



A new proof of Faber's intersection number conjecture [☆]

A. Buryak ^{a,b}, S. Shadrin ^{a,*}

^a Korteweg–de Vries Institute for Mathematics, University of Amsterdam, P.O. Box 94248,
1090 GE Amsterdam, The Netherlands

^b Department of Mathematics, Moscow State University, Leninskie gory, 19992 GSP-2 Moscow, Russia

Received 4 January 2010; accepted 3 May 2011

Available online 19 May 2011

Communicated by Ravi Vakil

Abstract

We give a new proof of Faber's intersection number conjecture concerning the top intersections in the tautological ring of the moduli space of curves \mathcal{M}_g . The proof is based on a very straightforward geometric and combinatorial computation with double ramification cycles.

© 2011 Elsevier Inc. All rights reserved.

Keywords: Moduli space of curves; Faber's conjecture; Tautological ring; Intersection numbers; Double ramification cycles

Contents

1.	Introduction	23
1.1.	Notations	23
1.2.	Faber's conjecture	23
1.3.	Top intersections	24
1.4.	Double ramification cycles	24
1.5.	Organization of the paper	25
1.6.	Acknowledgments	25
2.	Integrals over DR-cycles	26

[☆] A.B. is partially supported by the grants RFBR-10-01-00678, NSH-8462.2010.1 and by the Moebius Contest Foundation for Young Scientists. Both A.B. and S.S. are partly supported by the Vidi grant of NWO.

* Corresponding author.

E-mail addresses: a.y.buryak@uva.nl, buryaksh@mail.ru (A. Buryak), s.shadrin@uva.nl (S. Shadrin).

2.1.	Reduction to initial DR-cycles	26
2.2.	Expression for a ψ -class on the initial cycle	27
2.3.	Initial values	28
3.	Basic properties of integrals over DR-cycles	29
3.1.	A small simplification	29
3.2.	Polynomiality	29
3.3.	Divisibility	31
4.	Faber’s conjecture	34
4.1.	A reformulation of Faber’s conjecture	34
4.2.	A reformulation of Proposition 2.4	35
4.3.	Computation of the coefficients	35
4.4.	A proof of Faber’s conjecture	36
4.5.	Proofs of Lemmas 4.3–4.5	40
	References	42

1. Introduction

1.1. Notations

Let $\mathcal{M}_{g,n}$ be the moduli space of complex algebraic curves of genus g with n labelled marked points. We denote by $\overline{\mathcal{M}}_{g,n}$ the space of stable curves which is the Deligne–Mumford compactification of $\mathcal{M}_{g,n}$, and by $\mathcal{M}_{g,n}^{rt} \subset \overline{\mathcal{M}}_{g,n}$ the partial compactification of $\mathcal{M}_{g,n}$ by stable nodal curves with rational tails (that is, one irreducible component of a stable curve must still have geometric genus g).

Throughout the paper we work with tautological classes on these spaces. The tautological ring $R^*(\overline{\mathcal{M}}_{g,n})$ can be defined as the minimal system of subalgebras of $A^*(\overline{\mathcal{M}}_{g,n})$ that contains the classes ψ_1, \dots, ψ_n and is closed under push-forwards with natural maps between moduli spaces. The tautological classes on $\mathcal{M}_{g,n}^{rt}$ are defined as restrictions of the tautological classes on $\overline{\mathcal{M}}_{g,n}$.

For further definitions and a detailed discussion of the tautological ring and related topics in geometry of the moduli space of curves we refer the reader to [13], which is a good survey on the subject.

1.2. Faber’s conjecture

The conjecture of C. Faber [1] describes the structure of the tautological ring $R^*(\mathcal{M}_g)$, $g \geq 2$ ($\mathcal{M}_g = \mathcal{M}_{g,0}$). Let us mention the key ingredients of this conjecture.

- (1) (*Vanishing*) For any $i \geq g - 1$, $R^i(\mathcal{M}_g) = 0$.
- (2) (*Socle*) $R^{g-2}(\mathcal{M}_g) \cong \mathbb{Q}$.
- (3) (*Perfect pairing*) For any $0 \leq i \leq g - 2$, the cup product

$$R^i(\mathcal{M}_g) \times R^{g-2-i}(\mathcal{M}_g) \rightarrow R^{g-2}(\mathcal{M}_g)$$

is a perfect pairing.

- (4) (*Top intersections*) Let $\pi : \mathcal{M}_{g,n}^{rt} \rightarrow \mathcal{M}_g$ be the forgetful morphism. Assume $d_1 + \dots + d_n = g + n - 2$, $d_i \geq 1$, $i = 1, \dots, n$. Then the class

$$\pi_* \left(\prod_{i=1}^n \psi_i^{d_i} (2d_i - 1)!! \right) \in R^{g-2}(\mathcal{M}_g)$$

does not depend on d_1, \dots, d_n .

The vanishing and socle properties are proved in several different ways, see [1,8,5]. The perfect pairing is still an open question. The top intersections property, also known as Faber’s intersection number conjecture, we discuss in the next section.

1.3. Top intersections

Faber [1] observed that the class $\lambda_g \lambda_{g-1}$ is equal to zero on $\overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}^{rt}$, $n \geq 0$. Moreover, the linear functional $\int \cdot \lambda_g \lambda_{g-1} : R^{g-2}(\mathcal{M}_g) \rightarrow \mathbb{Q}$ is an isomorphism. Therefore, a reformulation of Faber’s intersection number conjecture states that

$$\int_{\overline{\mathcal{M}}_{g,n}} \prod_{i=1}^n \psi_i^{d_i} \lambda_g \lambda_{g-1} = \frac{(2g - 3 + n)!(2g - 3)!!}{(2g - 2)! \prod_{i=1}^n (2d_i - 1)!!} \int_{\overline{\mathcal{M}}_{g,1}} \psi_1^{g-1} \lambda_g \lambda_{g-1}.$$

In this form it is already proved in two different ways that we would like to discuss here.

First proof is based on an observation of Getzler and Pandharipande [3]. The $\lambda_g \lambda_{g-1}$ -integrals appear in the Gromov–Witten theory of $\mathbb{C}P^2$, and the degree zero Virasoro constraints imply Faber’s intersection number conjecture. The Virasoro constraints for the Gromov–Witten potential of $\mathbb{C}P^2$ were proved later on by Givental, see [4].

Second proof is due to Liu and Xu [7] via very skillful combinatorial computations. Mumford’s formula [9] expresses λ -classes in terms of ψ -, κ -, and boundary classes. Therefore, the whole problem is reduced to a computation of some non-trivial combinations of the integrals of ψ -classes. Witten’s conjecture [14] (proved by now in several different ways) allows to compute all integrals of ψ -classes using string, dilaton, and KdV equations.

There is a third approach to the same problem due to Goulden, Jackson, and Vakil. They apply relative to infinity localization to the moduli space of mappings to $\mathbb{C}P^1$ in order to obtain relations that involve more general so-called Faber–Hurwitz classes and double Hurwitz numbers in genus 0. This set of relations allows, in principle, to resolve Faber’s intersection number conjecture completely, but there are combinatorial difficulties that they have managed to overcome only for a small number of points.

We give a new proof of Faber’s intersection number conjecture. There are at least two reasons to do that. First, two existing proofs mentioned above involve too advanced technique and, second, they do not provide any geometric feeling for the structure of the tautological ring of \mathcal{M}_g . Meanwhile, the approach of Goulden, Jackson, and Vakil allows to understand much more from the low-level geometry of \mathcal{M}_g , but it is not a complete proof of the conjecture. Our approach is somewhat similar to the main idea of Goulden, Jackson, and Vakil, but all computations have appeared to be much simpler.

1.4. Double ramification cycles

A particular type of double ramification cycles that we need in this paper can be described in the following way. Let $a_1, \dots, a_n, n \geq 1$ and $b_1, \dots, b_k, 0 \leq k \leq g$, be positive integers. A subva-

riety $\mathcal{H}(a_1, \dots, a_n, b_1, \dots, b_k) \subset \mathcal{M}_{g,n+k+1}$ consists of curves $(C_g, x_0, x_1, \dots, x_n, y_1, \dots, y_k)$ such that $-(\sum_{i=1}^n a_i + \sum_{i=1}^k b_i)x_0 + \sum_{i=1}^n a_i x_i + \sum_{i=1}^k b_i y_i$ is a principle divisor. Let $\pi : \overline{\mathcal{M}}_{g,n+k+1} \rightarrow \overline{\mathcal{M}}_{g,n+1}$ be the map that forgets the points y_1, \dots, y_k . We denote by

$$DR_g \left(\prod_{i=1}^n m_{a_i} \prod_{i=1}^k \tilde{m}_{b_i} \right)$$

the push-forward $\pi_*[\mathcal{H}(a_1, \dots, a_n, b_1, \dots, b_k)]$ of the class of the closure of $\mathcal{H}(a_1, \dots, a_n, b_1, \dots, b_k)$ in $\overline{\mathcal{M}}_{g,1+n+k}$. Sometimes it is more convenient to consider the restriction of the Poincaré dual of the class $DR_g(\prod_{i=1}^n m_{a_i} \prod_{i=1}^k \tilde{m}_{b_i})$ to $\mathcal{M}_{g,1+n}^{rt}$; abusing notations we denote it by the same symbol. It is proved in [12] (a generalization of the argument in [9]), that $DR_g(\prod_{i=1}^n m_{a_i} \prod_{i=1}^k \tilde{m}_{b_i})$ has codimension $g - k$.

