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Infinite order differential operators in spaces of entire functions ☆

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Abstract

Differential operators $\varphi(\Delta_{\theta,\omega})$, where φ is an exponential type entire function of a single complex variable and $\Delta_{\theta,\omega} = (\theta + \omega z)D + zD^2$, $D = \partial/\partial z$, $z \in \mathbb{C}$, $\theta \geqslant 0$, $\omega \in \mathbb{R}$, acting in the spaces of exponential type entire function are studied. It is shown that, for $\omega \geqslant 0$, such operators preserve the set of Laguerre entire functions provided the function φ also belongs to this set. The latter consists of the polynomials possessing real nonpositive zeros only and of their uniform limits on compact subsets of the complex plane $\mathbb{C}$. The operator $\exp(a\Delta_{\theta,\omega})$, $a \geqslant 0$ is studied in more details. In particular, it is shown that it preserves the set of Laguerre entire functions for all $\omega \in \mathbb{R}$. An integral representation of $\exp(a\Delta_{\theta,\omega})$, $a > 0$ is obtained. These results are used to obtain the solutions to certain Cauchy problems employing $\Delta_{\theta,\omega}$.
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Keywords: Fréchet spaces; Exponential type entire functions; Laguerre entire functions; Nonpositive zeros; Integral representation; Cauchy problem

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1. Introduction

Differential operators of infinite order naturally appear in many applications (in a certain sense they constitute a total set of linear operators acting between spaces of differentiable functions [3]). As usual, such operators are constructed by means of finite order differential expressions substituted in the arguments of appropriate functions. If such a function admits power series expansion, the operator may be defined by imposing corresponding convergence conditions. In this article we consider differentiation with respect to a single complex variable $z \in \mathbb{C}$ and the operators constructed by means of the following differential expression

$$\Delta_{\theta,\omega} = \Delta_\theta + \omega z D \stackrel{\text{def}}{=} (\theta + zD)D + \omega z D, \tag{1.1}$$

where $D = \partial/\partial z$ and $\theta \geqslant 0$, $\omega \in \mathbb{R}$ are parameters. Given entire functions $\varphi, f : \mathbb{C} \to \mathbb{C}$, we set

$$\big(\varphi(\Delta_{\theta,\omega}) f\big)(z) = \sum_{k=1}^{\infty} \frac{1}{k!} \varphi^{(k)}(0) \big(\Delta_{\theta,\omega}^k f\big)(z). \tag{1.2}$$

In order for the above series to converge to an entire function, we impose growth restrictions on the functions φ and f by placing them into certain spaces of exponential type entire function. These spaces were introduced and studied in our recent work [4] where we used them to describe the operators $\varphi(\Delta_\theta)$, $\theta \geqslant 0$. Our present research is mainly based on the results of that paper.

In Section 2 below we give definitions and a number of facts regarding the operators $\varphi(\Delta_\theta)$, $\theta \geqslant 0$. Our main results are presented in Section 3 in a sequence of theorems characterizing the properties of $\varphi(\Delta_{\theta,\omega})$. Theorem 3.1 describes the operator $\exp(a\Delta_{\theta,\omega})$. In particular, we prove that this operator obeys the decomposition rule

$$\exp(a\Delta_{\theta,\omega}) = \exp(a\omega z D) \cdot \exp\big\{\omega^{-1}(e^{a\omega} - 1)\Delta_\theta\big\}, \tag{1.3}$$

