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New kinds of generalized variational-like inequality problems in topological vector spaces

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ABSTRACT

In this work, we consider a generalized nonlinear variational-like inequality problem, in topological vector spaces, and, by using the KKM technique, we prove an existence theorem. Our result extends a theorem of Ahmad and Irfan [R. Ahmad, S.S. Irfan, On the generalized nonlinear variational-like inequality problems, Appl. Math. Lett. 19 (2006) 294–297].

1. Introduction

Variational inequalities were introduced and considered by Stampacchia [1] in the early sixties. It has been shown that a wide class of linear and nonlinear problems arising in various branches of mathematical and engineering sciences can be studied within the unified and general framework of variational inequalities. Variational inequalities have been generalized and extended in several directions using novel techniques. The variational-like inequality, also known as the pre-variational inequality, is one of the generalized forms of variational inequalities; see [2,3] and the references therein. Variational-like inequalities and generalized variational-like inequalities are powerful tools for studying nonconvex optimization problems and nonconvex and nondifferentiable optimization problems, respectively; see, for example, [4,5,2,6] and the references therein.

In this work, we consider a generalized nonlinear variational-like inequality problem, in the setting of topological vector spaces, and we prove an existence theorem concerning its solution.

Let $\langle X, X^*\rangle$ be a dual system of Hausdorff topological vector spaces and K a nonempty convex subset of X. Given single-valued mappings $f, g, p : X^* \to X^*$, a bifunction $\eta : K \times K \to X$, multivalued maps $M, S, T : K \to 2^{X^*}$, and a map $h : K \times K \to \mathbb{R}$, we consider the following *generalized nonlinear variational-like inequality problem (GNVLIP)*:

$$\begin{cases} \text{Find } x \in K \text{such that } \forall y \in K \\ \exists u \in M(x),\ v \in S(x),\ \text{and } w \in T(x) \text{ satisfying } (GNVLIP) \\ \langle p(u) - (f(v) - g(w)), \eta(y, x)\rangle \geq h(x, y). \end{cases}$$

Examples of (GNVLIP):

(1) If $h(x, y) = \phi(x) - \phi(y)$, where $\phi : K \to \mathbb{R}$ and $(x, y) \in K \times K$,, then (GNVLIP) collapses to the problem studied by Ahmad and Irfan [7].

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(2) If $h(u,u) = 0$, $\eta(v,u) = v-u$, $f(v) = g(v) = 0$, for all $u, v \in K$ and $M(u) = \{F(u)\}$, where $F : K \to X^*$, $u \in K$, then (GNVLIP) collapses to the mixed quasi-variational inequality problem (for short, MQVIP) which consists of finding $u \in K$ such that

$$\langle F(u), v - u\rangle + h(v,u) \geq 0, \quad \forall v \in K.$$

Note that MQVIP was studied in [2,3,12].

In [7] the authors studied (GNVLIP) for $h(x,y) = \phi(x) - \phi(y)$, where $\phi : K \to \mathbb{R}$ and $(x,y) \in K \times K$, in the setting of locally convex spaces. They obtained an existence result for the solution of (GNVLIP) but the proof of their main result seems to be incomplete since in [7] the authors claim that the set $A = \{x_\lambda\} \cup \{x^*\}$ is compact, where the net $\{x_\lambda\}$ converges to x^*, but this is incorrect in general.

In the rest of this section, we recall some definitions and results which are needed in the sequel.

Definition 1.1. Let X and Y be two topological spaces. A set-valued mapping $T : X \to 2^Y$ is called:

(i) **upper semicontinuous** (u.s.c.) at $x \in X$ if for each open set V containing $T(x)$, there is an open set U containing x such that for each $t \in U$, $T(t) \subseteq V$; T is said to be u.s.c. on X if it is u.s.c. for each $x \in X$,
(ii) **compact** if $cl(T(X))$ is a compact subset of Y,
(iii) **closed** if the graph of T, denoted by $Gr(T)$, i.e., the set $Gr(T) = \{(x,y) : x \in X, y \in T(x)\}$, is a closed set in $X \times Y$.