An advantage of the double ramification cycles is that any tautological class can be expressed in terms of them [6] and there is a simple expression for a ψ -class restricted to a DR-cycle in terms of DR-cycles of higher codimension. All DR-cycles lie in the tautological ring [2].

The main idea of our approach to Faber’s intersection number conjecture can be described in the following way. The fundamental class of the moduli space of curves of genus g can be represented by a DR-class with $k = g$. Then any integral of ψ -classes over this cycle can be expressed in terms of integrals over DR-classes with $k = 0$ via the same argument as in the standard proof of the string equation. A lemma of E. Ionel [6] allows to find an expression for any monomial of ψ -classes (in $\mathcal{M}_{g,1+n}^{rt}$, $n \geq 1$) in terms of DR-cycles with $n = 1$ and $k = 0$, that is, $DR_g(m_a)$, $a \geq 2$. These classes are in the socle of the tautological ring of $\mathcal{M}_{g,2}^{rt}$, they are proportional to one particular class $DR_g(m_2)$ which is the hyperelliptic locus generating $R^g(\mathcal{M}_{g,2}^{rt})$.

This gives a combinatorial algorithm to compute explicitly any class involved in Faber’s conjecture. A relatively simple and straightforward analysis of this algorithm gives a new prove of Faber’s intersection number conjecture.

We hope that the technique of DR-cycles presented here can help with the rest of Faber’s conjecture, that is, with the prefect pairing, which is still the most mysterious part of it.

1.5. Organization of the paper

We split the argument into geometric (Section 2) and combinatorial (Sections 3 and 4) parts. In fact, the new ideas in this paper are only in combinatorial computation, while all geometric arguments are a sort of standard routine computations using the space of admissible covers or universal Jacobian. This sort of arguments is rather standard, so we decided to de-emphasize geometric part and we provide only sketches of the proofs there. Let us also mention here that all statements in Section 3 have a strong geometric flavor in the sense that there are some incomplete geometric arguments that could replace straightforward combinatorial proofs there.

1.6. Acknowledgments

The authors are very grateful to I. Goulden, D. Jackson, M. Kazarian, B. Moonen, R. Vakil, and D. Zvonkine for the plenty of fruitful discussions.

2. Integrals over DR-cycles

The goal of this section is to give an algorithm to compute an integral $\int_{\overline{\mathcal{M}}_{g,n+1}} \lambda_g \lambda_{g-1} \psi_0^0 \times \prod_{i=1}^n \psi_i^{d_i}$ for any non-negative integers d_1, \dots, d_n such that $\sum_{i=1}^n d_i = g + n - 1$. It is not exactly the integrals we need for Faber’s conjecture, however, there is an argument of Witten in [14] that explains how to use the string equation in order to recover the integrals with arbitrary (positive) powers of ψ -classes from these particular ones.

There are two different languages. One can either discuss the integrals of $\lambda_g \lambda_{g-1} \psi_0^0 \prod_{i=1}^n \psi_i^{d_i}$ over DR-cycles in $\overline{\mathcal{M}}_{g,1+n}$ (which is usually more convenient for particular computations), or we can say the same for the intersections of $\psi_0^0 \prod_{i=1}^n \psi_i^{d_i}$ with the restrictions of the Poincaré duals of DR-cycles to $R^*(\mathcal{M}_{g,1+n}^{rt})$ (which is more convenient for geometric arguments).

We introduce a new notation. Let $d_1, \dots, d_n, n \geq 1$, be non-negative integers such that $\sum_{i=1}^n d_i = n - 1$. Let a_1, \dots, a_n be arbitrary positive integers. Let

$$\left\langle \prod_{i=1}^n \begin{bmatrix} a_i \\ d_i \end{bmatrix} \right\rangle_g^{DR} := \int_{DR_g(\prod_{i=1}^n m_{a_i})} \lambda_g \lambda_{g-1} \psi_0^0 \prod_{i=1}^n \psi_i^{d_i}.$$

2.1. Reduction to initial DR-cycles

The initial DR-cycles are the cycles with no \tilde{m} -s in the notations of the previous section. There is a simple reduction formula for ψ -classes on the initial DR-cycles that we discuss in the next section. The goal of this section is to express any product of ψ -classes in $R^{g+n-1}(\mathcal{M}_{g,1+n}^{rt})$ in terms of the products of ψ -classes and initial DR-cycles.

There are two first observations that we are going to use.

Lemma 2.1. *In $R^0(\mathcal{M}_{g,1+n}^{rt})$ we have:*

$$DR_g \left(\prod_{i=1}^n m_{a_i} \prod_{i=1}^g \tilde{m}_{b_i} \right) = g! \prod_{i=1}^g b_i^2 [\overline{\mathcal{M}}_{g,1+n}].$$

Lemma 2.2. *Let $\pi : \mathcal{M}_{1+n}^{rt} \rightarrow \mathcal{M}_{1+(n-1)}^{rt}$ be the map that forgets the last marked point. Assume $k \leq g - 1$. Then*

$$\pi_* DR_g \left(\prod_{i=1}^n m_{a_i} \prod_{i=1}^k \tilde{m}_{b_i} \right) = DR_g \left(\prod_{i=1}^{n-1} m_{a_i} \tilde{m}_{a_n} \prod_{i=1}^k \tilde{m}_{b_i} \right).$$

Sketch of proofs. The first lemma is almost obvious, since the corresponding DR-cycle can be defined via an intersection in the universal Jacobian over $\mathcal{M}_{g,1+n}^{rt}$. Then the lemma follows from the fact that for any curve C of genus g with a chosen base point x_0 the map $C^g \rightarrow Jac(C)$, $(y_1, \dots, y_g) \mapsto \sum_{i=1}^g b_i y_i$, is of degree $g! \prod_{i=1}^g b_i^2$. The second lemma follows immediately from the definitions. \square

These two lemmas allow to express a monomial of ψ -classes in terms of intersections with the initial DR-cycles.

Proposition 2.3. *Let d_1, \dots, d_n be positive integers such that $\sum_{i=1}^n d_i = g + n - 1$. For any positive integers a_1, \dots, a_n and b_1, \dots, b_g , we have the following identity:*

$$\begin{aligned} & \left(g! \prod_{i=1}^g b_i^2 \right) \cdot \int_{\overline{\mathcal{M}}_{g,n+1}} \lambda_g \lambda_{g-1} \psi_0^n \prod_{i=1}^n \psi_i^{d_i} \\ &= \sum_{I_0 \sqcup \dots \sqcup I_n = \{1, \dots, g\}} (-1)^{g-|I_0|} \left\langle \prod_{i=1}^n \begin{bmatrix} a_i + \sum_{j \in I_i} b_j \\ d_i - |I_i| \end{bmatrix} \prod_{i \in I_0} \begin{bmatrix} b_i \\ 0 \end{bmatrix} \right\rangle_g^{DR}. \end{aligned} \tag{1}$$

Sketch of a proof. The argument that derives this proposition from Lemmas 2.1 and 2.2 is a straightforward application of the pull-back formula for ψ -classes, cf. proof of the string equation in [14]. See [10,11] for the same argument applied in some other cases that involve DR-cycles. \square

2.2. Expression for a ψ -class on the initial cycle

In general, an initial DR-cycle is the image of a particular space of admissible covers where one has a map to the target genus 0 curve. A lemma of Ionel [6] states that the ψ -class lifted from the DR-cycle is proportional to a ψ -class lifted from the moduli space of target genus 0 curves. This allows to use the genus 0 topological recursion relation for a ψ -class on double ramification cycles. The result of that can be described on the level of intersection numbers by the following proposition.

Proposition 2.4. *For any positive a, a_1, \dots, a_n and for any non-negative d, d_1, \dots, d_n , we have the following recursion relation:*

$$\begin{aligned} & a \cdot (2g + n) \cdot \left\langle \begin{bmatrix} a \\ d + 1 \end{bmatrix} \prod_{i=1}^n \begin{bmatrix} a_i \\ d_i \end{bmatrix} \right\rangle_g^{DR} \\ &= \sum_{I \sqcup J = \{1, \dots, n\}} \left(\left(a + \sum_{j \in J} a_j \right) \cdot |I| \cdot \left\langle \begin{bmatrix} a + \sum_{j \in J} a_j \\ 0 \end{bmatrix} \prod_{i \in I} \begin{bmatrix} a_i \\ d_i \end{bmatrix} \right\rangle_0^{DR} \left\langle \begin{bmatrix} a \\ d \end{bmatrix} \prod_{j \in J} \begin{bmatrix} a_j \\ d_j \end{bmatrix} \right\rangle_g^{DR} \right. \\ & \quad - \left(\sum_{j \in J} a_j \right) \cdot (2g + |J| - 1) \cdot \left\langle \begin{bmatrix} a \\ d \end{bmatrix} \left[\sum_{j \in J} a_j \right] \prod_{i \in I} \begin{bmatrix} a_i \\ d_i \end{bmatrix} \right\rangle_0^{DR} \left\langle \prod_{j \in J} \begin{bmatrix} a_j \\ d_j \end{bmatrix} \right\rangle_g^{DR} \\ & \quad + \left(a + \sum_{j \in J} a_j \right) \cdot (2g + |I|) \cdot \left\langle \begin{bmatrix} a + \sum_{j \in J} a_j \\ 0 \end{bmatrix} \prod_{i \in I} \begin{bmatrix} a_i \\ d_i \end{bmatrix} \right\rangle_g^{DR} \left\langle \begin{bmatrix} a \\ d \end{bmatrix} \prod_{j \in J} \begin{bmatrix} a_j \\ d_j \end{bmatrix} \right\rangle_0^{DR} \\ & \quad \left. - \left(\sum_{j \in J} a_j \right) \cdot (|J| - 1) \cdot \left\langle \begin{bmatrix} a \\ d \end{bmatrix} \left[\sum_{j \in J} a_j \right] \prod_{i \in I} \begin{bmatrix} a_i \\ d_i \end{bmatrix} \right\rangle_g^{DR} \left\langle \prod_{j \in J} \begin{bmatrix} a_j \\ d_j \end{bmatrix} \right\rangle_0^{DR} \right). \end{aligned}$$