which is then used to describe the operator $\varphi(\Delta_{\theta,\omega})$ with an arbitrary exponential type entire function φ (Theorem 3.2). A special role in our study is played by Laguerre entire functions being the polynomials of a single complex variable possessing real nonpositive zeros only or the limits of sequences of such polynomials taken in the topology of uniform convergence on compact subsets of $\mathbb{C}$. It turns out that these functions are of exponential type and possess corresponding infinite product representations. We prove (Theorem 3.3) that if both φ and f are Laguerre entire functions and if $\omega \geqslant 0$, the $\varphi(\Delta_{\theta,\omega})f$ is also a Laguerre entire function. The above theorems extend the results of [4] to nonzero values of ω. Then we consider again the operator $\exp(a\Delta_{\theta,\omega})$, $a \geqslant 0$, for which the decomposition (1.3) implies that it preserves the set of Laguerre entire functions *for all real* ω. Further, (1.3) is used to obtain (Theorem 3.5) an integral representation of $\exp(a\Delta_{\theta,\omega})$, $a > 0$. The latter result allows us to extend this operator to a wider class of functions. Theorem 3.6 gives additional information regarding the action of $\exp(a\Delta_{\theta,\omega})$ on the functions of the type of $\exp(uz)g(z)$. In Section 4 we use the above results to describe the solutions to the Cauchy problem

$$\frac{\partial f(t,z)}{\partial t} = (\theta + \omega z)\frac{\partial f(t,z)}{\partial z} + z\frac{\partial^2 f(t,z)}{\partial z^2}, \quad t > 0,$$
$$f(0,z) = g(z)$$

(Theorem 4.1). It enables us to describe (Theorem 4.2) the solutions also to the Cauchy problem of the following type (heat equation with a drift)

$$\frac{\partial F(t,x)}{\partial t} = \big(\Delta + b(x,\nabla)\big)F(t,x), \quad t > 0,$$
$$F(0,x) = G(x), \quad x \in \mathbb{R}^N,\ N \in \mathbb{N},$$

with the initial function G belonging to the class of isotropic (i.e., $O(N)$-invariant) analytic functions. Here Δ and ∇ stand for the N-dimensional Laplacian and gradient, respectively.

All propositions are given without proofs since either they are taken from other sources or the proofs are evident.

2. Preliminaria

Let $\mathcal{E}$ stand for the set of all entire functions $f:\mathbb{C} \to \mathbb{C}$ equipped with the topology $\mathcal{T}_{\mathbb{C}}$ of uniform convergence on compact subsets of $\mathbb{C}$. Thus, $(\mathcal{E}, \mathcal{T}_{\mathbb{C}})$ is a Fréchet space. For $b > 0$, we set

$$\mathcal{B}_b = \big\{ f \in \mathcal{E} \mid \|f\|_b < \infty \big\},$$

where

$$\|f\|_b = \sup_{k\in\mathbb{N}_0} \big\{ b^{-k} \big| f^{(k)}(0) \big| \big\}, \quad f^{(k)}(0) = (D^k f)(0), \tag{2.1}$$

and $\mathbb{N}_0$ stands for the set of all nonnegative integers. Every $(\mathcal{B}_b, \|\cdot\|_b)$ is a Banach space. For $a \geqslant 0$, let

$$\mathcal{A}_a = \bigcap_{b>a} \mathcal{B}_b = \big\{ f \in \mathcal{E} \mid (\forall b > a)\ \|f\|_b < \infty \big\}. \tag{2.2}$$

Equipped with the topology $\mathcal{T}_a$ defined by the family of norms $\{\|\cdot\|_b,\ b > a\}$, this set becomes a Fréchet space. To shorten our notation we write $\mathcal{E}$, $\mathcal{A}_a$, $\mathcal{B}_b$ instead of $(\mathcal{E}, \mathcal{T}_{\mathbb{C}})$, $(\mathcal{A}_a, \mathcal{T}_a)$, $(\mathcal{B}_b, \|\cdot\|_b)$, respectively.

