can set all \(c_{0r}\) and \(c_{1t}\) to their upper bounds \(C_{0r}^U\) and \(C_{1t}^U\), respectively. On the other hand, one should set all \(b_i\) to their lower bounds \(B_i^L\). Consequently, the two-level mathematical program in (10) can be simplified to the following conventional geometric program:

\[
Z^U_z = \max_{w} \prod_{t=1}^{s_0} \left( \frac{C_{0t}}{w_{0t}} \right)^{w_{0t}} \prod_{i=1}^{m} \prod_{t=1}^{s_i} \left( \frac{C_{1t} w_{1t}}{b_{iw_{it}}} \right)^{w_{it}}
\]

\[
s.t. \quad \sum_{t=1}^{s_0} w_{0t} = 1,
\]

\[
\sum_{i=1}^{m} \sum_{t=1}^{s_i} \gamma_{ij} w_{it} = 0, \quad j = 1, \ldots, n,
\]

\[
(A_{0ij})^L_z \leq a_{0ij} \leq (A_{0ij})^U_z, \quad \forall i, j,
\]

\[
w_{it} \geq 0, \quad \forall i, t,
\]

(11)

where \(\sum_{t=1}^{s_i} w_{it} = w_{i0}, \forall i\). Model (11) is a nonlinear program with a concave objective function [2,5] and nonlinear terms \(a_{0ij} w_{0t}\) in (11.2). The nonlinear constraints can be linearized by multiplying Constraint (11.3) by \(w_{0t}\) and substituting \(a_{0ij} w_{0t}\) by \(y_{ij}\) to obtain the following concave programming problem with linear constraints:
\[ Z^U_x = \text{Max}_{\mathbf{w}} \prod_{t=1}^{s_0} \left( \frac{(C_{0t})^U_{x}}{w_{0t}} \right)^{w_{0t}} \prod_{i=1}^{m} \prod_{t=1}^{s_i} \left( \frac{(C^U_{it} w_i)}{(B^L_{ij} w_{it})} \right)^{w_{it}} \] (12)

s.t. \[ \sum_{t=1}^{s_0} w_{0t} = 1, \]
\[ \sum_{t=1}^{s_0} y_{ij} + \sum_{i=1}^{m} \sum_{t=1}^{s_i} \gamma_{it} w_{it} = 0, \quad j = 1, \ldots, n, \]
\[ (A_{0j})^L_{x} \leq y_{ij} \leq (A_{0j})^U_{x}, \quad \forall t, j, \]
\[ w_{it} \geq 0, \quad \forall i, t. \]

We can derive the upper bound of the objective value by solving Model (12). Notably, if all the exponent \(a_{0j}\) in (11.2) are constants, rather than variables, then (11.3) is vanished and we can derive the objective value simply by solving Model (11).

3.2. Lower bound

Model (9b) is to find the smallest objective value among all the possible objective values. To derive the lower bound of the objective value in Model (9b), one can directly set \(c_{0t}\) to its lower bound \((C^L_{0t})_{x}\) in the objective function. Furthermore, since the value of right-hand side in Model (9b) is the constant 1, the lower the ratios of \(c_{it}/b_i\) in constraints, the larger the feasible region is. Therefore, the values of \(c_{it}\) and \(b_i\) should, respectively, set to its lower bound \((C^L_{it})_{x}\) and its upper bound \((B^U_{ij})_{x}\), \(\forall i, t\). Hence, Model (9b) can be rewritten as the following mathematical program:

\[ Z^L_x = \text{Min}_{\mathbf{X}} \left\{ \begin{array}{l}
\text{Min} \sum_{t=1}^{s_0} (C_{0t})^L_{x} \prod_{j=1}^{n} x_{t,j}^{a_{0j}} \\
\text{s.t.} \sum_{t=1}^{s_i} (C^L_{it}/(B^U_{ij}))_{x} \prod_{j=1}^{n} x_{t,j}^{\gamma_{ij}} \leq 1, \quad i = 1, \ldots, m, \\
\quad x_j > 0, \quad j = 1, \ldots, n.
\end{array} \right. \] (13)