Here we use the notation:

$$\begin{aligned} \left\langle \prod_{i=1}^n \begin{bmatrix} a_i \\ d_i \end{bmatrix} \right\rangle_0^{DR} &:= \int_{DR_0(\prod_{i=1}^n m_{a_i})} \psi_0^0 \prod_{i=1}^n \psi_i^{d_i} \\ &= \int_{\overline{\mathcal{M}}_{0,1+n}} \psi_0^0 \prod_{i=1}^n \psi_i^{d_i} = \begin{cases} \frac{(n-2)!}{d_1! \cdots d_n!}, & \text{if } d_1 + \cdots + d_n = n - 2, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Sketch of a proof. This proposition is a very closed relative of the similar formulas in [10,11] and is based on Ionel’s lemma in the way described above. We only take into account the components of the general expression of a ψ -class restricted to a DR-cycle that belong to $\mathcal{M}_{g,2+n}^{rt}$, the rest of the prove is identical to [10,11]. \square

There is a nice interpretation of this recursion in terms of generating vector fields for the intersection numbers over DR-cycles. Let β and $t_{a,d}$, $a \geq 1$, $d \geq 0$, be the formal variables. Define

$$\begin{aligned} V_g &:= \sum_{n=1}^{\infty} \frac{\beta^{2g+n-1}}{n!} \sum_{\substack{a_1, \dots, a_n \\ d_1 + \dots + d_n = n-1}} \left\langle \prod_{i=1}^n \begin{bmatrix} a_i \\ d_i \end{bmatrix} \right\rangle_g^{DR} \prod_{i=1}^n t_{a_i, d_i} \cdot \frac{(\sum_{i=1}^n a_i) \partial}{\partial t_{(\sum_{i=1}^n a_i), 0}}, \\ V_0 &:= \sum_{n=2}^{\infty} \frac{\beta^{n-1}}{n!} \sum_{\substack{a_1, \dots, a_n \\ d_1 + \dots + d_n = n-2}} \left\langle \prod_{i=1}^n \begin{bmatrix} a_i \\ d_i \end{bmatrix} \right\rangle_0^{DR} \prod_{i=1}^n t_{a_i, d_i} \cdot \frac{(\sum_{i=1}^n a_i) \partial}{\partial t_{(\sum_{i=1}^n a_i), 0}}. \end{aligned}$$

Then the recursion relation in Proposition 2.4 can be written as

$$\frac{a \partial^2 V_g}{\partial t_{a,d+1} \partial \beta} = \left[\frac{\partial V_g}{\partial t_{a,d}}, \frac{\partial V_0}{\partial \beta} \right] + \left[\frac{\partial V_0}{\partial t_{a,d}}, \frac{\partial V_g}{\partial \beta} \right]. \tag{2}$$

2.3. Initial values

Using Proposition 2.4 one can eliminate all ψ -classes. This reduces the problem of computation of an integral over a DR-cycle to the following set of initial values.

Proposition 2.5. *There is a constant C_g that depends only on genus g , such that for any $a \geq 1$*

$$\left\langle \begin{bmatrix} a \\ 0 \end{bmatrix} \right\rangle_g^{DR} = C_g \cdot (a^{2g} - 1).$$

Sketch of a proof. The proof of this proposition is based on the fact that $DR_g(m_a)$ is proportional to a generator of $R^g(\mathcal{M}_{g,2}^{rt})$ with the coefficient $a^{2g} - 1$. That can be proved by a universal Jacobian argument, see the proof in [5, Proof of Theorem 3.5]. \square

3. Basic properties of integrals over DR-cycles

Here we discuss how the integrals over DR-cycles $DR_g(\prod_{i=1}^n m_{a_i})$ depend on the multiplicities a_1, \dots, a_n .

3.1. A small simplification

Propositions 2.4 and 2.5 imply that the integral $\langle \prod_{i=1}^n \begin{bmatrix} a_i \\ d_i \end{bmatrix}_g \rangle_g^{DR}$ is a sum of two rational functions in a_1, \dots, a_n of degree $2g$ and 0 whose denominators divide $\prod_{i=1}^n a_i^{d_i}$. We know from Proposition 2.3 that in the computation of a particular integral over $\overline{\mathcal{M}}_{g,1+n}$ all degree 0 terms should cancel each other, so we can ignore them in the course of computation. An explicit statement about their values is the following:

Lemma 3.1. *Let $n \geq 1$. For any non-negative d_1, \dots, d_n , $d_1 + \dots + d_n = n - 1$, we consider the degree 0 part of the expression of the integral $\langle \prod_{i=1}^n \begin{bmatrix} a_i \\ d_i \end{bmatrix}_g \rangle_g^{DR}$ as a rational function in a_1, \dots, a_n . It is independent of a_1, \dots, a_n and is equal to $-C_g \cdot (n - 1)! / d_1! \dots d_n!$.*

Proof. It is proved by induction on n via a straightforward application of the recursion relation in Proposition 2.4. \square

One more observation is that all integrals that we consider are proportional to C_g , some basic constant that is related to the choice of a particular isomorphism $\int \cdot \lambda_g \lambda_{g-1} : R^{g-2} \rightarrow \mathbb{Q}$. For convenience we may assume that $C_g = 1$. Therefore, we can assume for simplicity that the initial values for our computational algorithm are given simply by $\langle \begin{bmatrix} a \\ 0 \end{bmatrix}_g \rangle_g^{DR} = a^{2g}$. We keep to this simplified assumption till the end of the paper.

3.2. Polynomiality

Taking into account the simplification in Section 3.1 we see that the integral $\langle \prod_{i=1}^n \begin{bmatrix} a_i \\ d_i \end{bmatrix}_g \rangle_g^{DR}$ is a rational function in a_1, \dots, a_n of degree $2g$ whose denominator divides $\prod_{i=1}^n a_i^{d_i}$. In fact, one can say more than that.

Proposition 3.2. *Let n be positive integer. For any non-negative integers d_1, \dots, d_n , $d_1 + \dots + d_n = n - 1$, the integral $\langle \prod_{i=1}^n \begin{bmatrix} a_i \\ d_i \end{bmatrix}_g \rangle_g^{DR}$ is a polynomial in a_1, \dots, a_n .*

Proof. The proposition in general follows from the particular case when $d_1 = n - 1$ and $d_2 = \dots = d_n = 0$. Indeed, applying the recursion relation in Proposition 2.4 to ψ -classes at all points but the first one we come to this particular case, and there is no occurrence of a_1 in the denominator so far. Hence, the whole integral is a polynomial in a_1 , and, therefore, in all a_i , $i = 1, \dots, n$. So, this special case is enough. It is proved below, in Lemma 3.5 based on Lemmas 3.3 and 3.4. \square

So, we consider the integral

$$I_n(a, a_1, \dots, a_n) := \left\langle \begin{bmatrix} a \\ n \end{bmatrix} \prod_{i=1}^n \begin{bmatrix} a_i \\ 0 \end{bmatrix} \right\rangle_g^{DR}, \quad n \geq 0.$$

It is a homogeneous function of degree g that can be expanded as $I_n = \sum_{i=-n}^{2g} a^i \cdot P_{n,2g-i}(a_1, \dots, a_n)$, where $P_{n,k}$ are some symmetric polynomials of degree k in n variables. Explicit computations with the recursion relation in Proposition 2.4 give the first few formulas for I_n :

$$I_0 = a^{2g}; \tag{3}$$

$$I_1 = a^{2g} + \sum_{i=0}^{2g-1} a^i \cdot \frac{2g}{2g+1} \binom{2g+1}{2g-i} a_1^{2g-i}. \tag{4}$$

Lemma 3.3. *For any $n \geq 1$, we have:*

$$I_n(a, a_1, \dots, a_{n-1}, 0) = I_{n-1}(a, a_1, \dots, a_{n-1}).$$

Proof. This lemma is an exercise on the recursion relation in Proposition 2.4. We prove it by induction. For $n = 1$, it follows from the formula for I_1 above. For an arbitrary n ,

$$\begin{aligned} I_n(a, a_1, \dots, a_n) &= \frac{1}{2g+n} \sum_{i=1}^n \frac{a + \sum_{j \neq i} a_j}{a} I_{n-1}(a, a_1, \dots, \hat{a}_i, \dots, a_n) \\ &\quad - \frac{1}{2g+n} \sum_{i < j} \frac{a_i + a_j}{a} I_{n-1}(a, a_1, \dots, \hat{a}_i, \dots, \hat{a}_j, \dots, a_n, a_i + a_j) \\ &\quad + \frac{2g}{2g+n} \frac{(a + \sum_{i=1}^n a_i)^{2g+1}}{a} - \frac{2g}{2g+n} \sum_{i=1}^n \frac{a_i^{2g+1}}{a}. \end{aligned} \tag{5}$$

We apply this recursion to $I_n(a, a_1, \dots, a_{n-1}, 0) - I_{n-1}(a, a_1, \dots, a_{n-1})$. The resulting formula turns to be equal to zero due to the induction assumption. \square

This lemma means that $I_n(a, a_1, \dots, a_n)$ splits into the terms that can be expressed in $I_{<n}$ and the terms that are divisible by $a_1 \cdots a_n$. For convenience, we introduce a new notation. We say that two polynomials in a_1, \dots, a_n , f and g , are equivalent (notation: $f \equiv g$) if $f - g$ doesn't contain monomials divisible by $a_1 \cdots a_n$.