Definition 2.1. A family $\mathcal{L}$ (respectively $\mathcal{L}_0$, $\mathcal{L}^+$, $\mathcal{L}^-$) consists of the entire functions possessing the representation

$$f(z) = Cz^m \exp(\alpha z) \prod_{j=1}^{\infty} (1 + \beta_j z),$$
$$C \in \mathbb{C},\ m \in \mathbb{N}_0,\ \beta_j \geqslant \beta_{j+1} \geqslant 0,\ \sum_{j=1}^{\infty} \beta_j < \infty, \tag{2.3}$$

with $\alpha \in \mathbb{R}$ (respectively $\alpha = 0$, $\alpha \geqslant 0$, and $\alpha < 0$).

The elements of $\mathcal{L}^+$ are known as Laguerre entire functions [1,4]. Let $\mathcal{P}^+$ stand for the set of polynomials belonging to $\mathcal{L}^+$. By Laguerre and Pólya (see, e.g., [1,5]),

Proposition 2.1. *The family $\mathcal{L}^+$ is exactly the closure of $\mathcal{P}^+$ in $\mathcal{T}_{\mathbb{C}}$.*

It is worth to note that every f being of the form (2.3) may be written $f(z) = \exp(\alpha z) \times h(z)$, where h is an entire function of order less than one or equal to one and, in the latter case, of minimal type. Consider the families

$$\mathcal{L}_a \stackrel{\text{def}}{=} \mathcal{L} \cap \mathcal{A}_a, \qquad \mathcal{L}_a^{\pm} \stackrel{\text{def}}{=} \mathcal{L}^{\pm} \cap \mathcal{A}_a. \tag{2.4}$$

Obviously, $\mathcal{P}$ and $\mathcal{P}^+$ are dense respectively in the sets $\mathcal{A}_a$ and $\mathcal{L}_a^+$ equipped with the topologies induced on them by $\mathcal{T}_{\mathbb{C}}$. However, the set $\mathcal{P}$ is not dense in any space $\mathcal{B}_b$. Thus, a priori it is not obvious whether or not the sets $\mathcal{P}$ and $\mathcal{P}^+$ are dense respectively in $\mathcal{A}_a$ (in its standard topology) and in $\mathcal{L}_a^+$ in the topology induced from $\mathcal{A}_a$. Fortunately, this density property holds in both cases. The following two statements, which we borrow from [4], give information regarding the topological properties of $\mathcal{A}_a$ and $\mathcal{L}_a^+$.

Proposition 2.2. *For every $a \geqslant 0$, the relative topology on a bounded subset of $\mathcal{A}_a$ coincides with the topology induced on it by $\mathcal{T}_{\mathbb{C}}$.*

Proposition 2.3. *For every $a \geqslant 0$,*

(i) *the set of all polynomials $\mathcal{P} \subset \mathcal{E}$ is dense in $\mathcal{A}_a$;*
(ii) *the set $\mathcal{P}^+$ is dense in $\mathcal{L}_a^+$ in the topology induced by $\mathcal{T}_a$.*

Below unless explicitly stated we consider $\mathcal{L}_a^+$, $a \geqslant 0$ as a topological space equipped with the topology induced by $\mathcal{T}_a$.

For $\theta \geqslant 0$, let $\Delta_\theta : \mathcal{E} \to \mathcal{E}$ be as in (1.1), that is $\Delta_\theta = \Delta_{\theta,0} = (\theta + zD)D$. Given entire functions φ and f, we define $\varphi(\Delta_\theta) f(z)$ by (1.2). Further, one has

$$\Delta_\theta^k z^m = q_\theta^{(m,k)} z^{m-k}, \quad q_\theta^{(m,k)} = \begin{cases} 0, & k > m, \\ \gamma_\theta(m)/\gamma_\theta(m-k), & 0 \leqslant k \leqslant m, \end{cases} \tag{2.5}$$

where

$$\gamma_\theta(m) = m!\Gamma(\theta + m).$$

Applying this in (1.2) one may prove the following statement [4].