Lemma 1.2 ([8]). *Let X and Y be two topological spaces. If $T : X \to 2^Y$ is a set-valued mapping, then:*

(i) *T is closed if and only if for any net $\{x_\alpha\}$, $x_\alpha \to x$, and any net $\{y_\alpha\}$, $y_\alpha \in T(x_\alpha)$, $y_\alpha \to y$, one has $y \in T(x)$,*
(ii) *if Y is compact and $T(x)$ is closed for each $x \in X$, then T is upper semicontinuous if and only if T is closed,*
(iii) *if for any $x \in X$, $T(x)$ is compact, and T is upper semicontinuous on X, then for any net $\{x_\alpha\} \subseteq X$ such that $x_\alpha \to x \in X$ and for every $y_\alpha \in T(x_\alpha)$, there exists $y \in T(x)$ and a subnet $\{y_\beta\}$ of y_α such that $\{y_\beta\} \to y$.*

Definition 1.3. Let X be a nonempty subset of a topological vector space E. A multifunction $F : X \to 2^E$ is said to be a KKM mapping if for each nonempty finite set $\{x_1, \ldots, x_n\} \subseteq X$, we have

$$\operatorname{conv}\{x_1, \ldots, x_n\} \subseteq \bigcup_{j=1}^{n} F(x_j).$$

The following version of the KKM principle is a special case of the Fan KKM principle [9].

Lemma 1.4. *Let X be a nonempty subset of a topological vector space E and $F : X \to 2^E$ be a KKM mapping with closed values. Assume that there exists a nonempty compact convex subset B of X such that $D = \bigcap_{x\in B} F(x)$ is compact. Then*

$$\bigcap_{x\in X} F(x) \neq \emptyset.$$

2. Main results

In this section we provide an existence theorem for (GNVLIP), in topological vector spaces. Throughout this work, we assume that the pairing $\langle ., .\rangle$ is upper semicontinuous.

Notation:

$$\psi_\Omega(x) = \{y \in \Omega : \forall u \in M(x),\ v \in S(x) \text{ and } w \in T(x) \text{ s.t. } \langle p(u) - (f(v) - g(w)), \eta(y,x)\rangle < h(x,y)\}$$

where $\Omega \subseteq K$ and $x \in K$.

Theorem 2.1. *Let X be a Hausdorff topological vector space and K be a nonempty convex subset of X. Let $M, S, T : K \to 2^{X^*}$, be upper semicontinuous mappings with nonempty compact values, $h : K \times K \to \mathbb{R}$, and $f, g, p : X^* \to X^*$ be continuous. Suppose the following conditions hold:*

(i) *the map $x \to h(x,y)$ is lower semicontinuous with $h(y,y) = 0$, for all $y \in K$;*
(ii) *$\eta : K \times K \to X$ is continuous in the second argument such that $\eta(x,x) = 0$, $\forall x \in K$;*
(iii) *the set $\psi_K(x)$ is convex, for all $x \in K$;*
(iv) *there exist a nonempty compact and convex subset B of K and a nonempty compact subset D of K such that $\psi_B(x) \neq \emptyset$ for all $x \in K \setminus D$.*

Then the solution set of (GNVLIP) is nonempty and compact.