Lemma 3.4. *For any $n \geq 1$, $-n \leq i \leq 2g - n$, we have:*

$$P_{n,2g-i}(a_1, \dots, a_n) \equiv \frac{2g}{n+i} \binom{2g}{i} (a_1 + \dots + a_n)^{2g-i}.$$

In particular, for $i < 0$, $P_{n,2g-i} \equiv 0$.

Proof. We prove it by induction on n . For $n = 1$ it follows from the explicit formula. For $n \geq 2$, Eq. (5) implies that

$$I_n(a, a_1, \dots, a_n) \equiv \frac{2g}{2g+n} \frac{(a + \sum_{i=1}^n a_i)^{2g+1}}{a} - \frac{1}{2g+n} \sum_{i < j} \frac{a_i + a_j}{a} \cdot I_{n-1}(a, a_1, \dots, \hat{a}_i, \dots, \hat{a}_j, \dots, a_n, a_i + a_j).$$

Using the induction assumption, we can continue this equivalence as

$$I_n(a, a_1, \dots, a_n) \equiv \sum_{i=-1}^{2g-n} a^i \left(\sum_{i=1}^n a_i \right)^{2g-i} \cdot \left(\frac{2g}{2g+n} \cdot \binom{2g+1}{i+1} - \frac{n-1}{2g+n} \cdot \frac{2g}{n-1+i+1} \cdot \binom{2g}{i+1} \right).$$

It is obvious that the coefficient of $a^{-1}(\sum_{i=1}^n a_i)^{2g+1}$ is equal to 0, and all other coefficients are exactly the same as in the statement of the lemma. \square

Finally, we are able to conclude with polynomiality.

Lemma 3.5. For any $n \geq 0$, $I_n(a, a_1, \dots, a_n)$ is a polynomial in a, a_1, \dots, a_n .

Proof. We know a priori that I_n is a polynomial in a_1, \dots, a_n whose coefficients are polynomials in a and a^{-1} . From Lemma 3.4 we know that I_n is equivalent to a polynomial \tilde{I}_n in a_1, \dots, a_n whose coefficients are polynomials in a . Meanwhile, from Lemma 3.3 we know that \tilde{I}_n can be chosen in such a way that $I_n - \tilde{I}_n$ is a linear combination of $I_{<n}$, so one can complete the proof by an induction argument. \square

3.3. Divisibility

One more fact about the integrals over DR-cycles that we use below in combinatorial computations is the following:

Proposition 3.6. For any non-negative integers d_1, \dots, d_n , $d_1 + \dots + d_n = n$, the polynomial in b, a_1, \dots, a_n given by the formula

$$\left\langle \left[\begin{matrix} b \\ 0 \end{matrix} \right] \prod_{i=1}^n \left[\begin{matrix} a_i \\ d_i \end{matrix} \right] \right\rangle_g^{DR} - \sum_{\substack{j=1 \\ d_j \neq 0}}^n \left\langle \left[\begin{matrix} a_j + b \\ d_j - 1 \end{matrix} \right] \prod_{\substack{i=1 \\ i \neq j}}^n \left[\begin{matrix} a_i \\ d_i \end{matrix} \right] \right\rangle_g^{DR} \tag{6}$$

is divisible by b^2 .

Remark 3.7. Observe that using Lemma 2.2 and the pull-back formula for ψ -classes one can rewrite this expression as

$$\int_{DR_g(\prod_{i=1}^n m_{a_i} \cdot \bar{m}_b)} \lambda_g \lambda_{g-1} \psi_0 \prod_{i=1}^n \psi_i^{d_i}.$$

Proof. Lemma 3.8 below allows us to consider a special case when $d_1 = n$ and $d_2 = \dots = d_n = 0$. In this case we have to prove that

$$I_{n+1}(a, b, a_1, \dots, a_n) - I_n(a + b, a_1, \dots, a_n)$$

is divisible by b^2 (we shift n to $n + 1$ for convenience and we use notations from the previous section). We can do it by induction on n . Explicit formulas (3) and (4) applied for $I_1(a, b) - I_0(a + b)$ prove it for $n = 0$. Lemma 3.3 allows to consider only the terms that are divisible by $b \cdot a_1 \cdots a_n$. Using Lemma 3.4 we see that it is enough to prove that the linear term in b in the expression

$$\begin{aligned} & \sum_{i=0}^{2g-n-1} \frac{2g}{n+1+i} \binom{2g}{i} \cdot a^i \cdot \left(b + \sum_{j=1}^n a_j \right)^{2g-i} \\ & - \sum_{i=1}^{2g-n} \frac{2g}{n+i} \binom{2g}{i} \cdot (a+b)^i \cdot \left(\sum_{j=1}^n a_j \right)^{2g-i} \end{aligned}$$

is equal to 0. The last statement follows from a direct computation. \square

Lemma 3.8. For any $n \geq 0, b, a', a'', a_1, \dots, a_n \geq 0, d > 0, d_1, \dots, d_n \geq 0, d + d_1 + \dots + d_n = n + 1$, we have:

$$\begin{aligned} & -a' \cdot \left(\left\langle \begin{bmatrix} b \\ 0 \end{bmatrix} \begin{bmatrix} a' \\ d+1 \end{bmatrix} \begin{bmatrix} a'' \\ 0 \end{bmatrix} \prod_{i=1}^n \begin{bmatrix} a_i \\ d_i \end{bmatrix} \right\rangle_g^{DR} - \left\langle \begin{bmatrix} a'+b \\ d \end{bmatrix} \begin{bmatrix} a'' \\ 0 \end{bmatrix} \prod_{i=1}^n \begin{bmatrix} a_i \\ d_i \end{bmatrix} \right\rangle_g^{DR} \right. \\ & \left. - \sum_{\substack{j=1 \\ d_j \neq 0}}^n \left\langle \begin{bmatrix} a' \\ d+1 \end{bmatrix} \begin{bmatrix} a'' \\ 0 \end{bmatrix} \begin{bmatrix} a_j+b \\ d_j-1 \end{bmatrix} \prod_{\substack{i=1 \\ i \neq j}}^n \begin{bmatrix} a_i \\ d_i \end{bmatrix} \right\rangle_g^{DR} \right) \\ & + a'' \cdot \left(\left\langle \begin{bmatrix} b \\ 0 \end{bmatrix} \begin{bmatrix} a' \\ d \end{bmatrix} \begin{bmatrix} a'' \\ 1 \end{bmatrix} \prod_{i=1}^n \begin{bmatrix} a_i \\ d_i \end{bmatrix} \right\rangle_g^{DR} - \left\langle \begin{bmatrix} a'+b \\ d-1 \end{bmatrix} \begin{bmatrix} a'' \\ 1 \end{bmatrix} \prod_{i=1}^n \begin{bmatrix} a_i \\ d_i \end{bmatrix} \right\rangle_g^{DR} \right. \\ & \left. - \left\langle \begin{bmatrix} a' \\ d \end{bmatrix} \begin{bmatrix} a''+b \\ 0 \end{bmatrix} \prod_{i=1}^n \begin{bmatrix} a_i \\ d_i \end{bmatrix} \right\rangle_g^{DR} - \sum_{\substack{j=1 \\ d_j \neq 0}}^n \left\langle \begin{bmatrix} a' \\ d+1 \end{bmatrix} \begin{bmatrix} a'' \\ 0 \end{bmatrix} \begin{bmatrix} a_j+b \\ d_j-1 \end{bmatrix} \prod_{\substack{i=1 \\ i \neq j}}^n \begin{bmatrix} a_i \\ d_i \end{bmatrix} \right\rangle_g^{DR} \right) \end{aligned}$$

is divisible by b^2 .

Remark 3.9. The meaning of this lemma is that in the proof of Proposition 3.6 for any n it is enough to consider only one particular choice of $d_1, \dots, d_n, d_1 + \dots + d_n = n$.

Remark 3.10. Though this lemma looks a bit cumbersome and not so natural, in fact it has a clear geometric origin. Indeed, a particular consequence of Ionel’s lemma in [6] is that the difference

of two ψ -classes weighted by multiplicities at the corresponding points on one side of a DR-cycle should be a nice expression that doesn't involve any multiplicities coming from the count of simple critical values of the corresponding meromorphic functions.

Proof of Lemma 3.8. We prove this lemma by induction on n . The assumption of induction is that Proposition 3.6 is true for any number of points that is less than $n + 2$. We apply the recursion relation in Proposition 2.4 for the ψ -class at the points of multiplicity a' in the first summand and a'' in the second summand and collect all terms into the similar sums.