Proposition 2.4. *For all $\theta \geqslant 0$ and for arbitrary $a > 0$ and $b > 0$, such that $ab < 1$, $(\varphi, f) \mapsto \varphi(\Delta_\theta) f$ is a continuous bilinear map from $\mathcal{B}_a \times \mathcal{B}_b$ (respectively from $\mathcal{A}_a \times \mathcal{A}_b$ with $a \geqslant 0$, $b \geqslant 0$) into $\mathcal{B}_c$ (respectively $\mathcal{A}_c$), where $c = b(1 - ab)^{-1}$. Moreover,*

$$\left\| \varphi(\Delta_\theta) f \right\|_c \leqslant (1 - ab)^{-\theta} \|\varphi\|_a \|f\|_b.$$

The action of $\varphi(\Delta_\theta)$ on the Laguerre entire functions is described by the following statement, which was proven in [4].

Proposition 2.5. *For all* $\theta \geqslant 0$ *and for arbitrary* $a \geqslant 0$ *and* $b \geqslant 0$, *such that* $ab < 1$, $(\varphi, f) \mapsto \varphi(\Delta_\theta) f$ *is a continuous map from* $\mathcal{L}_a^+ \times \mathcal{L}_b^+$ *into* $\mathcal{L}_c^+$, *where* $c = b(1 - ab)^{-1}$.

Given $\omega \in \mathbb{R}$ and $f \in \mathcal{E}$, we set

$$\exp(\omega z D) f(z) = \sum_{k=0}^{\infty} \frac{\omega^k}{k!} \big((zD)^k f\big)(z),$$

which readily yields

$$\exp(\omega z D) f(z) = f(e^{\omega} z). \tag{2.6}$$

Proposition 2.6. *For any* $\omega \in \mathbb{R}$, $\exp(\omega z D)$ *is a continuous linear map from* $\mathcal{B}_c$ *with* $c > 0$ (*respectively from* $\mathcal{A}_c$ *with* $c \geqslant 0$) *into* $\mathcal{B}_d$ (*respectively* $\mathcal{A}_d$), *where* $d = e^{\omega} c$.

The identity (2.6) also implies that

Proposition 2.7. *For any* $\omega \in \mathbb{R}$ *and* $c \geqslant 0$, $\exp(\omega z D)$ *is a continuous map from* $\mathcal{L}_c^+$ (*respectively from* $\mathcal{L}_c^-$) *into* $\mathcal{L}_d^+$ (*respectively* $\mathcal{L}_d^-$), *where* $d = e^{\omega} c$.

The following representation of $\exp(a\Delta_\theta)$ was obtained in [4].

Proposition 2.8. *For every* $\theta \geqslant 0$, *for arbitrary* $a > 0$, $b \geqslant 0$, *such that* $ab < 1$, *and for all* $f \in \mathcal{A}_b$,

$$\begin{aligned} \big(\exp(a\Delta_\theta) f\big)(z) &= \exp\left(-\frac{z}{a}\right) \int_0^{+\infty} w_\theta\left(\frac{sz}{a}\right) f(as) s^{\theta-1} e^{-s}\, \mathrm{d}s \\ &\stackrel{\text{def}}{=} \int_0^{+\infty} K_\theta\left(\frac{z}{a}, s\right) f(as) s^{\theta-1} e^{-s}\, \mathrm{d}s, \end{aligned} \tag{2.7}$$

where

$$K_\theta(z, s) = e^{-z} w_\theta(zs), \quad w_\theta(\xi) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} \frac{\xi^k}{\gamma_\theta(k)}. \tag{2.8}$$

This representation may be used for extending the operator $\exp(a\Delta_\theta)$. The assertion below, also taken from [4], describes a property of such an extended operator.