Proof. We define $\Gamma : K \to 2^K$ as follows:

$$\Gamma(y) = \{x \in K : \exists u \in M(x), v \in S(x), w \in T(x) \text{ such that } \langle p(u) - (f(v) - g(w)), \eta(y, x)\rangle \geq h(x, y)\}.$$

We first show that Γ is a KKM mapping. Suppose the contrary, i.e. suppose there exist $y_1, y_2, \ldots, y_n \in K$ and $z \in \text{conv}\{y_1, y_2, \ldots, y_n\}$ such that $z \notin \cup_{i=1}^n \Gamma(y_i)$. Hence, for all $u \in M(y_i)$, for all $v \in S(y_i)$, and for all $w \in T(y_i)$, we have

$$\langle p(u) - (f(v) - g(w)), \eta(y_i, x)\rangle < h(z, y_i).$$

Hence by (iii) and $\eta(z, z) = 0$ (note that $h(z, z) = 0$), we deduce that $0 < 0$ which is a contradiction. Next we show that $\Gamma(y)$ is a closed subset of X for each $y \in K$. To see this, let (x_α) be a net in $\Gamma(y)$ which converges to $x_0 \in X$. Since $x_\alpha \in \Gamma(y)$, by the definition of $\Gamma(y)$, there exist nets (u_α), (v_α), and (w_α) with $u_\alpha \in M(x_\alpha)$, $v_\alpha \in S(x_\alpha)$ and $w_\alpha \in T(x_\alpha)$ such that

$$\langle p(u_\alpha) - (f(v_\alpha) - g(w_\alpha)), \eta(y, x_\alpha)\rangle \geq h(x_\alpha, y). \tag{1}$$

Since the multivalued maps M, S and T are upper semicontinuous with compact values, then by Lemma 1.2, (u_α), (v_α) and (w_α) have convergent subnets with limits, say u_0, v_0 and w_0. Without loss of generality we may assume that (u_α) converges to u_0, (v_α) converges to v_0, and (w_α) converges to w_0. Then by the upper semicontinuity of M, S and T, we have $u_0 \in M(x_0)$, $v_0 \in S(x_0)$ and $w_0 \in T(x_0)$ (note that M, S, T are closed maps since M, S, T are u.s.c. with compact values and X is Hausdorff). The continuity of f, g, p and the upper semicontinuity of $\langle ., .\rangle$ and (1) yield

$$\begin{aligned}\langle p(u_0) - (f(v_0) - g(w_0)), \eta(y, x_0)\rangle &\geq \limsup_\alpha \langle p(u_\alpha) - (f(v_\alpha) - g(w_\alpha)), \eta(y, x_\alpha)\rangle \\ &\geq \liminf h(x_\alpha, y) \geq h(x_0, y)\end{aligned}$$

(note that the last inequality follows from (i)). Consequently $x_0 \in \Gamma(y)$. Hence $\Gamma(y)$ is closed in K for all $y \in K$. By (iv) the set $\bigcap_{x\in B} \Gamma(x)$ is compact. Therefore Γ satisfies all the assumptions of Lemma 1.4. Hence by Lemma 1.4, there exists $\bar{y} \in K$ such that $\bar{y} \in \cap_{x\in K} \Gamma(x)$ and so $\bar{y}$ is a solution of (GNVLIP) (note that the solution set of (GNVLIP) is equal to $\cap_{x\in K} \Gamma(x)$). Hence the solution set of (GNVLIP) is nonempty. By (iv) it is clear that the solution set of (GNVLIP)is a closed subset of the compact set D. This completes the proof. □

As an application of Theorem 2.1, we establish an existence theorem for the following problem which consists of finding $x \in K$, $u \in M(x)$ and $v \in S(x)$, such that

$$\langle u - v, y - x\rangle \geq 0 \quad \text{for all } y \in K. \tag{2}$$

This problem was studied by Verma [10], in the setting of a real Hilbert spaces. For the next result we need the following lemma.

Lemma 2.2 ([11]). *Let D be a convex, compact set and K be a convex set. Let $f : D \times K \to \mathbb{R}$ be concave and upper semicontinuous in the first variable, and convex in the second variable. Assume that*

$$\max_{x\in D} f(x, y) \geq 0 \quad \forall y \in K.$$

Then there exists $\bar{x} \in D$ such that $f(\bar{x}, y) \geq 0$ $\forall y \in K$.