It is convenient to rewrite everything in terms of generating functions defined in Section 2.2. Let

$$U_b := \frac{\partial}{\partial t_{b,0}} - \beta \sum_{a,d \geq 1} t_{a,d} \frac{\partial}{\partial t_{a+b,d-1}}.$$

Then Proposition 3.6 can be reformulated as $Lie_{U_b} V_g = O(b^2)$. The statement of this lemma can be reformulated as

$$\left(-\frac{a' \partial^2}{\partial t_{a',d+1} \partial t_{a'',0}} + \frac{a'' \partial^2}{\partial t_{a',d} \partial t_{a'',1}} \right) Lie_{U_b} V_g = O(b^2).$$

A useful observation is that $Lie_{U_b} V_0 = \beta \sum_{a>0} t_{a,0} \frac{(a+b)\partial}{\partial t_{a+b,0}}$. The recursion relation (2) implies that

$$-\frac{a' \partial^2}{\partial t_{a',d+1} \partial t_{a'',0}} V_g + \frac{a'' \partial^2}{\partial t_{a',d} \partial t_{a'',1}} V_g = -\left[\frac{\partial V_g}{\partial t_{a',d}}, \frac{\partial V_0}{\partial t_{a'',0}} \right] + \left[\frac{\partial V_g}{\partial t_{a'',0}}, \frac{\partial V_0}{\partial t_{a',d}} \right].$$

Observe also that $[U_b, \frac{\partial}{\partial t_{a,d}}] = \beta \frac{\partial}{\partial t_{a+b,d-1}}$ and $Lie_{V_g} U_b = Lie_{V_0} U_b = 0$.

We use these observations in order to obtain the following formulas:

$$\begin{aligned} & \left(-\frac{a' \partial^2}{\partial t_{a',d+1} \partial t_{a'',0}} + \frac{a'' \partial^2}{\partial t_{a',d} \partial t_{a'',1}} \right) Lie_{U_b} V_g \\ &= Lie_{U_b} \left(-\frac{a' \partial^2}{\partial t_{a',d+1} \partial t_{a'',0}} + \frac{a'' \partial^2}{\partial t_{a',d} \partial t_{a'',1}} \right) V_g \\ & \quad + \beta \left(\frac{a' \partial^2}{\partial t_{a'+b,d} \partial t_{a'',0}} - \frac{a'' \partial^2}{\partial t_{a',d} \partial t_{a''+b,0}} - \frac{a'' \partial^2}{\partial t_{a'+b,d-1} \partial t_{a'',1}} \right) V_g; \\ & Lie_{U_b} \left(-\left[\frac{\partial V_g}{\partial t_{a',d}}, \frac{\partial V_0}{\partial t_{a'',0}} \right] + \left[\frac{\partial V_g}{\partial t_{a'',0}}, \frac{\partial V_0}{\partial t_{a',d}} \right] \right) \\ &= -\left[\frac{\partial}{\partial t_{a',d}} Lie_{U_b} V_g, \frac{\partial}{\partial t_{a'',0}} V_0 \right] + \left[\frac{\partial}{\partial t_{a'',0}} Lie_{U_b} V_g, \frac{\partial}{\partial t_{a',d}} V_0 \right] \\ & \quad + \beta \left(\frac{(a'' + b) \partial^2}{\partial t_{a',d} \partial t_{a''+b,0}} - \frac{(a' + b) \partial^2}{\partial t_{a'+b,d} \partial t_{a'',0}} + \frac{a'' \partial^2}{\partial t_{a'+b,d-1} \partial t_{a'',1}} \right) V_g. \end{aligned}$$

Therefore,

$$\begin{aligned} & \left(-\frac{a'\partial^2}{\partial t_{a',d+1} \partial t_{a'',0}} + \frac{a''\partial^2}{\partial t_{a',d} \partial t_{a'',1}} \right) Lie_{U_b} V_g \\ &= -\left[\frac{\partial}{\partial t_{a',d}} Lie_{U_b} V_g, \frac{\partial}{\partial t_{a'',0}} V_0 \right] + \left[\frac{\partial}{\partial t_{a'',0}} Lie_{U_b} V_g, \frac{\partial}{\partial t_{a',d}} V_0 \right] \\ &+ \beta \cdot b \cdot \left(\frac{\partial^2}{\partial t_{a',d} \partial t_{a''+b,0}} - \frac{\partial^2}{\partial t_{a'+b,d} \partial t_{a'',0}} \right) V_g. \end{aligned}$$

Here the first two summands in the right-hand side are divisible by b^2 by induction assumption. Indeed, we are interested in terms of homogeneous degree n . In both summands these terms are obtained as some product with the components of $Lie_{U_b} V_g$ of degree $\leq n + 1$. The last summand is divisible by b^2 for the obvious reason. \square

4. Faber’s conjecture

In this section we apply the properties of the integrals over DR-cycles obtained in the previous sections in order to prove Faber’s intersection number conjecture.

Theorem 4.1. *For any positive integers $d_1, \dots, d_n, d_1 + \dots + d_n = g + n - 2$, we have:*

$$\int_{\overline{\mathcal{M}}_{g,n}} \prod_{i=1}^n \psi_i^{d_i} \lambda_g \lambda_{g-1} = \frac{(2g - 3 + n)!(2g - 3)!!}{(2g - 2)! \prod_{i=1}^n (2d_i - 1)!!} \int_{\overline{\mathcal{M}}_{g,1}} \psi_1^{g-1} \lambda_g \lambda_{g-1}.$$

We prove this theorem in four steps. First, we reformulate Faber’s conjecture in a way that is better compatible with DR-cycles (that is, we need a special point with no ψ -classes). Second step is an explicit expression of the integral in Faber’s conjecture in terms of coefficients of the polynomials $\langle \prod_{i=1}^n \left[\begin{smallmatrix} d_i \\ d_i \end{smallmatrix} \right]_g^{DR} \rangle$. Third step is an explicit formula for these coefficients. Finally, we combine these results into a proof of Faber’s conjecture.

4.1. A reformulation of Faber’s conjecture

There is a string equation for the integrals of ψ -classes with $\lambda_g \lambda_{g-1}$ over the moduli space of curves (see, e.g., [5]). In particular for any positive integers $d_1, \dots, d_n, d_1 + \dots + d_n = g + n - 1$, we have:

$$\int_{\overline{\mathcal{M}}_{g,1+n}} \lambda_g \lambda_{g-1} \psi_0 \prod_{i=1}^n \psi_i^{d_i} = \frac{(2g - 2 + n)!(2g - 1)!!}{(2g - 1)! \prod_{i=1}^n (2d_i - 1)!!} \int_{\overline{\mathcal{M}}_{g,2}} \lambda_g \lambda_{g-1} \psi_0^0 \psi_1^g. \tag{7}$$

In fact, this equation is equivalent to Faber’s conjecture. One can prove that via the same argument as Witten used in [14] for the inversion of string equation.

4.2. A reformulation of Proposition 2.4

We introduce a new notation for the coefficients of the polynomial $\langle \prod_{i=1}^m [a_i]_{d_i} \rangle_g^{DR}$. Let

$$\left\langle \prod_{i=1}^n \left[\begin{matrix} a_i \\ d_i \end{matrix} \right]_g \right\rangle_g^{DR} := \sum_{\substack{p_1, \dots, p_n \geq 0 \\ p_1 + \dots + p_n = 2g}} \left\langle \prod_{i=1}^n \left[\begin{matrix} p_i \\ d_i \end{matrix} \right]_g \right\rangle_g^{coeff} \frac{(2g)!}{p_1! \dots p_n!} \prod_{i=1}^n a_i^{p_i}.$$

In these terms, we can rewrite Eq. (1) as

$$\begin{aligned} \int_{\overline{\mathcal{M}}_{g,n+1}} \lambda_g \lambda_{g-1} \psi_0^n \prod_{i=1}^n \psi_i^{d_i} \\ = \frac{(2g)!}{g! 2^g} \sum_{\substack{i_0, \dots, i_n \geq 0 \\ i_0 + i_1 + \dots + i_n = g \\ i_j \leq d_j, j=1, \dots, n}} \frac{(-1)^{g-i_0} g!}{i_0! \dots i_n!} \left\langle \prod_{j=1}^n \left[\begin{matrix} 2i_j \\ d_j - i_j \end{matrix} \right]_g \middle| \prod_{j=1}^{i_0} \left[\begin{matrix} 2 \\ 0 \end{matrix} \right]_g \right\rangle_g^{coeff}. \end{aligned} \tag{8}$$

Note that in this formula we use only coefficients $\langle \prod_{i=1}^m [a_i]_{c_i} \rangle_g^{coeff}$ with $p_i + c_i \geq 1, i = 1, \dots, m$.

4.3. Computation of the coefficients

We express the coefficients $\langle \prod_{i=1}^m [a_i]_{c_i} \rangle_g^{coeff}$ in terms of the counting of some paths in the integral lattice.

Consider the lattice \mathbb{Z}^m . Let $\{e_1, \dots, e_m\}$ be the standard basis of \mathbb{Z}^m . A path in the space \mathbb{Z}^m is a sequence of points $p_j \in \mathbb{Z}^m, j = 1, \dots, N$ such that $p_j - p_{j+1} = e_k$ for some k . We associate to each subset $I \subset \{1, \dots, m\}$ a special point in the lattice that we denote by $\bar{\mathbf{1}}_I := \sum_{i \in I} e_i$.