Proposition 2.9. *Given* $a > 0$ *and* $u \in \mathbb{R}$, *let* $b \in [0, -u + 1/a)$. *Then, for every* $g \in \mathcal{A}_b$, *the operator* (2.7) *may be applied to the function*

$$f(z) = \exp(uz) g(z), \tag{2.9}$$

yielding

$$\big(\exp(a\Delta_\theta) f\big)(z) = (1 - ua)^{-\theta} \exp\left(\frac{uz}{1 - ua}\right) h(z), \tag{2.10}$$

where

$$h(z) = \left[\exp\left(\frac{a}{1-ua}\Delta_\theta\right)g\right]\left(\frac{z}{(1-ua)^2}\right) = \exp\bigl[a(1-ua)\Delta_\theta\bigr]g\left(\frac{z}{(1-ua)^2}\right). \tag{2.11}$$

Moreover, $h \in \mathcal{A}_c$ *with* $c = b(1-ua)^{-1}[1-a(u+b)]^{-1}$.

The extensions of $\exp(a\Delta_\theta)$ on the base of the integral representation (2.7) to the spaces of integrable functions are given in [2].

3. Main results

Theorem 3.1. *For all* $\theta \geqslant 0$, *for arbitrary* $\omega \in \mathbb{R}$, $a \geqslant 0$, *and* $b > 0$, *obeying the condition* $b\omega^{-1}[e^{a\omega} - 1] < 1$, *the series expansion* (1.2) *defines a continuous linear operator* $\exp(a\Delta_{\theta,\omega})$ *acting from the space* $\mathcal{B}_b$ (*respectively from* $\mathcal{A}_b$ *with* $b \geqslant 0$) *into the space* $\mathcal{B}_c$ (*respectively* $\mathcal{A}_c$), *where*

$$c = c(b) \stackrel{\text{def}}{=} be^{a\omega}\bigl(1 - b\omega^{-1}[e^{a\omega} - 1]\bigr)^{-1}. \tag{3.1}$$

This operator obeys the decomposition rule (1.3).

Remark 3.1. This theorem holds also for $\omega = 0$, since $\lim_{\omega\to 0}[e^{a\omega} - 1]\omega^{-1} = a$, where it coincides with a partial case of Proposition 2.4. In the sequel the case $\omega = 0$ will be understood in this sense.

Theorem 3.2. *For all* $\theta \geqslant 0$, *for arbitrary* $a > 0$, $\omega \in \mathbb{R}$, $b > 0$, *obeying the condition* $b\omega^{-1}[e^{a\omega} - 1] < 1$, $(\varphi, f) \mapsto \varphi(\Delta_{\theta,\omega})f$, *is a continuous bilinear map from* $\mathcal{B}_a \times \mathcal{B}_b$ (*respectively from* $\mathcal{A}_a \times \mathcal{A}_b$ *with* $a \geqslant 0$ *and* $b \geqslant 0$) *into the space* $\mathcal{B}_c$ (*respectively* $\mathcal{A}_c$), *where* $c = c(b)$ *is given by* (3.1). *Moreover,*

$$\bigl\|\varphi(\Delta_{\theta,\omega})f\bigr\|_c \leqslant \bigl(1 - b\omega^{-1}[e^{a\omega} - 1]\bigr)^{-\theta}\|\varphi\|_a\|f\|_b.$$

Theorem 3.3. *For all* $\theta \geqslant 0$, *for arbitrary* $\omega \geqslant 0$, $a \geqslant 0$, $b \geqslant 0$, *obeying the condition* $b\omega^{-1}[e^{a\omega} - 1] < 1$, $(\varphi, f) \mapsto \varphi(\Delta_{\theta,\omega})f$ *is a continuous map from* $\mathcal{L}_a^+ \times \mathcal{L}_b^+$ *into* $\mathcal{L}_c^+$, *where* $c = c(b)$ *is given by* (3.1).

Theorem 3.4. *For all* $\theta \geqslant 0$, *for arbitrary* $\omega \in \mathbb{R}$, $a \geqslant 0$, $b \geqslant 0$, *obeying the condition* $b\omega^{-1}[e^{a\omega} - 1] < 1$, $\exp(a\Delta_{\theta,\omega})$ *is a continuous map from* $\mathcal{L}_b^+$ *into* $\mathcal{L}_c^+$, *where* $c = c(b)$ *is given by* (3.1).