In the objective function of Model (13), if \(0 < x_j < 1\), then \(x_{t,j}^{a_{0j}}\) is a decreasing function; conversely, if \(x_j > 1\), then \(x_{t,j}^{a_{0j}}\) is an increasing function. Certainly, if \(x_j = 1\), then any value of \(a_{0j}\) has no effect on the objective value. Therefore, we can classify type of decision variables as \(0 < x_j < 1\) and \(x_j \geq 1\). Let \(J = \{1, 2, \ldots, n\}, \quad P = \{j|x_j^{*} \geq 1\}\), and \(Q = \{j|0 < x_j^{*} < 1\}\). We have \(P \cup Q = J\). With the two types of decision variables, Model (13) can be transformed into the problem:

\[ Z^L_x = \text{Min}_{\mathbf{X}} \sum_{t=1}^{s_0} (C_{0t})^L_{x} \prod_{j \in P} x_{j}^{a_{0j}} \prod_{j \in Q} x_{j}^{a_{0j}} \]
\[ \quad \text{s.t.} \sum_{t=1}^{s_i} (C^L_{it}/(B^U_{ij}))_{x} \prod_{j \in P} x_{t,j}^{\gamma_{ij}} \prod_{j \in Q} x_{t,j}^{\gamma_{ij}} \leq 1, \quad i = 1, \ldots, m, \]
\[ \quad (A_{0j})^L_{x} \leq a_{0j} \leq (A_{0j})^U_{x}, \]
\[ \quad x_j > 0, \quad j = 1, \ldots, n. \] (14)
Since \( x_j > 1 \), \( a_{0j} \) is an increasing function. The value of \( a_{0j} \) decreases when the value of \( a_{0j} \) decreases. For deriving the lower bound of the objective value, we should specify \( a_{0j} \) to its lower bound \( A_{0j}^L \). On the other hand, if \( 0 < x_j < 1 \), then \( a_{0j} \) is a decreasing function. The value of \( a_{0j} \) decreases when the value of \( a_{0j} \) increases. In this case, we should set \( a_{0j} \) to its upper bound \( A_{0j}^U \) to obtain the lower bound of the objective value. In other words, we can simplify Model (14) to the following mathematical form:

\[
Z^L_x = \min_x \sum_{t=1}^{s} \left( C_{0t}^L \right) \prod_{j \in \mathcal{P}} x_j^{A_{0j}^L} \prod_{j \in \mathcal{Q}} x_j^{A_{0j}^U}
\]

\[
\text{s.t.} \sum_{t=1}^{s} \left( C_{it}^L \right) \prod_{j \in \mathcal{P}} x_j^{B_{it}^L} \prod_{j \in \mathcal{Q}} x_j^{B_{it}^U} \leq 1, \quad i = 1, \ldots, m,
\]

\[
x_j > 0, \quad j = 1, \ldots, n.
\]

However, the value of \( x_j \) in (15) is still unknown and we need to solve (14) in advance to derive the value of \( x_j \). Based on which, the value of exponent \( a_{0j} \) can be specified appropriately. This causes some difficulty in assigning the correct value to \( a_{0j} \). The dual form of Model (15) is the following geometric program:

\[
Z^L_x = \max_w \prod_{t=1}^{s} \left( \left( C_{0t}^L \right) w_{0t} \right)^{w_{0t}} \prod_{t=1}^{m} \prod_{i=1}^{s} \left( C_{it}^L w_{it} \right)^{w_{it}} \prod_{t=1}^{m} \prod_{i=1}^{s} \left( B_{it}^U w_{it} \right)^{w_{it}}
\]

\[
\text{s.t.} \sum_{t=1}^{s} w_{0t} = 1,
\]

\[
\sum_{t=1}^{s} \left( A_{0j}^L \right) w_{0t} + \sum_{i=1}^{m} \sum_{t=1}^{s} \gamma_{it} w_{it} = 0, \quad j \in \mathcal{P},
\]

\[
\sum_{t=1}^{s} \left( A_{0j}^U \right) w_{0t} + \sum_{i=1}^{m} \sum_{t=1}^{s} \gamma_{it} w_{it} = 0, \quad j \in \mathcal{Q},
\]

\[
w_{it} \geq 0, \quad \forall i, t,
\]