Consider a point $\bar{c} = (c_1, \dots, c_m) \in \mathbb{Z}^m, c_i \geq 0, i = 1, \dots, m$. Let $w_I(\bar{c})$ be the number of paths (p_1, \dots, p_N) such that $p_1 = \bar{c}, p_N = \bar{\mathbf{1}}_I$, and the points $p_i, i = 1, \dots, N$ are disjoint from $\bar{\mathbf{1}}_J$ for all $J \neq I$.

Proposition 4.2. Let p_1, \dots, p_m and $c_1, \dots, c_m, m \geq 1$, be non-negative integers such that $p_i + c_i \geq 1, i = 1, \dots, m$. Then we have:

$$\left\langle \prod_{i=1}^m \left[\begin{matrix} p_i \\ c_i \end{matrix} \right]_g \right\rangle_g^{coeff} = \sum_{\substack{I \subset \{1, \dots, m\} \\ I \neq \emptyset}} \frac{\prod_{i=1}^{|I|} (2g + i - 1)}{\prod_{i \in I} (p_i + c_i)} w_I(\bar{c}).$$

This proposition is based on the following three lemmas that we prove in Section 4.5.

Lemma 4.3. Let $p_m \geq 1$. Then

$$\left\langle \prod_{i=1}^{m-1} \left[\begin{matrix} p_i \\ 1 \end{matrix} \right]_g \middle| \left[\begin{matrix} p_m \\ 0 \end{matrix} \right]_g \right\rangle_g^{coeff} = \prod_{i=1}^{m-1} \frac{2g + i - 1}{p_i + 1}.$$

Lemma 4.4. Let $p_i + c_i \geq 1, i = 1, \dots, m - 2$, and $p_{m-1}, p_m \geq 1$. Then

$$\left\langle \prod_{i=1}^{m-2} \binom{p_i}{c_i} \cdot \binom{p_{m-1} + 1}{0} \binom{p_m}{0} \right\rangle_g^{coeff} = \left\langle \prod_{i=1}^{m-2} \binom{p_i}{c_i} \cdot \binom{p_{m-1}}{0} \binom{p_m + 1}{0} \right\rangle_g^{coeff}.$$

Lemma 4.5. Let $p_i + c_i \geq 1, i = 1, \dots, m - 1$. Then

$$\left\langle \prod_{i=1}^{m-1} \binom{p_i}{c_i} \cdot \binom{1}{0} \right\rangle_g^{coeff} = \sum_{i=1}^{m-1} \left\langle \prod_{\substack{j=1 \\ j \neq i}}^{m-1} \binom{p_j}{c_j} \cdot \binom{p_i + 1}{c_i - 1} \right\rangle_g^{coeff}.$$

Proof of Proposition 4.2. Since $\langle \prod_{i=1}^m \binom{p_i}{c_i} \rangle_g^{coeff} \neq 0$ only for $\sum_{i=1}^m c_i = m - 1$, we have at least one of the indices c_i equal to zero. Assume that there is exactly one index equal to zero, say, $c_i = 0$. Then all other indices $c_j, j \neq i$, are equal to 1. In this case, the proposition follows from Lemma 4.3. Indeed, in this case $w_I(\bar{c})$ is equal to 0 for all $I \subset \{1, \dots, m\}$ except for $I = \{1, \dots, m\} \setminus \{i\}$, where $w_I(\bar{c}) = 1$.

If we have at least two zeros among the indices $c_i, i = 1, \dots, m$, we can apply the following corollary of Lemmas 4.4 and 4.5. If $p_i + c_i \geq 1$ for $i = 1, \dots, m - 2$, and $p_{m-1}, p_m \geq 1$, then

$$\left\langle \prod_{i=1}^{m-2} \binom{p_i}{c_i} \cdot \binom{p_{m-1}}{0} \binom{p_m}{0} \right\rangle_g^{coeff} = \sum_{i=1}^{m-2} \left\langle \prod_{\substack{j=1 \\ j \neq i}}^{m-2} \binom{p_j}{c_j} \cdot \binom{p_i + 1}{c_i - 1} \binom{p_{m-1} + p_m - 1}{0} \right\rangle_g^{coeff}. \tag{9}$$

This relation is compatible with the definition of the number of paths. Applying this relation sufficiently many times we come to the situation when all indices but one are equal to 1. This corresponds to a point $\bar{1}_I$ for some I in the lattice \mathbb{Z}^m , and Lemma 4.3 implies that the coefficient at this endpoint is exactly $\frac{\prod_{i=1}^{|I|} (2g+i-1)}{\prod_{i \in I} (p_i + c_i)}$. \square

4.4. A proof of Faber’s conjecture

In this section, we prove Faber’s intersection number conjecture.

Proof of Theorem 4.1. We are going to compute explicitly both sides of Eq. (7) using Proposition 4.2.

We denote by \vec{i} the vector $(i_1, \dots, i_n) \in \mathbb{Z}^n$. Proposition 4.2 and Eq. (8) imply that

$$\begin{aligned} & \int_{\bar{\mathcal{M}}_{g,n+1}} \lambda_g \lambda_{g-1} \psi_0 \prod_{i=1}^n \psi_i^{d_i} \\ &= \frac{(2g)!}{g! 2^g} \sum_{\substack{i_0, \dots, i_n \geq 0 \\ i_0 + i_1 + \dots + i_n = g \\ i_j \leq d_j, j=1, \dots, n \\ I \subset \{1, \dots, n\}, I \neq \emptyset}} \frac{(-1)^{g-i_0} g! \prod_{j=1}^{|I|} (2g + j - 1)}{i_0! \dots i_n! \prod_{j \in I} (d_j + i_j)} w_I(\vec{d} - \vec{i}). \end{aligned} \tag{10}$$

This allows us to compute the integral in the right-hand side of Eq. (7). Indeed,

$$\int_{\overline{\mathcal{M}}_{g,2}} \lambda_g \lambda_{g-1} \psi_1^g = \frac{(2g)!}{g!2^g} \sum_{i=0}^g (-1)^i \binom{g}{i} \frac{2g}{g+i} = \frac{g!}{2^{g-1}}. \tag{11}$$

Eqs. (10) and (11) imply that Eq. (7) is equivalent to

$$\begin{aligned} & \sum_{\substack{i_0, \dots, i_n \geq 0 \\ i_0 + i_1 + \dots + i_n = g \\ i_j \leq d_j, j=1, \dots, n \\ I \subset \{1, \dots, n\}, I \neq \emptyset}} \frac{(-1)^{g-i_0}}{i_0! \dots i_n!} \frac{\prod_{j=1}^{|I|} (2g+j-1)}{\prod_{j \in I} (d_j + i_j)} w_I(\bar{d} - \bar{i}) \\ &= \prod_{i=1}^{n-1} (2g+i-1) \prod_{i=1}^n \frac{(d_i - 1)!}{(2d_i - 1)!}. \end{aligned} \tag{12}$$

We prove in Lemma 4.6 below that for all subsets $I \subset \{1, \dots, n\}$ such that $|I| \leq n - 2$ the corresponding summands on the left-hand side of this formula vanish. Before that, let us introduce a new definition that would allow us to count the number of paths in the lattice in a convenient way.

Let $\bar{c} \in \mathbb{Z}^n$. We denote by $w_0(\bar{c})$ the number of paths (p_1, \dots, p_N) in \mathbb{Z}^n , such that $p_1 = \bar{c}$ and $p_N = (0, \dots, 0)$. Observe that for any non-empty $I \subset \{1, \dots, n\}$,

$$\begin{aligned} w_I(\bar{c}) &= \begin{cases} 1, & \text{if } \bar{c} = \bar{\mathbf{1}}_I, \\ \sum_{k \in I} w_0(\bar{c} - \bar{\mathbf{1}}_I - \bar{\mathbf{1}}_{\{k\}}), & \text{otherwise;} \end{cases} \\ w_0(\bar{c}) &= \begin{cases} (\sum_{i=1}^n c_i)! / \prod_{j=1}^n c_j!, & \text{if } c_i \geq 0, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

We also introduce two auxiliary functions. Let

$$\begin{aligned} f_{a,b}(x) &:= \sum_{i=0}^a (-1)^i \frac{x^{a-i}}{b+i} \binom{a}{i}, \\ g_{a,b}(x) &:= \int x^a (1-x)^b dx. \end{aligned}$$

We list some properties of these functions:

$$\begin{aligned} f_{a,b} &= (-1)^{a+1} x^{a+b} g_{-a-b-1,a}, & \frac{d}{dx} f_{a,b} &= a f_{a-1,b}, \\ g_{a,b}(1) &= \frac{b!}{(a+1)(a+2) \dots (a+b+1)}, & f_{a,b}(0) &= \frac{(-1)^a}{a+b}. \end{aligned}$$

Lemma 4.6. *Let I be a subset of $\{1, \dots, n\}$ such that $0 < |I| \leq n - 2$. Then the corresponding summand of the left-hand side of Eq. (12) is equal to zero, that is,*

$$\sum_{\substack{i_0, \dots, i_n \geq 0 \\ i_0 + i_1 + \dots + i_n = g \\ i_j \leq d_j, j=1, \dots, n}} \frac{(-1)^{g-i_0}}{i_0! \cdots i_n!} \frac{w_I(\bar{d} - \bar{i})}{\prod_{j \in I} (d_j + i_j)} = 0.$$

Proof. Let $k \in I$. An explicit calculation shows that

$$\begin{aligned} & \sum_{\substack{i_0, \dots, i_n \geq 0 \\ i_0 + i_1 + \dots + i_n = g \\ i_j \leq d_j, j=1, \dots, n}} \frac{(-1)^{g-i_0}}{i_0! \cdots i_n!} \frac{w_0(\bar{d} - \bar{i} - \bar{\mathbf{1}}_I - \bar{\mathbf{1}}_{\{k\}})}{\prod_{j \in I} (d_j + i_j)} \\ &= \left(\frac{d}{dx}\right)^{n-2-|I|} \left(\frac{f_{d_k-2, d_k}}{(d_k - 2)!} \prod_{\substack{j \in I \\ j \neq k}} \frac{f_{d_j-1, d_j}}{(d_j - 1)!} \prod_{j \notin I} \frac{(x - 1)^{d_j}}{d_j!} \right) \Big|_{x=1}. \end{aligned}$$

The derivative at $x = 1$ in the right-hand side of this equation is equal to zero. Indeed, $(x - 1)$ enters the numerator with the multiplicity $\sum_{j \notin I} d_j \geq \sum_{j \notin I} 1 = n - |I| > n - 2 - |I|$.

