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Green's Function and Stability of a Linear Partial Difference Scheme

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Abstract—This paper is concerned with a linear partial difference equation which arises from discretizing a heat and a first order wave equations. A Green's function for this equation is derived. Various representation theorems, monotonicity, and oscillatory properties of the Green's functions are obtained. Then by means of the representation theorems, we derive several stability criteria for arbitrary solutions of this equation.

Keywords—Heat equation, Partial difference equation, Green's function, Stability.

1. INTRODUCTION

Before we develop the general theme of this paper, we shall consider two examples. First consider an initial value problem involving the heat equation [1]:

$$u_t = u_{xx}, \qquad t > 0, \tag{1.1}$$

$$u(x,0) = f(x), \qquad -\infty < x < +\infty. \tag{1.2}$$

By means of standard finite difference methods [2–5], we set up a grid in the x, t plane with grid spacings Δx and Δt, then we replace the second derivative u_{xx} with a central difference, and replace u_t with a forward difference. By writing $x_m = m\Delta x$, $t_n = n\Delta t$, $f_m = f(x_m)$, and $u_m^n \approx u(x_m, t_n)$, a finite difference scheme for the heat equation is obtained:

$$\frac{u_m^{n+1} - u_m^n}{\Delta t} = \frac{1}{(\Delta x)^2}\left(u_{m-1}^n - 2u_m^n + u_{m+1}^n\right)$$

or

$$u_m^{n+1} = ru_{m-1}^n + (1-2r)u_m^n + ru_{m+1}^n, \tag{1.3}$$

where $r = \Delta t/(\Delta x)^2 > 0$, with the initial condition (1.2) replaced by

$$u_m^0 = f_m.$$

Next, we consider an initial value problem involving a hyperbolic equation

$$u_t + au_x = 0, \qquad t > 0,$$

$$u(x,0) = g(x), \qquad -\infty < x < +\infty.$$

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By means of the forward-time forward-space finite difference scheme [4], we obtain similarly

$$u_m^{n+1} = (1+\lambda a)u_m^n - a\lambda u_{m+1}^n, \tag{1.4}$$

where $\lambda = \Delta t/\Delta x$.

In this paper, we will consider a partial difference equation which is more general than either (1.3) or (1.4):

$$u_m^{n+1} = au_{m-1}^n + bu_m^n + cu_{m+1}^n, \qquad (m,n) \in \Omega, \tag{1.5}$$

where a, b, c are real numbers and Ω is the upper half lattice plane

$$\Omega = \{(m,n) \mid m = 0, \pm 1, \pm 2, \dots;\ n = 0, 1, 2, \dots\}.$$

A solution of (1.5) is a real double sequence $u = \{u_m^n\}_{(m,n)\in\Omega}$ which satisfies (1.5). It is clear that under the initial condition

$$u_m^0 = f_m, \qquad m \in Z, \tag{1.6}$$

where Z is the set of integers, we can calculate

$$u_0^1;\ u_{-1}^1, u_1^1, u_0^2;\ u_{-2}^1, u_{-1}^2, u_1^2, u_2^1, u_0^3;\ \dots \tag{1.7}$$

successively in a unique manner. An existence and uniqueness theorem for a solution of (1.5) subject to (1.6) is thus easily formulated and proved. Note that there are two interesting consequences of the computational scheme. First of all, the value u_0^n is completely determined by the values $f_{-n}, f_{-n+1}, \dots, f_n$; and similarly, u_m^n by f_k, $m-n \le k \le m+n$. For this reason, we say that the integers $m-n, \dots, m+n$ are the domain of dependence for the value u_m^n. Second, if $f_m = 0$ for $m \ge k$, then the values u_m^n will vanish for $m \ge n+k$.

EXAMPLE 1.1. Suppose $b = 0$. Let $f_m = 0$ when m is odd and $f_m = (-1)^{m/2}$ when m is even. Then by means of the computational scheme stated above, we obtain the sequence

$$0;\ -(a-c), (a-c), -(a-c)^2;\ 0,0,0,0,0;\ (a-c), (a-c)^2, (a-c)^3,\ \dots,$$

which shows that u_m^n is either equal to 0 or to $\pm(a-c)^n$.

It will be convenient to view the solution $u = \{u_m^n\}$ of (1.5),(1.6) as a set of vectors in the linear space l of all real sequences $\{x_m\}_{m\in Z}$. More precisely, for each fixed $n = 0, 1, 2, \dots$, we may view $\{u_m^n\}_{m\in Z}$ as a vector in l. This vector, denoted by $u^{(n)}$, will be called the n^{th} horizontal vector of the solution u. In terms of the horizontal vectors, we may also rewrite (1.5) and (1.6) as

$$u^{(n+1)} = Tu^{(n)}, \qquad n = 0, 1, 2, \dots,$$

and

$$u^{(0)} = f, \qquad f = \{f_m\}_{m\in Z},$$

respectively, where $T : l \to l$ has an infinite tridiagonal matrix representation:

$$\begin{bmatrix} \cdot & & & & & \\ & \cdot & & & & \\ & a & b & c & & \\ & & a & b & c & \\ & & & a & b & c \\ & & & & & \cdot \\ & & & & & & \cdot \end{bmatrix}.$$

We will be concerned with the stability behavior of solutions of the initial value problems (1.5), (1.6). As is well known, stability behavior of solutions of difference schemes such as the one described here is of fundamental importance because it is related to questions of convergence and growth of numerical errors. Indeed, the same concern has been raised in [2] and it is shown in [2, p. 92] that the well-known condition $r \le 1/2$ is a necessary and sufficient condition for equation (1.3) to be "stable".

To achieve our goal, we will first discuss the behavior of a special solution which will be called the Green's function of (1.5). Then by means of a convolution theorem, we will be able to derive stability criteria for more general solutions. There are many concepts which are related to the stability of solutions of partial difference equations. We will adopt the concept of stability as discussed in [2, Section 5.1]. More specifically, equation (1.5), or, the trivial solution of (1.5), is said to be stable if there exists a positive constant Γ such that for every initial sequence $f = \{f_m\}_{m=-\infty}^{\infty}$, the corresponding solution of (1.5),(1.6) will satisfy

$$|u_m^n| \le \Gamma \|f\|_{mn}, \qquad (m,n) \in \Omega,$$

where

$$\|f\|_{mn} = \max\{|f_k| : \ m-n \le k \le m+n\}.$$

Note that we have taken the domain of dependence into consideration in making the above definition. Clearly, if (1.5) is stable, then

$$\sup_{-\infty<k<\infty} |f_k| \le \delta$$

will imply $|u_m^n| \le \Gamma\delta$ for all m and n.