where \( \sum_{i=1}^{s} w_{it} = w_{0t} \). At a specified \( x \)-level, Model (16) is a conventional geometric program. Nevertheless, we are not able to solve without knowing the sets \( \mathcal{P} \) and \( \mathcal{Q} \). One idea is to guess all decision variables \( x_j \geq 1 \) in Model (15) and formulate a geometric program according to Model (16). The constraints define one possible set of feasible region. We then solve this program. If the optimal values for all \( x_j^* \geq 1 \), as we guess in the initial stage, then we have found the lower bound of the objective value \( Z^L \). If, on the other hand, not all \( x_j^* \geq 1 \), then we need to modify the index sets \( \mathcal{P} \) and \( \mathcal{Q} \) according to the values of the decision variables and formulate another geometric program per Model (16). The following procedure describes the solution method in an algorithmic way.

Step 0: Define \( \mathcal{P} = \{1, 2, \ldots, n\} \) and \( \mathcal{Q} = \emptyset \).

Step 1: Formulate a geometric program according to Model (16) and solve to get the optimal solution \( \mathbf{w}^* \) and the objective value.

Step 2: Transform \( \mathbf{w}^* \) into \( \mathbf{x}^* \) for Model (15) by using the method introduced in [2]. If all \( j \) such that \( x_j^* \geq 1 \) belong to \( \mathcal{P} \) and all \( j \) such that \( 0 < x_j^* < 1 \) belong to \( \mathcal{Q} \), then the optimal solution is found. Otherwise, continue Step 3.
Step 3: Define the index sets $P = \{ j \in J \mid x_j \geq 1 \}$ and $Q = \{ j \in J \mid 0 < x_j < 1 \}$. Go to Step 1.

After performing the algorithm, we can derive the lower bound of the objective value $Z^L_x$. Similar to Model (11), if all $a_{0j}$ are constants, then we can directly solve model (16) to obtain the objective value $Z^L_x$. Together with $Z^U_x$ solved from Model (12), $[Z^L_x, Z^U_x]$ constitutes the interval that the objective value lies.

For two possibility levels $x_1$ and $x_2$ such that $0 < x_2 < x_1 \leq 1$, the feasible regions defined by $x_1$ in Models (8a) and (8b) are smaller than those defined by $x_2$. Consequently, $Z^L_{x_1} \geq Z^L_{x_2}$ and $Z^U_{x_1} \leq Z^U_{x_2}$; in other words, the left shape function $L(z)$ is nondecreasing and the right shape function $R(z)$ is nonincreasing. This property assures the convexity of $Z$. From $L(z)$ and $R(z)$, the membership function $\mu_z$ is constructed as

$$\mu_z = \begin{cases} 
L(z), & Z^L_{x=0} \leq z \leq Z^L_{x=1}, \\
1, & Z^L_{x=1} \leq z \leq Z^U_{x=1}, \\
R(z), & Z^U_{x=1} \leq z \leq Z^U_{x=0}.
\end{cases}$$

The numerical solutions for $Z^L_x$ and $Z^U_x$ at different possibility level $z$ can be collected to approximate the shapes of $L(z)$ and $R(z)$.

4. Numerical examples

In this section, we present two engineering design examples to illustrate the solution method proposed in this paper. The notation used here is $(p, q, r, s)$ for a trapezoidal fuzzy number with $p$, $q$, $r$, and $s$ as the coordinates of the four vertices of the trapezoid and $(x, y, z)$ for a triangular fuzzy number with $x$, $y$, and $z$ as the coordinates of the three vertices of the triangle.

**Example 1.** Consider the design problem of a journal bearing. The design of journal bearing is an inverse problem, where for a given load and speed, the eccentricity ratio and attitude angle are determined. The engineers have no experiences in designing this newest type of journal bearing. Therefore, some parameters of the design are approximately known and are estimated by engineers. Let $x_1$ be radial clearance, $x_2$ be fluid force, $x_3$ be journal diameter, $x_4$ be journal rotation speed, and $x_5$ be the length to diameter ratio. The following mathematical form can describe the design problem:

$$\begin{align*}
\text{Min} & \quad 0.5x_1^2x_2x_4x_5 + (1, 1.1, 1.3)x_1^{-1}x_2^{-1}x_3^{-1} \\
\text{s.t.} & \quad (8.0, 8.4, 8.7)x_1x_2^{-1}x_3^{-1}x_4^{-1}x_5 \leq (4.0, 4.1, 4.2), \\
& \quad 0.5x_2x_3 + (0.9, 1.0, 1.1)x_1x_4^{-1}x_5^{-1} + (1.3, 1.6, 1.8)x_3x_4 \leq 1, \\
& \quad x_1, x_2, x_3, x_4, x_5 > 0,
\end{align*}$$

where the numbers in parentheses are the estimated values of the parameters. Since the exponents of the decision variables are all crisp numbers, Models (11) and (16) can be employed to obtain the upper bound and lower bound of the objective value, respectively.
According to Models (11) and (16), the upper bound and lower bound of the objective value at possibility level \( \alpha \) can be solved as

\[
Z^U_\alpha = \max_w \left( \frac{0.5}{w_{01}} \right)^{w_{01}} \left( \frac{1.3 - 0.2\alpha}{w_{02}} \right)^{w_{02}} \left( \frac{8.7 - 0.3\alpha}{4.0 + 0.1\alpha} \right)^{w_{11}} \left( \frac{0.5w_{20}}{w_{21}} \right)^{w_{21}} \left( \frac{(1.1 - 0.1\alpha)w_{20}}{w_{22}} \right)^{w_{22}} \left( \frac{(1.8 - 0.2\alpha)w_{20}}{w_{23}} \right)^{w_{23}}
\]

s.t. \( w_{01} + w_{02} = 1 \),
\( 2w_{01} - w_{02} + w_{11} + w_{22} = 0 \),
\( w_{01} - w_{02} - w_{11} + w_{21} = 0 \),
\( -w_{02} - w_{11} + w_{21} + w_{23} = 0 \),
\( w_{01} + w_{11} - w_{22} + w_{23} = 0 \),
\( w_{01} + w_{11} - w_{22} = 0 \),
\( w_{01}, w_{02}, w_{11}, w_{21}, w_{22}, w_{23} \geq 0 \),

where \( w_{21} + w_{22} + w_{23} = w_{20} \).

\[
Z^L_\alpha = \max_w \left( \frac{0.5}{w_{01}} \right)^{w_{01}} \left( \frac{1 + 0.1\alpha}{w_{02}} \right)^{w_{02}} \left( \frac{8.0 + 0.4\alpha}{4.2 - 0.1\alpha} \right)^{w_{11}} \left( \frac{0.5w_{20}}{w_{21}} \right)^{w_{21}} \left( \frac{(0.9 + 0.3\alpha)w_{20}}{w_{22}} \right)^{w_{22}} \left( \frac{(1.3 + 0.3\alpha)w_{20}}{w_{23}} \right)^{w_{23}}
\]

s.t. \( w_{01} + w_{02} = 1 \),
\( 2w_{01} - w_{02} + w_{11} + w_{22} = 0 \),
\( w_{01} - w_{02} - w_{11} + w_{21} = 0 \),
\( -w_{02} - w_{11} + w_{21} + w_{23} = 0 \),
\( w_{01} + w_{11} - w_{22} + w_{23} = 0 \),
\( w_{01} + w_{11} - w_{22} = 0 \),
\( w_{01}, w_{02}, w_{11}, w_{21}, w_{22}, w_{23} \geq 0 \),

where \( w_{21} + w_{22} + w_{23} = w_{20} \).

Using the logarithmic form of the dual objective function, this problem is a concave programming problem with linear constraints [2]. We can derive the global optimum solution from solving the problem. Since this problem has zero degree of difficulty, there is only one set of solution for \( w^* \). We can easily find that \( w_{01}^* = 0.2 \), \( w_{02}^* = 0.8 \), \( w_{11}^* = 0.1 \), \( w_{21}^* = 0.7 \), \( w_{22}^* = 0.3 \), and \( w_{23}^* = 0.2 \). The upper bound and lower bound of the objective value are determined by the value of possibility level \( \alpha \). Table 1 lists the \( \alpha \)-cuts of the objective value at 11 distinct \( \alpha \) values: \( 0, 0.1, 0.2, \ldots, 1.0 \). Using the method described in [2]
to transform $w^*$ to $x^*$, the values of decision variables $x_1$, $x_2$, $x_3$, $x_4$, and $x_5$ can be recovered. At $\alpha$-level $= 0$, the value of $Z_{x=0}^U = 4.314$ occurs at $x_1^* = 0.323$, $x_2^* = 11.652$, $x_3^* = 0.100$, $x_4^* = 0.925$, $x_5^* = 1.536$ and the value of $Z_{x=0}^L = 3.045$ occurs at $x_1^* = 0.352$, $x_2^* = 7.763$, $x_3^* = 0.150$, $x_4^* = 0.853$, $x_5^* = 1.458$. At $\alpha$-level $= 1$, the value of $Z_{x=1}^U = Z_{x=1}^L = 3.561$ occurs at $x_1^* = 0.331$, $x_2^* = 9.824$, $x_3^* = 0.0119$, $x_4^* = 0.877$, $x_5^* = 1.509$.