In order to complete the proof, we just observe that the sum over all $k \in I$ of the left-hand side of this formula is exactly the expression in the statement of the lemma. \square

This lemma implies that the left-hand side of Eq. (12) is equal to $S_0 + \sum_{l=1}^n S_l$, where

$$\begin{aligned} S_0 &= \sum_{\substack{i_0, \dots, i_n \geq 0 \\ i_0 + i_1 + \dots + i_n = g \\ i_j \leq d_j, j=1, \dots, n}} \frac{(-1)^{g-i_0}}{i_0! \cdots i_n!} \frac{\prod_{j=1}^n (2g + j - 1)}{\prod_{j=1}^n (d_j + i_j)} w_{\{1, \dots, n\}}(\bar{d} - \bar{i}), \\ S_l &= \sum_{\substack{i_0, \dots, i_n \geq 0 \\ i_0 + i_1 + \dots + i_n = g \\ i_j \leq d_j, j=1, \dots, n}} \frac{(-1)^{g-i_0}}{i_0! \cdots i_n!} \frac{\prod_{j=1}^{n-1} (2g + j - 1)}{\prod_{j \neq l} (d_j + i_j)} w_{\{1, \dots, n\} \setminus \{l\}}(\bar{d} - \bar{i}). \end{aligned}$$

Recall that $\sum_{i=1}^n d_i = g + n - 1$. Using the expression of w_I in terms of w_0 in this particular case, we see that S_0 can be represented in the following way:

$$\begin{aligned} S_0 &= \prod_{j=1}^n \frac{2g + j - 1}{(d_j - 1)!} \int_0^1 \prod_{j=1}^n f_{d_j-1, d_j} dx \\ &= \frac{\prod_{j=1}^{n-1} (2g + j - 1)}{\prod_{j=1}^n (d_j - 1)!} (-1)^{g+n-1} \int_0^1 \prod_{j=1}^n g_{-2d_j, d_j-1} dx x^{2g+n-1} \end{aligned}$$

$$\begin{aligned}
 &= \prod_{i=1}^{n-1} (2g + i - 1) \prod_{i=1}^n \frac{(d_i - 1)!}{(2d_i - 1)!} \\
 &\quad - \frac{\prod_{j=1}^{n-1} (2g + j - 1)}{\prod_{j=1}^n (d_j - 1)!} \sum_{l=1}^n (-1)^{d_l} \int_0^1 (1-x)^{d_l-1} \prod_{j \neq l} f_{d_j-1, d_j} dx.
 \end{aligned}$$

In order to complete the proof of Eq. (12), and, therefore, the proof of Theorem 4.1, it is sufficient to show that

$$S_l = \frac{\prod_{j=1}^{n-1} (2g + j - 1)}{\prod_{j=1}^n (d_j - 1)!} (-1)^{d_l} \int_0^1 (1-x)^{d_l-1} \prod_{j \neq l} f_{d_j-1, d_j} dx$$

for all $l = 1, \dots, n$. The right-hand side of this formula is equal to

$$\begin{aligned}
 & - \frac{\prod_{j=1}^{n-1} (2g + j - 1)}{d_l! \prod_{j \neq l} (d_j - 1)!} \int_0^1 \prod_{j \neq l} f_{d_j-1, d_j} d(x-1)^{d_l} \\
 &= \sum_{k \neq l} \frac{\prod_{j=1}^{n-1} (2g + j - 1)}{d_l! (d_k - 2)! \prod_{j \neq l, k} (d_j - 1)!} \int_0^1 (x-1)^{d_l} f_{d_k-2, d_k} \prod_{j \neq l, k} f_{d_j-1, d_j} dx \\
 &\quad + \frac{(-1)^g \prod_{j=1}^{n-1} (2g + j - 1)}{d_l! \prod_{j \neq l} (d_l - 1)! \prod_{j \neq l} (2d_j - 1)}.
 \end{aligned}$$

Meanwhile, using the expression of w_l in terms of w_0 , we see that

$$\begin{aligned}
 S_l = & \sum_{\substack{i_0, \dots, i_n \geq 0 \\ i_0 + i_1 + \dots + i_n = g \\ i_j \leq d_j, j=1, \dots, n}} \left(\frac{(-1)^{g-i_0} \prod_{j=1}^{n-1} (2g + j - 1)}{\prod_{j=0}^n i_j! \prod_{j \neq l} (d_j + i_j)} \cdot \sum_{k \neq l} w_0(\bar{d} - \bar{i} - \bar{\mathbf{1}}_{\{1, \dots, n\} \setminus \{l\}} - \bar{\mathbf{1}}_{\{k\}}) \right) \\
 & + \frac{(-1)^g \prod_{j=1}^{n-1} (2g + j - 1)}{d_l! \prod_{j \neq l} (d_l - 1)! \prod_{j \neq l} (2d_j - 1)},
 \end{aligned}$$

and an explicit calculation shows that

$$\begin{aligned}
 & \sum_{\substack{i_0, \dots, i_n \geq 0 \\ i_0 + i_1 + \dots + i_n = g \\ i_j \leq d_j, j=1, \dots, n}} \frac{(-1)^{g-i_0} w_0(\bar{d} - \bar{i} - \bar{\mathbf{1}}_{\{1, \dots, n\} \setminus \{l\}} - \bar{\mathbf{1}}_{\{k\}})}{\prod_{j=0}^n i_j! \prod_{j \neq l} (d_j + i_j)} \\
 &= \frac{\int_0^1 (x-1)^{d_l} f_{d_k-2, d_k} \prod_{j \neq l, k} f_{d_j-1, d_j} dx}{d_l! (d_k - 2)! \prod_{j \neq l, k} (d_j - 1)!}. \quad \square
 \end{aligned}$$

4.5. Proofs of Lemmas 4.3–4.5

Proof of Lemma 4.3. Let us reformulate the lemma. We want to prove that the coefficient of the monomial $a_1^{p_1} \cdots a_m^{p_m}$ in $\langle \prod_{i=1}^{m-1} \begin{bmatrix} a_i \\ 1 \end{bmatrix} \begin{bmatrix} a_m \\ 0 \end{bmatrix} \rangle_g^{DR}$ is equal to the coefficient of the same monomial in

$$\frac{2g}{2g + m - 1} \frac{(a_1 + \cdots + a_m)^{2g+m-1}}{a_1 \cdots a_{m-1}}. \tag{13}$$

We prove it by induction on m . The base of induction, $m = 1$, is obvious. Let $m \geq 2$. The induction assumption that the recursion relation in Proposition 2.4 implies that the coefficient of the monomial $a_1^{p_1} \cdots a_m^{p_m}$ in $\langle \prod_{i=1}^{m-1} \begin{bmatrix} a_i \\ 1 \end{bmatrix} \begin{bmatrix} a_m \\ 0 \end{bmatrix} \rangle_g^{DR}$ is equal to the coefficient of the same monomial in

$$\begin{aligned} & \sum_{I \subset \{2, \dots, m-1\}} \left(\frac{2g + m - |I| - 2}{2g + m - 1} \cdot \frac{a_1 + a_m + \sum_{i \in I} a_i}{a_1} \right. \\ & \cdot \frac{2g}{2g + m - |I| - 2} \cdot \frac{(\sum_{i=1}^m a_i)^{2g+m-|I|-2}}{\prod_{i \in \{2, \dots, m-1\} \setminus I} a_i} \cdot |I|! \Big) \\ & = \frac{2g}{2g + m - 1} \cdot \frac{1}{\prod_{i=1}^{m-1} a_i} \\ & \cdot \sum_{I \subset \{2, \dots, m-1\}} |I|! \left(a_1 + a_m + \sum_{i \in I} a_i \right) \left(\prod_{i \in I} a_i \right) \left(\sum_{i=1}^m a_i \right)^{2g+m-|I|-2}. \end{aligned}$$

The right-hand side of this equation coincides with (13) by the following combinatorial observation:

$$\sum_{I \subset \{2, \dots, k\}} |I|! \left(x_1 + \sum_{i \in I} x_i \right) \left(\prod_{i \in I} x_i \right) \left(\sum_{i=1}^k x_i \right)^{k-|I|-1} = \left(\sum_{i=1}^k x_i \right)^k$$

(we assume $k \geq 1$). \square

Proof of Lemma 4.4. First, let us introduce some notations. Let P and Q be polynomials in the variables a_1, \dots, a_m . Let $I \subset \{1, \dots, m\}$. We write $P \stackrel{a_I}{\equiv} Q$ iff the polynomial $P - Q$ doesn't have monomials divisible by $\prod_{i \in I} a_i$. Let $J \subset \{1, \dots, m\}$. We will write $P(a_J)$ in order to specify that the polynomial P depends only on variables a_i for $i \in J$.