Three other important and related concepts of stability can also be defined as follows. First, equation (1.5), or, the trivial solution of (1.5), is said to be exponentially asymptotically stable if there is a real number $\xi \in (0,1)$ and a positive number Γ such that for every initial sequence $f = \{f_m\}_{m=-\infty}^{\infty}$, the corresponding solution will satisfy

$$|u_m^n| \le \Gamma\xi^m \|f\|_{mn}, \qquad (m,n) \in \Omega.$$

Second, we say that a solution, or more generally, a double sequence $u = \{u_m^n\}_{(m,n)\in\Omega}$ is subexponential if there are positive constants M, α, and β such that

$$|u_m^n| \le M\alpha^m\beta^n, \qquad (m,n) \in \Omega.$$

Third, a double sequence $u = \{u_m^n\}_{(m,n)\in\Omega}$ is said to be doubly subexponential if there are positive constants M, α, and β such that

$$|u_m^n| \le M\frac{\alpha^m + \alpha^{-m}}{2}\beta^n, \qquad (m,n) \in \Omega,$$

or equivalently, if there are positive constants N, σ, τ such that

$$|u_m^n| \le N\sigma^{|m|}\tau^n, \qquad (m,n) \in \Omega.$$

The latter two concepts are no doubt generalizations of the concepts of a subexponential sequence $\{x_i\}_{i=0}^{\infty}$ and a doubly subexponential sequence $\{y_i\}_{i=-\infty}^{\infty}$ defined by

$$|x_i| \le M\alpha^i, \qquad i = 0,1,2,\ldots,$$

and

$$|y_i| \le M\frac{\alpha^i + \alpha^{-i}}{2}, \qquad i \in Z,$$

respectively, where M and α are positive constants.

In the next section, we first derive an explicit formula for the Green's function; some specific examples will also be given. Then in Section 3, various properties of the Green's function will be derived. These include various representation theorems, monotonicity properties as well as some simple oscillatory properties. In Section 4, stability criteria will be derived which are centered around the stability concepts introduced above. In the final section, numerical computations will be provided for comparison with the theoretical results.

2. THE GREEN'S FUNCTION

For the sake of convenience, we restate our initial value problem (1.5),(1.6) as follows:

$$u_m^{n+1} = au_{m-1}^n + bu_m^n + cu_{m+1}^n, \qquad (m,n) \in \Omega, \tag{2.1}$$

$$u_m^0 = f_m, \qquad m \in Z, \tag{2.2}$$

where Z is the set of integers $\{\dots,-1,0,1,\dots\}$ and $f = \{f_m\}_{m\in Z}$ is a real sequence. A solution of (2.1) will be called the Green's function of (2.1) if it satisfies the initial condition

$$u_m^0 = \delta_{m0}, \qquad m \in Z, \tag{2.3}$$

where δ_{ij} is the Kronecker delta function. Such a solution exists and is unique and will be denoted by $G = \{G_m^n\}$. When $a,b,c \geq 0$ and $a+b+c=1$, it is stated in Berman and Fryer [6, p. 237] that G_m^n is given by the coefficient of the term x^m in the expansion of

$$\left(ax + b + cx^{-1}\right)^n \tag{2.4}$$

for $|m| \leq n$. We can show that this statement is still valid when the restrictions on the coefficients a, b, and c are removed. While we shall find it useful to assign specific nonzero values to x in the sequel, the primary role of x is as an ordering parameter since it is the coefficients of powers of x which are significant; thus, the case $x = 0$ is of no interest. Note also that in view of the second remark following (1.7), the Green's function $\{G_m^n\}$ is identically zero for $|m| > n$.

THEOREM 2.1. *The $(m,n)^{th}$ component G_m^n of the Green's function $G = \{G_m^n\}$ is the coefficient of the term x^m in the expansion of the rational function $(ax+b+cx^{-1})^n$ when $|m| \leq n$ and is zero otherwise.*

PROOF. We will prove our assertion by induction. First of all, when $n = 0$, the function $(ax + b + cx^{-1})^n = 1$ and the 0^{th} horizontal vector of G is $\{\dots,0,1,0,\dots\}$. Thus our assertion holds when $n = 0$. The case when $n = 1$ is also easy to see. Assume by induction that the case $n = j$ holds, that is,

$$\left(ax+b+cx^{-1}\right)^j = G_{-j}^j x^{-j} + G_{-j+1}^j x^{-j+1} + \cdots + G_{j-1}^j x^{j-1} + G_j^j x^j.$$

Then for $n = j+1$,

$$\begin{aligned}
&\left(ax+b+cx^{-1}\right)^{j+1} \\
&= \left\{G_{-j}^j x^{-j} + G_{-j+1}^j x^{-j+1} + \cdots + G_{j-1}^j x^{j-1} + G_j^j x^j\right\}\left(cx^{-1}+b+ax\right) \\
&= cG_{-j}^j x^{-(j+1)} + \left(cG_{-j+1}^j + bG_{-j}^j\right)x^{-j} + \left(cG_{-j+2}^j + bG_{-j+1}^j + aG_{-j}^j\right)x^{-j+1} \\
&\quad + \cdots + \left(cG_j^j + bG_{j-1}^j + aG_{j-2}^j\right)x^{j-1} + \left(bG_j^j + aG_{j-1}^j\right)x^j + aG_j^j x^{j+1} \\
&= \left(aG_{-j-2}^j + bG_{-j-1}^j + cG_{-j}^j\right)x^{-(j+1)} + \cdots + \left(aG_{i-1}^j + bG_i^j + cG_{i+1}^j\right)x^i \\
&\quad + \cdots + \left(aG_j^j + bG_{j+1}^j + cG_{j+2}^j\right)x^{j+1} \\
&= G_{-(j+1)}^{j+1} x^{-(j+1)} + \cdots + G_i^{j+1} x^i + \cdots + G_{j+1}^{j+1} x^{j+1}.
\end{aligned}$$

The proof is complete.