Theorem 2.3. *Let X be a Hausdorff topological vector space and K be a nonempty convex subset of X. Let $M, S : K \to 2^{X^*}$, with nonempty compact convex values, be such that:*

(a) *there exist a nonempty compact and convex subset B of K and a nonempty compact subset D of K such that for all $x \in K \setminus D$ there exists $y \in B$ satisfying*

$$\langle u - v, y - x\rangle < 0, \quad \forall u \in M(x),\ \forall v \in S(x).$$

Then the solution set of (2) is nonempty.

Proof. One can easily see that the mappings $M, S, T : K \to 2^{X^*}$, $p, f, g : X^* \to X^*$, $h : K \times K \to \mathbb{R}$, and $\eta : K \times K \to X$ defined by $p(u) = u$, $f(v) = v$, $g(w) = 0$, $\eta(y, x) = y - x$, $h(x, y) = 0$ and $T(x) = \{0\}$, for each $(u, v, w) \in X^* \times X^* \times X^*$ and $(x, y) \in K \times K$, satisfy all the hypotheses of Theorem 2.1. Then by Theorem 2.1 there exists $\bar{x} \in K$ such that

$$\forall y \in K, \exists (u, v) \in M(\bar{x}) \times S(\bar{x}) \quad \text{with } \langle u - v, y - \bar{x}\rangle \geq 0. \tag{3}$$

We now define a mapping $P : (M(\bar{x}) \times S(\bar{x})) \times K \to \mathbb{R}$ by

$$P(u, v, y) = \langle u - v, y - \bar{x}\rangle,$$

for each $(u, v, y) \in M(\bar{x}) \times S(\bar{x}) \times K$. We now show:

(i) for each $y \in K$, the mapping $(u, v) \to P(u, v, y)$ is concave and upper semicontinuous (even continuous),

(ii) for each $(u, v) \in M(\bar{x}) \times S(\bar{x})$, the mapping $y \to P(u, v, y)$ is convex,

(iii) $\max_{(u,v,y)\in M(\bar{x})\times S(\bar{x})\times K} P(u, v, y) \geq 0$.

To see (i), let $(u_j, v_j) \in M(\bar{x}) \times S(\bar{x})$, for $j = 1, 2$, $\lambda \in]0, 1[$, and fixed $y \in K$. Thus we have

$$P(\lambda u_1 + (1-\lambda)u_2, \lambda v_1 + (1-\lambda)v_2, y) = \langle \lambda u_1 + (1-\lambda)u_2 - (\lambda v_1 + (1-\lambda)v_2), y - \bar{x}\rangle$$
$$= \lambda\langle u_1 - v_1, y - \bar{x}\rangle + (1-\lambda)\langle u_2 - v_2, y - \bar{x}\rangle = \lambda P(u_1, v_1, y) + (1-\lambda)P(u_2, v_2, y).$$

Hence the mapping $(u, v) \to P(u, v, y)$ is concave. It is clear that the mapping $(u, v) \to P(u, v, y) = \langle u - v, y - \bar{x}\rangle$ is continuous and hence upper semicontinuous. The following equality:

$$\langle u - v, \lambda y_1 + (1-\lambda)y_2 - \bar{x}\rangle = \lambda\langle u - v, y_1 - \bar{x}\rangle + (1-\lambda)\langle u - v, y_2 - \bar{x}\rangle,$$

where $(y_1, y_2) \in K \times K$, $\lambda \in]0, 1[$ and $(u, v) \in M(\bar{x}) \times S(\bar{x})$, shows that (ii) holds. Now (3) guarantees (iii). Thus the mapping P satisfies all the assumptions of Lemma 2.2 and so there exists $(\overline{u}, \overline{v}) \in M(\bar{x}) \times S(\bar{x})$ such that $P(\overline{u}, \overline{v}, y) = \langle \overline{u} - \overline{v}, y - \bar{x}\rangle \geq 0$ for all $y \in K$. This completes the proof. □

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