The lemma is equivalent to the following statement. If $c_1, \dots, c_m, c_1 + \cdots + c_m = m - 1$ are non-negative integers and $E = \{i \in \{1, \dots, m\} \mid c_i = 0\}$, then there exists a polynomial $P(a_{\{1, \dots, m\} \setminus E}, x)$ such that

$$\left\langle \prod_{i=1}^m \begin{bmatrix} a_i \\ c_i \end{bmatrix} \right\rangle_g^{DR} \stackrel{a_E}{\equiv} P \left(a_{\{1, \dots, m\} \setminus E}, \sum_{i \in E} a_i \right). \tag{14}$$

We prove this by induction on m . The base of induction $m = 1$ is obvious. Let $m \geq 2$. If $|E| = 1$ then there is nothing to prove. Suppose that $|E| \geq 2$. Then there exists $i \in I_m$ such that $c_i \geq 2$.

Without loss of generality we can assume that $i = m$. We are going to apply the recursion relation in Proposition 2.4.

Let $m = n + 1$, $c_m = d + 1$, $a_m = a$, $d_j = c_j$, $1 \leq j \leq n$. There are four summands in the right-hand side of the recursion relation. Let us denote them by S_1, S_2, S_3, S_4 in the same order as they are listed in Proposition 2.4. We prove that $S_i \stackrel{aE}{\equiv} P_i(a, a_{\{1, \dots, n\} \setminus E}, \sum_{j \in E} a_j)$ for some P_i separately for each i . It is easy to see that $S_1 \stackrel{aE}{\equiv} S_2 \stackrel{aE}{\equiv} 0$.

Let us discuss S_3 . It can be represented as a sum

$$S_3 = \sum_{J_1 \subset \{1, \dots, n\} \setminus E} \sum_{k \leq |E|} S_{J_1, k},$$

where

$$S_{J_1, k} = \sum_{\substack{J_2 \subset E \\ |J_2|=k}} \left(a + \sum_{j \in J_1 \sqcup J_2} a_j \right) \cdot (2g + |\{1, \dots, n\} \setminus (J_1 \sqcup J_2)|) \cdot \left\langle \left[a + \sum_{j \in J_1 \sqcup J_2} a_j \right]_0 \right\rangle \prod_{i \in \{1, \dots, n\} \setminus (J_1 \sqcup J_2)} \left[\begin{matrix} a_i \\ d_i \end{matrix} \right]_g^{DR} \left\langle \left[a \right]_d \prod_{j \in J_1 \sqcup J_2} \left[\begin{matrix} a_j \\ d_j \end{matrix} \right]_0 \right\rangle^{DR}.$$

Let us prove that $S_{J_1, k} \stackrel{aE}{\equiv} P_{J_1, k}(a, a_{\{1, \dots, n\} \setminus E}, \sum_{j \in E} a_j)$ for some $P_{J_1, k}$. Note that $\left\langle \left[a \right]_d \prod_{j \in J_1 \sqcup J_2} \left[\begin{matrix} a_j \\ d_j \end{matrix} \right]_0 \right\rangle^{DR}$ and $(2g + |\{1, \dots, n\} \setminus (J_1 \sqcup J_2)|)$ are just some constants that depend only on the subset J_1 and the number $k = |J_2|$. The induction assumption implies that

$$\left\langle \left[\begin{matrix} b \\ 0 \end{matrix} \right]_{i=1}^{|E|-k} \prod_{i=1}^{|E|-k} \left[\begin{matrix} x_i \\ 0 \end{matrix} \right] \prod_{i \in \{1, \dots, n\} \setminus (E \sqcup J_1)} \left[\begin{matrix} a_i \\ d_i \end{matrix} \right]_g \right\rangle^{DR} \stackrel{b, x}{\equiv} Q_{J_1, k} \left(a_{\{1, \dots, n\} \setminus (E \sqcup J_1)}, b + \sum_{i=1}^{|E|-k} x_i \right)$$

for some polynomial $Q_{J_1, k}$. Hence

$$\begin{aligned} & \left(a + \sum_{j \in J_1 \sqcup J_2} a_j \right) \left\langle \left[a + \sum_{j \in J_1 \sqcup J_2} a_j \right]_0 \right\rangle \prod_{i \in \{1, \dots, n\} \setminus (J_1 \sqcup J_2)} \left[\begin{matrix} a_i \\ d_i \end{matrix} \right]_g^{DR} \\ & \stackrel{aE}{\equiv} \left(a + \sum_{j \in J_1 \sqcup J_2} a_j \right) Q_{J_1, k} \left(a_{\{1, \dots, n\} \setminus (E \sqcup J_1)}, a + \sum_{i \in J_1 \sqcup E} a_i \right). \end{aligned}$$

Notice that

$$\sum_{\substack{J_2 \subset E \\ |J_2|=k}} \left(a + \sum_{j \in J_1 \sqcup J_2} a_j \right) = \binom{|E|}{k} \left(a + \sum_{j \in J_1} a_j \right) + \binom{|E|-1}{k-1} \sum_{j \in E} a_j.$$

Therefore,

$$Q_{J_1, k} \left(a_{\{1, \dots, n\} \setminus (E \sqcup J_1)}, a + \sum_{i \in J_1 \sqcup E} a_i \right) \cdot \sum_{\substack{J_2 \subset E \\ |J_2|=k}} \left(a + \sum_{j \in J_1 \sqcup J_2} a_j \right)$$

can be represented as a polynomial that depends only on $a, a_i, i \in \{1, \dots, n\} \setminus E$, and $\sum_{j \in E} a_j$. The same argument can be applied to S_4 . This concludes the proof of the lemma. \square

Proof of Lemma 4.5. The lemma follows immediately from Proposition 3.6. Indeed, the statement of the lemma is equivalent to the fact that the polynomial (6) doesn't have terms linear in the variable b . \square

References

- [1] C. Faber, A conjectural description of the tautological ring of the moduli space of curves, in: *Moduli of Curves and Abelian Varieties*, in: *Aspects Math.*, vol. E33, Vieweg, Braunschweig, 1999, pp. 109–129.
- [2] C. Faber, R. Pandharipande, Relative maps and tautological classes, *J. Eur. Math. Soc. (JEMS)* 7 (1) (2005) 13–49.
- [3] E. Getzler, R. Pandharipande, Virasoro constraints and the Chern classes of the Hodge bundle, *Nuclear Phys. B* 530 (3) (1998) 701–714.
- [4] A. Givental, Gromov–Witten invariants and quantization of quadratic hamiltonians, *Mosc. Math. J.* 1 (4) (2001) 551–568.
- [5] I.P. Goulden, D.M. Jackson, R. Vakil, The moduli space of curves, double Hurwitz numbers, and Faber's intersection number conjecture, arXiv:math/0611659v1, 45 pp.
- [6] E.-N. Ionel, Topological recursive relations in $H^{2g}(\mathcal{M}_{g,n})$, *Invent. Math.* 148 (3) (2002) 627–658.
- [7] K. Liu, H. Xu, A proof of the Faber intersection number conjecture, arXiv:0803.2204, 13 pp.
- [8] E. Looijenga, On the tautological ring of \mathcal{M}_g , *Invent. Math.* 121 (2) (1995) 411–419.
- [9] D. Mumford, Towards an enumerative geometry of the moduli space of curves, in: *Arithmetic and Geometry*, vol. II: *Geometry*, in: *Progr. Math.*, vol. 36, Birkhäuser, Boston, 1983, pp. 271–328.
- [10] S. Shadrin, Geometry of meromorphic functions and intersections on moduli spaces of curves, *Int. Math. Res. Not. IMRN* 2003 (38) (2003) 2051–2094.
- [11] S. Shadrin, Combinatorics of binomial decompositions of the simplest Hodge integrals, in: T. Jarvis, T. Kimura, A. Vaintrob (Eds.), *Gromov–Witten Theory of Spin Curves and Orbifolds*, in: *Contemp. Math.*, vol. 403, Amer. Math. Soc., Providence, RI, 2006, pp. 153–165.
- [12] S. Shadrin, D. Zvonkine, Intersection numbers with Witten's top Chern class, *Geom. Topol.* 12 (4) (2008) 713–745.
- [13] R. Vakil, The moduli space of curves and Gromov–Witten theory, in: *Enumerative Invariants in Algebraic Geometry and String Theory*, in: *Lecture Notes in Math.*, vol. 1947, Springer, Berlin, 2008, pp. 143–198.
- [14] E. Witten, Two dimensional gravity and intersection theory on moduli space, in: *Surveys in Differential Geometry*, vol. 1, Lehigh Univ., Bethlehem, PA, 1991, pp. 243–310.