By means of the Green's function G, we will be able to show that the solution of the initial value problem (2.1),(2.2) can be expressed as a convolution. This result can be regarded as a discrete Poisson representation formula. Here, the convolution of two sequences $f = \{f_m\}_{m\in Z}$ and $g = \{g_m\}_{m\in Z}$ is the sequence $f * g$ defined by

$$(f * g)_m = \sum_{k=-\infty}^{\infty} f_k g_{m-k}, \qquad m \in Z.$$

Note that such a sequence exists whenever one of the sequences f and g has a finite number of nonzero terms.

THEOREM 2.2. *The solution $u = \{u_m^n\}$ of the initial value problem (2.1),(2.2) is given by*

$$u_m^n = \sum_{k=-\infty}^{\infty} G_k^n f_{m-k} = \sum_{k=-n}^{n} G_k^n f_{m-k}, \ (m,n) \in \Omega.$$

In other words, the n^{th} horizontal vector of the solution u is the convolution of the n^{th} horizontal vector of the Green's function with the initial vector f. The proof is accomplished by direct verification. Indeed,

$$\sum_{k=-\infty}^{\infty} G_k^0 f_{m-k} = \sum_{k=-\infty}^{\infty} \delta_{k0} f_{m-k} = f_m, \qquad m \in Z,$$

and

$$a \sum_{k=-\infty}^{\infty} G_k^n f_{m-1-k} + b \sum_{-\infty}^{\infty} G_k^n f_{m-k} + c \sum_{k=-\infty}^{\infty} G_k^n f_{m+1-k}$$
$$= \sum_{t=-\infty}^{\infty} \left\{ aG_{t-1}^n f_{m-t} + bG_t^n f_{m-t} + cG_{t+1}^n f_{m-t} \right\} = \sum_{t=-\infty}^{\infty} G_t^{n+1} f_{m-t}.$$

We remark that there is no convergence problem in the above derivations since $G_i^j = 0$ for $|i| > j$.

When the coefficients a, b, and c take on special values, the corresponding Green's function can be calculated relatively easily by expanding the corresponding function $(ax + b + cx^{-1})^n$. In the sequel, C_α^β will denote the binomial coefficient, i.e.,

$$C_\alpha^\beta = \begin{cases} \dfrac{\beta!}{((\beta-\alpha)!\alpha!)}, & 0 \le \alpha \le \beta;\ \alpha, \beta \in Z, \\ 0, & \text{otherwise.} \end{cases}$$

EXAMPLE 2.1. If $b = 0$, then by means of

$$\left(ax + cx^{-1}\right)^n = \sum_{k=0}^{n} C_k^n a^k c^{n-k} x^{-n+2k},$$

we see that $G_m^n = 0$ when $m + n$ is odd and

$$G_m^n = C_{(n-|m|)/2}^n a^{(n+m)/2} c^{(n-m)/2}, \qquad |m| \le n$$

when $m + n$ is even.

EXAMPLE 2.2. If $a = c$ and $b = 2a$, then by means of the equalities

$$\left(ax + 2a + ax^{-1}\right)^n = a^n \left(x + 2 + x^{-1}\right)^n = a^n \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^{2n},$$

it is easily seen that

$$G_m^n = C_{n-m}^{2n} a^n, \qquad |m| \le n.$$

EXAMPLE 2.3. If $a = 0$, then

$$G_m^n = C_{-m}^n b^{m+n} c^{-m}, \qquad m \le 0, \quad |m| \le n,$$

and $G_m^n = 0$ otherwise.

EXAMPLE 2.4. If $c = 0$, $a = b = 1$, then

$$G_m^n = C_m^n, \qquad |m| \le n.$$

This shows that the binomial coefficient is a Green's function of the initial value problem

$$\begin{aligned} u_m^{n+1} &= u_{m-1}^n + u_m^n, & (m,n) &\in \Omega, \\ u_m^0 &= \delta_{m0}, & m &\in Z. \end{aligned}$$

EXAMPLE 2.5. When $a = c \neq 0$ and $b = 1 - 2a$, then

$$\begin{aligned} \left(ax - 2a + ax^{-1} + 1\right)^n &= \left(a\left(x - 2 + x^{-1}\right) + 1\right)^n \\ &= \left(a\left(x^{1/2} - x^{-1/2}\right)^2 + 1\right)^n \\ &= \sum_{k=0}^n C_k^n a^k \left(x^{1/2} - x^{-1/2}\right)^{2k} \\ &= \sum_{k=0}^n \sum_{i=0}^{2k} (-1)^i C_k^n C_i^{2k} a^k x^{k-i}, \end{aligned}$$

which may be regarded as a sum of values each of which corresponds to a point in the subset $\{(i,k) : 0 \le i \le 2k,\ 0 \le k \le n\}$ of lattice points. To find G_0^n, we need to look for the coefficient of the term x^0 in the last sum. But this is just the sum of the values corresponding to the subset $\{(i,k) : i = k, 0 \le k \le n\}$:

$$G_0^n = \sum_{k=0}^n (-1)^k C_k^n C_k^{2k} a^k.$$

Similarly, we may calculate G_m^n when $m \neq 0$ and obtain the Green's function in [7]:

$$G_m^n = \sum_{k=|m|}^n (-1)^{k-m} C_k^n C_{k-m}^{2k} a^k, \qquad |m| \le n.$$

3. PROPERTIES OF THE GREEN'S FUNCTION

Widder [1] has listed many properties of the Green's function $G(x,t)$ for the heat equation (1.1). In what follows, we will derive several properties of our Green's function which are similar to those of its continuous analog.

There are several immediate consequences of Theorem 2.1. First, we see that each horizontal vector of the Green's function is nontrivial when one of the coefficients a, b, and c is nonzero. Second, if $a = c$, then since the coefficients of the terms x^m and x^{-m} are identical, we see that

$$G_m^n = G_{-m}^n, \qquad (m,n) \in \Omega.$$

Third, it is clear by substituting $x = 1$ and $x = -1$ into the rational function $(ax + b + cx^{-1})^n$ that

$$\sum_{k=-n}^n G_k^n = (a+b+c)^n, \tag{3.1}$$

and

$$\sum_{k=-n}^{n} G_k^n(-1)^k = \sum_{k=-n}^{n} G_k^n(-1)^{-k} = (-1)^n(a-b+c)^n. \tag{3.2}$$

In particular, if $a = c$ and $b = 1 - 2a$ (cf. (1.3)), then

$$\sum_{i=-n}^{n} G_i^n = 1.$$

This result may be regarded as a discrete analog of Theorem 2.D in [1, p. 31] which states that the integral of the source solution of the heat equation (1.1) over the space domain is one.