The $\alpha$ value indicates the level of possibility and the degree of uncertainty of the obtained information. The greater the $\alpha$ value, the greater the level of possibility and the lower the degree of uncertainty is. Since the fuzzy objective value lies in a range, different $\alpha$-cuts shows the different intervals and the uncertainty level of the objective value. Specifically, $\alpha = 0$ has the widest interval indicating that the objective value will definitely fall into this range. At the other extreme end, the possibility level $\alpha = 1$ is the most possible value of the objective value. In this example, the objective value is impossible to exceed 4.314 or fall below 3.045 and its most possible value is 3.561. When the uncertain parameters are represented by crisp values, the objective value is believed to be a single value of 3.561, rather than an interval estimation in the range of 4.314 and 3.045.

**Example 2.** This example is an engineering design problem of cofferdam. A cofferdam is a temporary structure built to enclose an ordinary submerged area to permit construction of a permanent structure on the site. Cofferdam function in a random environment is characterized by fluctuations in surrounding water levels. The designers work with a dam height $x_1$, a rectangular section of length $x_2$, and an average width $x_3$, and would like to know the possible total cost for decision-making. The corresponding mathematical program is as follows:

$$
\text{Min}_x \quad (7.0, 7.2, 7.3)x_1^{(-1.2, -1.1, -1.0)}x_2^{-1}x_3^{-1} + 2x_1^2x_2^{(1.0, 1.0, 1.1)}x_3^{(-0.40, -0.38, -0.36)}
$$

s.t. \quad (4.0, 4.1, 4.3, 4.5)x_1x_2^{-1}x_3^{-1} \leq 1,

(2.1, 2.4, 2.6, 2.8)x_2^2x_3 + (12, 15, 17)x_1^{1.5}x_3 \leq (9.5, 9.7, 10.0, 10.3),

$x_1, x_2, x_3 \geq 0$.

According to Model (12), the upper bound of the objective value $Z_x^U$ can be solved as

$$
Z_x^U = \text{Max}_w \left( \frac{7.3 - 0.1x}{w_01} \right)^{w_{01}} \left( \frac{2}{w_{02}} \right)^{w_{02}} \left( \frac{4.5 - 0.2x}{w_{11}} \right)^{w_{11}} \left( \frac{(2.8 - 0.2x)w_{20}}{(9.5 + 0.2x)w_{22}} \right)^{w_{22}}
$$