There are several immediate consequences of Theorems 2.1 and 2.2 also. For instance, if $f = \{1\}_{m\in Z}$, then the solution $\{u_m^n\}$ of (2.1) subject to (2.2) is given by

$$u_m^n = \sum_{k=-n}^{n} G_k^n = (a+b+c)^n.$$

If $f = \{(-1)^m\}_{m\in Z}$, then the solution $\{u_m^n\}$ of (2.1) subject to (2.2) is given by

$$u_m^n = \sum_{k=-n}^{n} G_k^n(-1)^{m-k} = (-1)^{m-n}(a-b+c)^n.$$

Several representation theorems also follow from Theorem 2.1 (some of these have already been listed in the previous section). To this end, we shall explicitly calculate G_m^n from the expansion of the rational function $(ax + b + cx^{-1})^n$ and obtain

$$G_m^n = \sum_{i=|m|}^{n} \delta^{(i+m)} C_i^n C_{(i-|m|)/2}^i a^{(i+m)/2} b^{n-i} c^{(i-m)/2}, \tag{3.3}$$

where

$$\delta^{(k)} = \frac{1+(-1)^k}{2}, \qquad k \in Z. \tag{3.4}$$

Note that $\delta^{(k)}$ equals 0 when k is an odd integer, and equals 1 when k is even.

THEOREM 3.1. *The Green's function $G = \{G_m^n\}$ of (2.1) is given by (3.3).*

PROOF. By means of the binomial theorem, we have

$$\begin{aligned}\left(ax+b+cx^{-1}\right)^n &= \sum_{i=0}^{n} C_i^n b^{n-i}\left(ax+cx^{-1}\right)^i \\ &= \sum_{i=0}^{n}\sum_{k=0}^{i} C_i^n C_k^i a^{i-k} b^{n-i} c^k x^{i-2k}.\end{aligned}$$

By writing

$$\left(ax+b+cx^{-1}\right)^n = \sum_{m=-n}^{n} G_m^n x^m$$

and then utilizing the same reasoning used in Example 2.5, it is not difficult to conclude the truth of our assertion.

In the above proof, we have expanded the rational function $(ax + b + cx^{-1})^n$ in a specific manner. We may also expand the rational function $(ax + b + cx^{-1})^n$ in different manner and obtain

$$G_m^n = \sum_{i=\lceil (n-m)/2\rceil}^{n-m} C_i^n C_{n-m-i}^i a^{n-i} b^{2i-n+m} c^{n-m-i}, \qquad m \geq 0,$$

and

$$G_m^n = \sum_{i=\lceil (n-m)/2 \rceil}^{n} C_i^n C_{n-m-i}^i a^{n-i} b^{2i-n+m} c^{n-m-i}, \qquad m < 0,$$

where $\lceil k \rceil$ stands for the smallest integer not less than k.

The next result states that the convolution of two different horizontal vectors of the Green's function is again a horizontal vector of the same Green's function.

THEOREM 3.2. *The Green's function $G = \{G_m^n\}$ of (2.1) satisfies the following "additivity" property:*

$$\{G_m^s\}_{m\in Z} * \{G_m^t\}_{m\in Z} = \{G_m^{s+t}\}_{m\in Z}.$$

The proof is not difficult and follows from the fact that

$$\left(ax + b + cx^{-1}\right)^s \left(ax + b + cx^{-1}\right)^t$$

is of the form

$$\sum_{m=-(s+t)}^{s+t} d_m x^m,$$

where d_m is given by the convolution of the two sequences of coefficients of the rational functions $(ax + b + cx^{-1})^s$ and $(ax + b + cx^{-1})^t$, respectively.

As we will see below, there are several monotonicity properties of the Green's function.

THEOREM 3.3. *Suppose $a = c \neq 0$. Then the Green's function $G = \{G_m^n\}$ satisfies*

$$G_{m+1}^{n+1} \geq G_m^n, \qquad m, n = 0, 1, 2, \dots, \tag{3.5}$$

if, and only if, $a \geq 1$ and $b \geq 0$.

PROOF. Assuming that $a = c \geq 1$ and $b \geq 0$, then for $m, n \geq 0$, we see that $G_{m+1}^{n+1} - G_m^n$ is equal to

$$\begin{aligned}
&\sum_{k=m+1}^{n+1} \delta^{(k+m+1)} C_k^{n+1} C_{(k-m-1)/2}^k a^k b^{n+1-k} - \sum_{k=m}^{n} \delta^{(k+m)} C_k^n C_{(k-m)/2}^k a^k b^{n-k} \\
&= \sum_{k=m}^{n} \delta^{(k+m)} C_{k+1}^{n+1} C_{(k-m)/2}^{k+1} a^{k+1} b^{n-k} - \sum_{k=m}^{n} \delta^{(k+m)} C_k^n C_{(k-m)/2}^k a^k b^{n-k} \\
&= \sum_{k=m}^{n} \delta^{(k+m)} \left\{ C_k^{n+1} C_{(k-m)/2}^{k+1} a - C_k^n C_{(k-m)/2}^k \right\} a^k b^{n-k} \\
&\geq \sum_{k=m}^{n} \delta^{(k+m)} \left\{ C_k^{n+1} C_{(k-m)/2}^{k+1} - C_k^n C_{(k-m)/2}^k \right\} a^k b^{n-k} \geq 0.
\end{aligned}$$

Conversely, there are two cases to consider. Suppose first that $a < 1$. Then $G_m^m = a^m$ so that the statement $G_{m+1}^{m+1} \geq G_m^m$ cannot be satisfied for all $m \geq 0$. Next suppose $a \geq 1$ and $b < 0$. Then $G_m^{m+1} = (m+1)ba^m$ so that the statement that $G_{m+1}^{m+2} \geq G_m^{m+1}$ cannot be satisfied for all $m \geq 0$. The proof is complete.

We remark that when $a = c$, the Green's function G_m^n satisfies $G_m^n = G_{-m}^n$. Thus the condition (3.5) in the above theorem can be replaced by the condition

$$G_{-m-1}^{n+1} \geq G_{-m}^n, \qquad m, n = 0, 1, 2, \ldots.$$

As an immediate consequence, when $a = c \geq 1$ and $b \geq 0$, then the maximum of the Green's function over a lattice rectangle $\{\alpha, \alpha+1, \ldots, \beta\} \times \{\sigma, \sigma+1, \ldots, \tau\}$ occurs at the "lateral edges" of the rectangle. This result can be regarded as a maximum principle for equation (2.1).

It is clear from the representation Theorem 3.1 that G is nonnegative when the coefficients a, b, and c are nonnegative. In case some of the coefficients a, b, and c are negative, the Green's function will have negative values and a variety of oscillatory behaviors can also be shown.

THEOREM 3.4. *The Green's function of (2.1) is nonnegative if, and only if,* $a, b, c \geq 0$.

PROOF. If $b < 0$ and $a = c = 0$, then $G_0^n = b^n$ for $n \geq 0$ so that $G_0^1 = b < 0$. If $b < 0$, $a > 0$, and $c \geq 0$, then $G_m^{m+1} = (m+1)ba^m < 0$ for $m \geq 0$. If $a < 0$, $b > 0$, and $c \geq 0$then $G_m^m = a^m$ for $m \geq 0$ so that $G_1^1 = a < 0$. The other cases are dealt with in similar manners.

Next, we assert that when $a+b+c < 0$, then adjacent horizontal vectors of the Green's function must have components with different signs.

THEOREM 3.5. *If $a + b + c < 0$, then there exist integers m_1 and m_2 such that*

$$G_{m_1}^n G_{m_2}^{n+1} < 0, \qquad n = 0, 1, \ldots.$$

PROOF. Since

$$(a+b+c)^n (a+b+c)^{n+1} < 0, \qquad n \geq 0,$$

thus by Theorem 2.1,

$$\left\{\sum_{k=-n}^{n} G_k^n\right\}\left\{\sum_{k=-(n+1)}^{n+1} G_k^{n+1}\right\} < 0,$$

which implies our assertion.

We may be more specific when $a, c < 0$, and $b \leq 0$.

THEOREM 3.6. *Suppose $a, c < 0$, and $b \leq 0$. If $G_m^n \neq 0$ for some lattice point (m, n), then $G_m^n > 0$ when n is even and $G_m^n < 0$ when n is odd.*

PROOF. First suppose $a, b, c < 0$. Then by the representation Theorem 3.1,

$$G_m^n = (-1)^n \sum_{i=|m|}^{n} \delta^{(i+m)} C_i^n C_{(i-|m|)/2}^i |a|^{(i+m)/2} |b|^{n-i} |c|^{(i-m)/2}.$$

Thus if G_m^n is not zero, then G_m^n is positive when n is even and negative when n is odd. Next suppose $a, c < 0$, and $b = 0$. Since (see Example 2.1)

$$G_m^n = C_{(n-|m|)/2}^n a^{(n+m)/2} c^{(n-m)/2} = (-1)^n C_{(n-|m|)/2}^n |a|^{(n+m)/2} |c|^{(n-m)/2}$$

when $m + n$ is even and $|m| \leq n$, and zero otherwise, the same conclusions may again be drawn.

4. STABILITY CRITERIA

In this section, we will derive several stability criteria for the solutions of the initial value problem (2.1),(2.2).

THEOREM 4.1. *Let $u = \{u_i^j\}$ be a solution of the initial value problem (2.1),(2.2). Then*

$$|u_m^n| \le (|a| + |b| + |c|)^n \, \|f\|_{mn}, \qquad (m,n) \in \Omega. \tag{4.1}$$

PROOF. In view of Theorem 2.2,

$$u_m^n = \sum_{k=-n}^{n} G_k^n f_{m-k}, \qquad (m,n) \in \Omega,$$

so that by Theorem 3.1,

$$\begin{aligned} |u_m^n| &\le \sum_{j=-n}^{n} |G_j^n| \, |f_{m-j}| \le \|f\|_{mn} \sum_{j=-n}^{n} |G_j^n| \\ &\le \|f\|_{mn} \sum_{j=-n}^{n} \sum_{i=|j|}^{n} \delta^{(i+j)} C_i^n C^i_{(i-|j|)/2} \, |a|^{(i+j)/2} \, |b|^{n-i} \, |c|^{(i-j)/2} \\ &= \|f\|_{mn} \, (|a| + |b| + |c|)^n, \qquad (m,n) \in \Omega. \end{aligned}$$

The proof is complete.

As an immediate consequence, we see that if $|a| + |b| + |c| \le 1$, then

$$|u_m^n| \le \|f\|_{mn}, \qquad (m,n) \in \Omega, \tag{4.2}$$

that is, the trivial solution of (2.1) is stable.

THEOREM 4.2. *If $|a| + |b| + |c| \le 1$, then the trivial solution of (2.1) is stable. The converse also holds either when $abc = 0$, or, when $abc \ne 0$ and $ac > 0$.*

PROOF. Suppose $|a| + |b| + |c| > 1$, $b \ge 0$ and $a, c > 0$. Let $f = \{d\}_{m \in Z}$ where $d > 0$. Then by (3.1), the corresponding solution of (2.1) is given by

$$u_m^n = \sum_{k=-n}^{n} G_k^n d = (a+b+c)^n d = (a+b+c)^n \, \|f\|_{mn}.$$

Since $a + b + c > 1$, for any $\Gamma > 0$, there exists an integer n such that $(a+b+c)^n > \Gamma$. Thus

$$u_m^n > \Gamma \, \|f\|_{mn},$$

which shows that the trivial solution of (2.1) is not stable. Next suppose $|a| + |b| + |c| > 1$, $a > 0$, $b < 0$ as well as $c > 0$. Let $f = \{(-1)^m d\}_{m=-\infty}^{\infty}$ where $d > 0$. Then in view of (3.2), the corresponding solution of (2.1) is given by

$$u_m^n = \sum_{k=-n}^{n} G_k^n (-1)^{m-k} d = (-1)^{m+n} d \, (a-b+c)^n.$$

Again, for any $\Gamma > 0$, there exists an integer n such that $(a-b+c)^n > \Gamma$. Thus

$$|u_m^n| > \Gamma \, \|f\|_{mn},$$

which shows that the trivial solution of (2.1) is not stable.

Next, suppose $|a| + |b| + |c| > 1$, $b = 0$ and $ac < 0$. Let $f = \{(-1)^{m/2} \delta^{(m)} d\}_{m \in Z}$ where $d > 0$, i.e., $f_m = 0$ when i is odd and $f_m = \pm d$ when m is even. Then in view of Example 1.1, it is easily checked that u_m^n is either equal to zero or to $\pm(a-c)^n d$. The same conclusion can thus be drawn as seen in the previous two cases.