s.t. \quad w_{01} + w_{02} = 1,

$y_{11} + 2w_{02} + w_{11} + 1.5w_{22} = 0$,

$-w_{01} + y_{22} - w_{11} + 2w_{21} = 0$,

$-w_{01} + y_{23} - w_{11} + 2w_{21} + w_{22} = 0$,

$(-1.2 + 0.1x)w_{01} \leq y_{11} \leq (-1.0 - 0.1x)w_{01}$,

$w_{02} \leq y_{22} \leq (1.1 - 0.1x)w_{02}$,

$(-0.4 + 0.02x)w_{02} \leq y_{23} \leq (-0.36 - 0.02x)w_{02}$,

$w_{01}, w_{02}, w_{11}, w_{21}, w_{22} \geq 0$,

where $w_{21} + w_{22} = w_{20}$. 
At \( z \)-level \( z=0 \), \( Z^U_{z=0} = 16.021 \), \( w^*_0 = 0.897 \), \( w^*_1 = 0.103 \), \( w^*_1 = 0.639 \), \( w^*_2 = 0.711 \), \( w^*_2 = 0.155 \) with \( a_{011} = -1.2 = (A_{011})^L_{z=0} \), \( a_{022} = 1.1 = (A_{022})^L_{z=0} \), and \( a_{023} = -0.4 = (A_{023})^L_{z=0} \). By transforming \( \mathbf{w}^* \) to \( \mathbf{x}^* \), the corresponding primal solution is \( x^*_1 = 0.371 \), \( x^*_2 = 3.783 \), and \( x^*_3 = 0.441 \). The upper bound of the objective value \( Z^U_z \) at 11 distinct \( z \) values: 0, 0.1, 0.2, \ldots, 1.0 are calculated and presented in the first row of Table 2.

Conceptually, based on Model (14), the lower bound of the objective value \( Z^L_z \) can be formulated as

\[
\begin{align*}
\text{Min} \quad & (7.0 + 0.2z)x_1^{a_{011}}x_2^{-1}x_3^{-1} + 2x_1^{a_{022}}x_2^{a_{023}} \\
\text{s.t.} \quad & (4.0 + 0.1z)x_1x_2^{-1}x_3^{-1} \leq 1, \\
& (2.1 + 0.3z)x_2^2x_3^2 + (12 + 3z)x_1^2x_3 \leq 1, \\
& (-1.2 + 0.1z) \leq a_{011} \leq (-1.0 - 0.1z), \\
& (-0.4 + 0.02z) \leq a_{023} \leq (-0.36 - 0.02z), \\
& 1 \leq a_{022} \leq (1.1 - 0.1z), \\
& x_1, x_2, x_3 \geq 0.
\end{align*}
\]

Using the algorithm discussed in previous section, we are able to calculate the lower bound of the objective value \( Z^L_z \) by the following steps:

**Iteration 1.**

Prior to initiating algorithm, we first let \( \mathbf{P} = \{1, 2, 3\}, \mathbf{Q} = \emptyset \), i.e., we guest that \( x^*_i \geq 1, \quad i=1, \ldots, 3 \). In this case, we should set \( a_{011} = (A_{011})^L_z = (-1.2 + 0.1z) \), \( a_{022} = (A_{022})^L_z = 1 \), and \( a_{023} = (A_{023})^L_z = (-0.4 + 0.02z) \), respectively. We substitute these values into Model (16) and formulate the following geometric program:

\[
\begin{align*}
\text{Max} \quad & \left( 7.0 + 0.2z \right) w_0^1 \left( 2 \right) w_0^2 \left( 4.0 + 0.1z \right) w_1^1 \left( 12 + 3z \right) w_2^1 \left( 10.3 - 0.3z \right) w_2^2 \\
\text{s.t.} \quad & w_0^1 + w_0^2 = 1, \\
& (-1.2 + 0.1z)w_0^1 + 2w_0^2 + w_1 + 1.5w_2^2 = 0, \\
& w_0^1 + w_0^2 - w_1 + w_2^2 = 0, \\
& w_0^1 + (-0.4 + 0.02z)w_0^2 - w_1 + 2w_2^1 + w_2^2 = 0, \\
& w_0^1, w_0^2, w_1, w_2^1, w_2^2 \geq 0,
\end{align*}
\]

where \( w_2^1 + w_2^2 = w_2 \). At the specific level \( z = 0 \), the objective value for this problem \( Z^* = 10.435 \), which occurs at \( w_0^1 = 0.856 \), \( w_0^2 = 0.144 \), \( w_1^1 = 0.438 \), \( w_2^1 = 0.576 \), \( w_2^2 = 0.201 \). The corresponding primal solution is \( x^*_1 = 0.477 \), \( x^*_2 = 2.822 \), and \( x^*_3 = 0.676 \). Since \( 0 < x^*_1 < 1 \), \( x^*_2 \geq 1 \), and \( 0 < x^*_3 < 1 \), they did not satisfy the initial assumption that \( x^*_1 \geq 1 \), \( x^*_2 \geq 1 \), \( x^*_3 \geq 1 \). The current solution is not the lower bound of the objective value \( Z^L_z \).
Iteration 2.