Next, suppose $c = 0$ and $ab \ne 0$. We may take $f = \{1\}_{m \in Z}$ and arrive at the same conclusion. The other cases are similarly proved. The proof is complete.

We remark that when $a = b = r > 0$ and $b = 1 - 2r$, the condition $|a| + |b| + |c| \le 1$ is equivalent to $r \le 1/2$. Thus the above theorem is an extension of the result stated in Section 1 concerning the discrete heat equation (1.3).

As another consequence of Theorem 4.1, we see that if $|a| + |b| + |c| = \xi < 1$, then

$$|u_m^n| \le \xi^n \|f\|_{mn}, \qquad (m,n) \in \Omega, \tag{4.3}$$

that is, the trivial solution of (2.1) is exponentially asymptotically stable.

THEOREM 4.3. *If $|a|+|b|+|c| < 1$, then the trivial solution of (2.1) is exponentially asymptotically stable. The converse also holds either when $abc = 0$, or, $abc \neq 0$ and $ac > 0$.*

The proof of the converse is similar to that of Theorem 4.2 and is thus omitted.

Our final result in this section provides a stability criterion for subexponential solutions of (2.1).

THEOREM 4.4. *Suppose $f = \{f_m\}_{m\in Z}$ is subexponential. Then the solution $u = \{u_m^n\}$ of (2.1), (2.2) is also subexponential.*

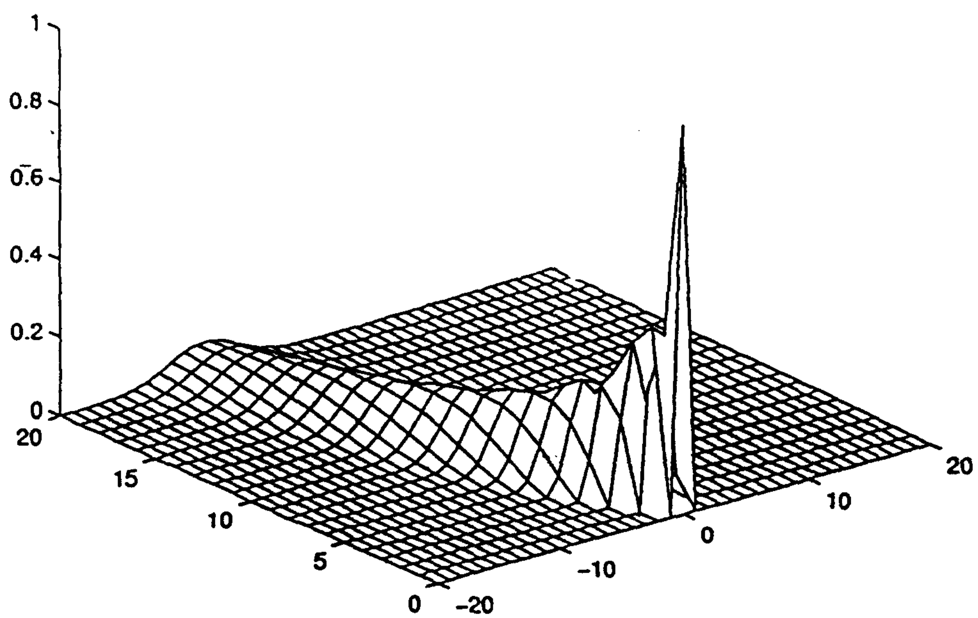
PROOF. Suppose

$$|f_m| \le M\alpha^m, \qquad m \in Z$$

for some positive numbers M and α. In view of Theorems 2.2 and 3.1, we have

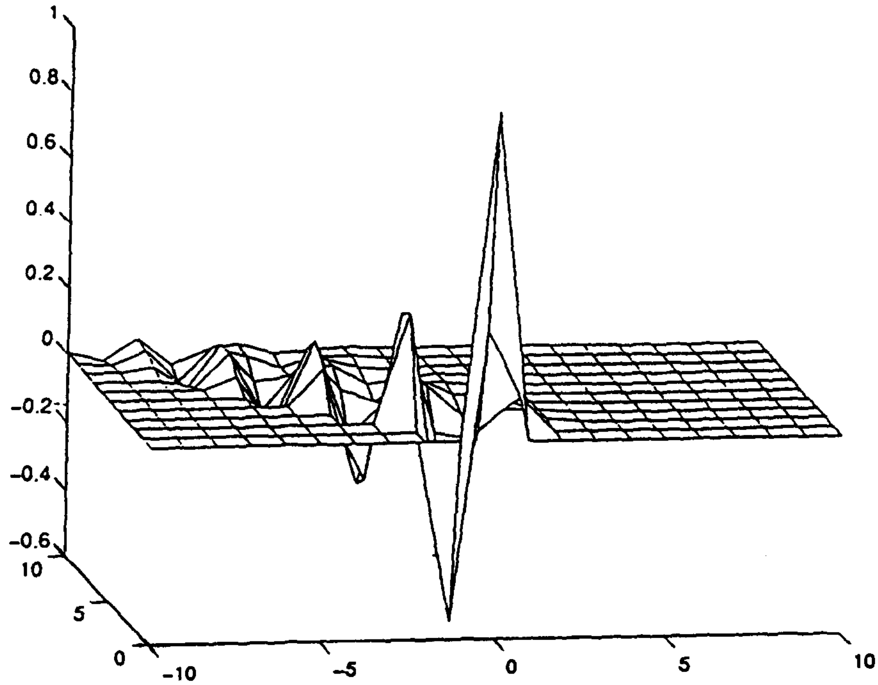
$$\begin{aligned}
|u_m^n| &\le \sum_{j=-n}^{n} |G_j^n|\,|f_{m-j}| \le M\alpha^m \sum_{j=-n}^{n} |G_j^n|\,\alpha^{-j} \\
&\le M\alpha^m \sum_{j=-n}^{n} \sum_{i=|j|}^{n} \delta^{(i+j)} C_i^n C_{(i-|j|)/2}^i \left|\frac{a}{\alpha}\right|^{(i+j)/2} |b|^{n-i}\,|c\alpha|^{(i-j)/2} \\
&= M\alpha^m \left(\frac{|a|}{\alpha} + |b| + |c|\,\alpha\right)^n .
\end{aligned}$$

The proof is complete.



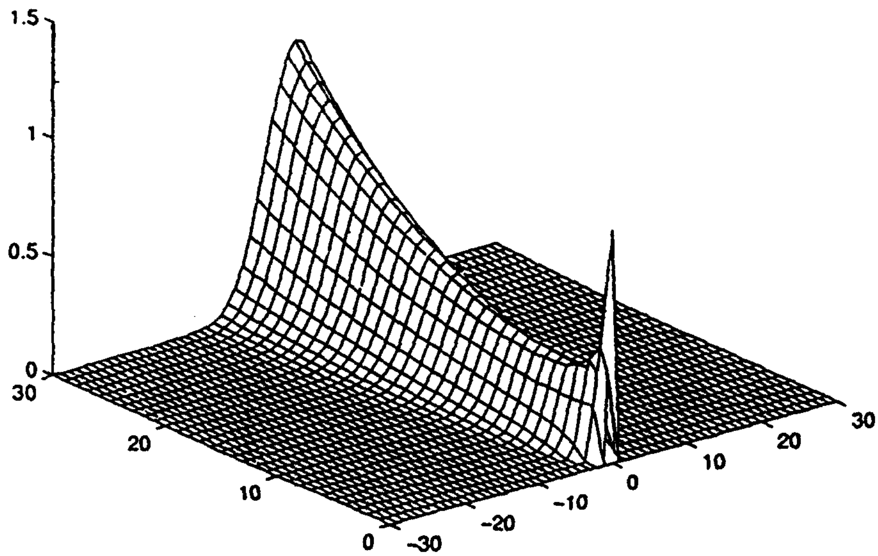
$a = 0.1949,\ b = 0.2822,\ c = 0.4225,\ |a| + |b| + |c| = 0.8987.$

Figure 1.



$a = -0.5753,\ b = 0.3027,\ c = 0.099,\ |a| + |b| + |c| = 0.9775.$

Figure 2.



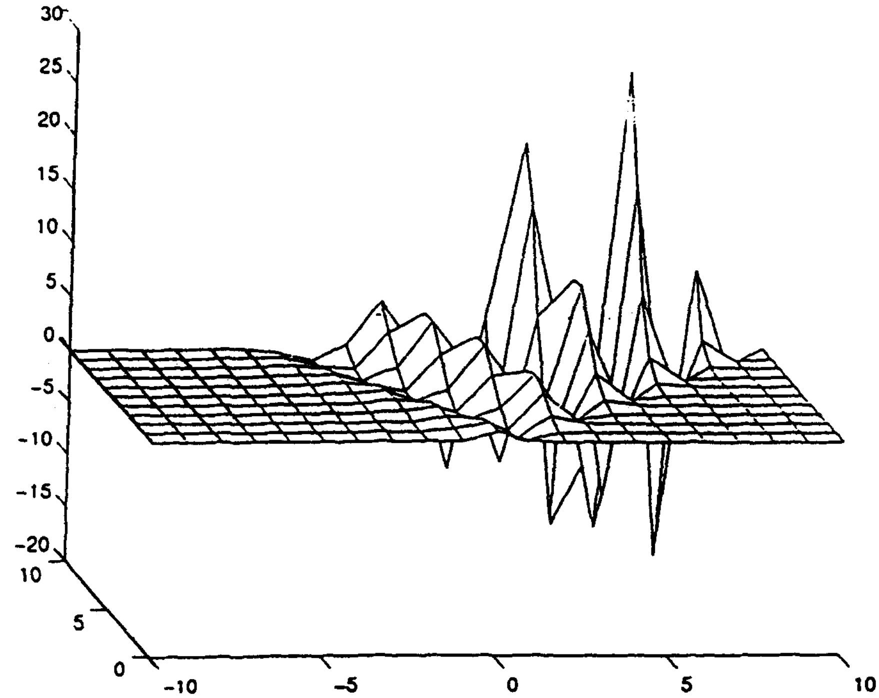
$a = 0.2401,\ b = 0.4861,\ c = 0.3581,\ |a| + |b| + |c| = 1.0843.$

Figure 3.

Similar arguments also lead to a stability criteria for the doubly subexponential solution of (2.1): if f is doubly subexponential, then the solution of (2.1),(2.2) is also doubly subexponential.

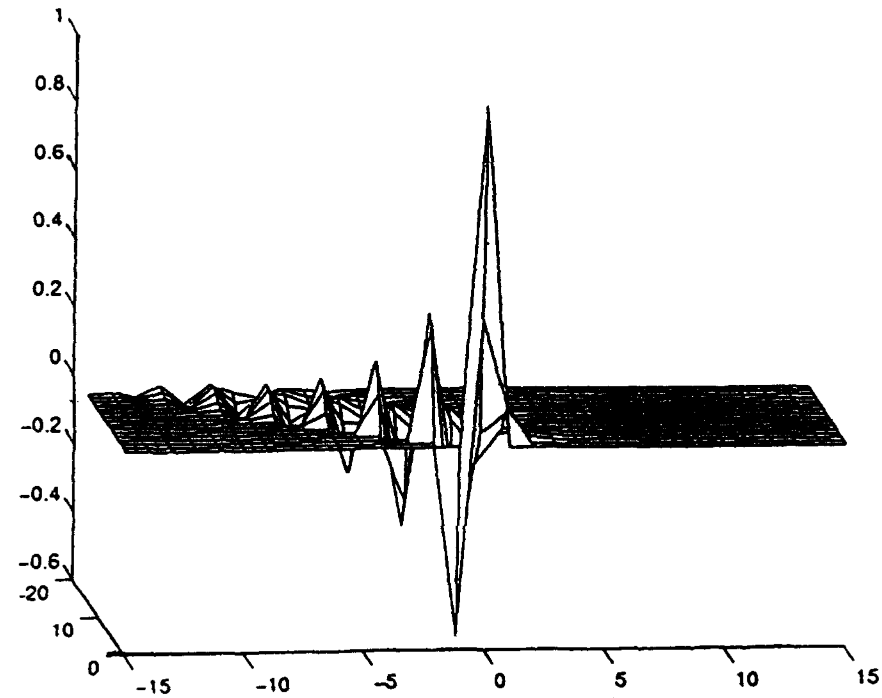
5. ILLUSTRATIVE EXAMPLES

In this section, we present several numerical examples. The first one is the Green's function of (2.1) when $a = 0.1940$, $b = 0.2822$, and $c = 0.4225$. The graph of this function is presented in Figure 1. Note that since $a, b, c > 0$, and $a + b + c = 0.8987$, it is expected that G is nonnegative and exponentially asymptotically stable in view of Theorem 4.3.