Based on the derived values of $x^*_i$ in the previous iteration, we redefine the index sets $P = \{2\}$ and $Q = \{1,3\}$. That is, we should set $a_{022} = (A_{022})^L_x = 1$, $a_{011} = (A_{011})^U_x = (-1.0 - 0.1x)$, and $a_{023} = (A_{023})^U_x = (-0.36 - 0.02x)$. From (16), the geometric program becomes

$$\begin{align*}
\text{Max} & \quad (7.0 + 0.2x) w_0, \\
\text{s.t.} & \quad w_0 + w_0 = 1, \\
& \qquad (-1.0 - 0.1x)w_0 + 2w_2 + w_1 + 1.5w_2 = 0, \\
& \qquad -w_0 + w_0 - w_1 + 2w_2 = 0, \\
& \qquad -w_0 + (-0.36 - 0.02x)w_0 - w_1 + 2w_2 + w_2 = 0, \\
& \qquad w_0, w_0, w_1, w_2, w_2 \geq 0.
\end{align*}$$

By solving this problem at $x = 0$, the optimal solution is $Z^* = 9.162$, $w_0^* = 0.859$, $w_0^* = 0.141$, $w_1^* = 0.291$, $w_2^* = 0.505$, and $w_2^* = 0.191$. Since we need to find the smallest objective value of the problem, the objective value derived in this iteration is better than that in the previous iteration. After transformation of $w^*$, we derive the primal solution $x^*_1 = 0.471$, $x^*_2 = 2.588$, and $x^*_3 = 0.723$. Now, since $0 < x^*_1 < 1$, $x^*_2 \geq 1$, and $0 < x^*_3 < 1$, they coincide with our guess $P = \{2\}$ and $Q = \{1,3\}$. We have found the lower bound of the objective value $Z^L_{x=0} = 9.169$.

With the same solution procedure, we can derive $Z^L_x$ for different values of possibility level $x$. The value of $Z^L_x$ at 11 distinct $x$ values: 0,0.1,0.2,\ldots,1.0 are calculated and shown in the second row of Table 2. This example shows that the objective value is impossible to fall below 9.162 or exceed 16.021 and the most possible value is to lie within 11.627 and 12.783.

5. Conclusion

Geometric programming is a methodology for solving algebraic nonlinear optimization problem. Its elegant theoretical basis has led to wide applications in engineering design. This paper develops a method that is able to find the membership function of the fuzzy objective value when the exponents of decision variables in the objective function, the cost and the constraint coefficients, and the right-hand sides are fuzzy numbers. The idea is based on Zadeh’s extension principle to transform the fuzzy geometric programming problem to a pair of two-level mathematical programs. Based on duality algorithm and a simple algorithm, the pair of two-level mathematical programs is transformed into a pair of conventional geometric programs. Solving the pair of geometric programs produces the upper bound and lower bound of the objective value at specific $x$ level. The membership function is approximated via different $x$-levels of the objective values. In performing the algorithm proposed in this paper, one may solve the problem first at $x$ level $= 1$, and utilize the derived values of the decision variables as initial guess points for solving the geometric programs that are given other $x$ levels. This could help end up the solution procedure more rapidly by solving less geometric programming problems.

The illustrated examples show that the solution is indeed able to solve fuzzy engineering optimization problems with geometric programming form. The geometric program
discussed in this paper is in posynomial forms, and a global minimizing problem solution to that problem can be derived. Since the degrees of difficulty in the examples are small, one can easily find the optimal solution by the method proposed in this paper. However, as the degree of difficulty increases, solution becomes harder. The studies [2,9,15] that comprehensively discuss algorithms and computational aspects for geometric programming problems can be referred to tackle the problem.

Geometric programming has already shown its power in practice in the past. In real-world applications, the parameters in the geometric program may not be known precisely due to insufficient information. When some parameters are only approximately known, the averages or the most likely values are used to find a point solution. Since only one point value is obtained, much valuable information is lost. With the additional ability of calculating fuzzy objective value developed in this paper, it might help lead to wider applications in the future.

Acknowledgements

Research is supported by the National Science Council of Republic of China under Contract NSC94-2416-H-238-001. The author is indebted to Editor Thierry Denoeux and the referees for their helpful and constructive comments that improved the quality of the paper.

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