$a = -0.6457$, $b = 0.5303$, $c = 0.3696$, $|a| + |b| + |c| = 1.5456$.

Figure 4.

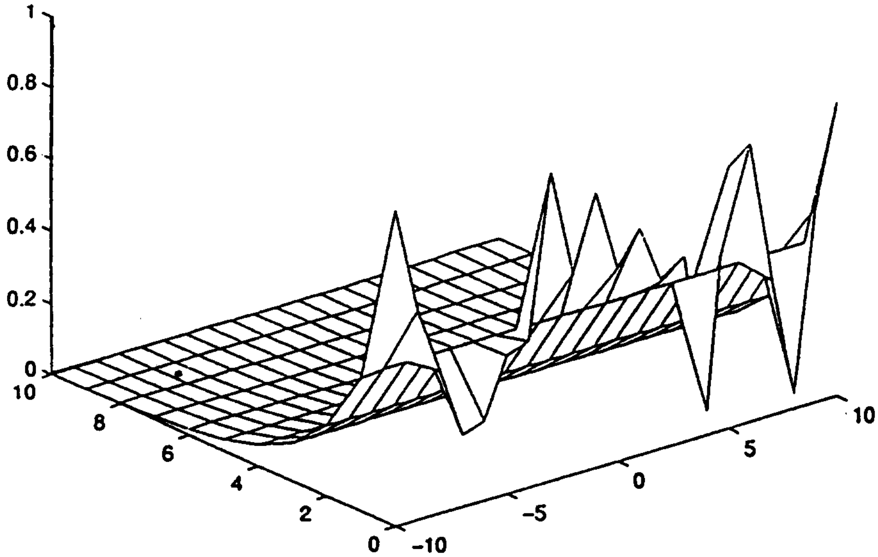


$a = 0.0995$, $b = 0.3627$, $c = -0.5753$, $|a| + |b| + |c| = 1.0175$.

Figure 5.

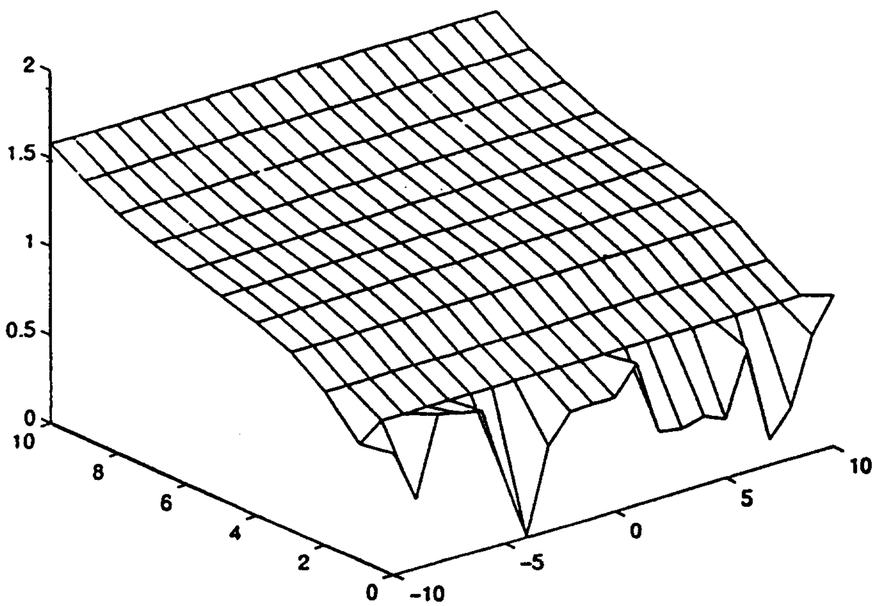
The second one is the Green's function of (2.1) when $a = -0.5753$, $b = 0.3027$, and $c = 0.0995$. The graph of this function is presented in Figure 2. Note that since $a < 0$, and $b, c > 0$ as well as $|a| + |b| + |c| = 0.9775 < 1$, the Green's function is oscillatory but exponentially asymptotically stable.

An unstable and nonnegative Green's function is plotted in Figure 3, while an unstable and oscillatory Green's function in Figure 4.



$a = 0.2557,\ b = 0.2283,\ c = 0.0571,\ |a| + |b| + |c| = 0.5419.$

Figure 6.



$a = 0.1963,\ b = 0.3391,\ c = 0.5753,\ |a| + |b| + |c| = 1.1107.$

Figure 7.

An interesting example is the Green's function of (2.1) when $a = 0.0995$, $b = 0.3627$, and $c = -0.5753$. Since $|a| + |b| + |c| > 1$, it is expected that the Green's function is unstable (see Theorem 4.2). However, Figure 5 shows that the contrary case appears to be true. The exact reason is not clear at this moment, but rounding errors may be the cause of this phenomenon.

The last two examples (Figures 6 and 7) are solutions of (2.1) subject to initial conditions generated by a built-in computer random number generator. Their graphs agree with our theoretical conclusions.

REFERENCES

1. D.V. Widder, *The Heat Equation*, Academic Press, New York, (1975).
2. G.E. Forsythe and W.R. Wasow, *Finite Difference Methods for Partial Differential Equations*, Wiley, New York, (1960).
3. G.D. Smith, *Numerical Solution of Partial Differential Equations: Finite Difference Methods*, 2^{nd} edition, Clarendon Press, Oxford, (1978).
4. G.H. Golub and J.M. Ortega, *Scientific Computing and Differential Equations: An Introduction to Numerical Methods*, Academic Press, New York, (1992).
5. J.C. Strikwerda, *Finite Difference Schemes and Partial Differential Equations*, Wadsworth, (1989).
6. G. Berman and K.D. Fryer, *Introduction to Combinatorics*, Academic Press, New York, (1972).
7. D.H. Mugler, *International Series of Numerical Mathematics*, Volume 65, pp. 543–554, Birkhauser Verlag, Basel, (1984.).
8. D.H. Mugler and R.A. Scott, Fast Fourier transform method for partial differential equations, case study: The 2-D diffusion equation, *Computers Math. Applic.* **16** (3), 221–228, (1988).