Theorem 3.5. *For all* $\theta \geqslant 0$, *for arbitrary* $\omega \in \mathbb{R}$, $a > 0$, *and* $b \geqslant 0$, *obeying the condition* $b\omega^{-1}[e^{a\omega} - 1] < 1$, *and for any* $f \in \mathcal{A}_b$,

$$\bigl(\exp(a\Delta_{\theta,\omega})f\bigr)(z) = \exp(-\nu z)\int_0^{+\infty} w_\theta(\nu s z)f\bigl(\omega^{-1}[e^{a\omega} - 1]s\bigr)s^{\theta-1}e^{-s}\,\mathrm{d}s$$

$$= \int_0^{+\infty} K_\theta(\nu z, s) f\big(\omega^{-1}[e^{a\omega} - 1]s\big) s^{\theta-1} e^{-s}\, \mathrm{d}s, \tag{3.2}$$

where

$$\nu = \nu(a, \omega) \stackrel{\text{def}}{=} \frac{\omega e^{a\omega}}{e^{a\omega} - 1}, \tag{3.3}$$

and w_θ *and* K_θ *are defined by* (2.8).

The above theorem allows us to extend the operator $\exp(a\Delta_{\theta,\omega})$ to the functions for which the integrals in the right-hand side of (3.2) converge. In the sequel we understand this operator in such an extended (integral) version.

Theorem 3.6. *Given* $a \geqslant 0$, $\omega \in \mathbb{R}$, $u \in \mathbb{R}$, *and* $b \geqslant 0$, *obeying the condition* $b\omega^{-1} \times [e^{a\omega} - 1] < 1 - u\omega^{-1}[e^{a\omega} - 1]$, *the operator* $\exp(a\Delta_{\theta,\omega})$ *may be applied to the function*

$$f(z) = \exp(uz)g(z), \tag{3.4}$$

with arbitrary $g \in \mathcal{A}_b$, *yielding*

$$\big(\exp(a\Delta_{\theta,\omega}) f\big)(z) = \big(1 - u\omega^{-1}[e^{a\omega} - 1]\big)^{-\theta} \exp\left(\frac{ue^{a\omega}z}{1 - u\omega^{-1}[e^{a\omega} - 1]}\right) h(z), \tag{3.5}$$

where

$$\begin{aligned} h(z) &= \left[\exp\left(\frac{a\omega}{\omega - u[e^{a\omega} - 1]}\Delta_\theta + a\omega z D\right) g\right]\left(\frac{\omega^2 z}{(\omega - u[e^{a\omega} - 1])^2}\right) \\ &= \left[\exp\left(\frac{e^{a\omega} - 1}{\omega - u[e^{a\omega} - 1]}\Delta_\theta\right) g\right]\left(\frac{\omega^2 e^{a\omega} z}{(\omega - u[e^{a\omega} - 1])^2}\right). \end{aligned} \tag{3.6}$$

Moreover, the latter function belongs to $\mathcal{A}_c$ *with*

$$c = be^{a\omega}\big(1 - u\omega^{-1}[e^{a\omega} - 1]\big)^{-1}\big(1 - (u + b)\omega^{-1}[e^{a\omega} - 1]\big)^{-1}.$$

Corollary 3.1. *For all* $\theta \geqslant 0$, $\omega \in \mathbb{R}$, $a \geqslant 0$, *and* $b \geqslant 0$, *the operator* $\exp(a\Delta_{\theta,\omega})$ *is a continuous map from*

(i) $\mathcal{L}_b^+$ *into* $\mathcal{L}_c^+$, *where* a, b, *and* ω *satisfy the condition* $b\omega^{-1}[e^{a\omega} - 1] < 1$, *and* $c = c(b)$ *is given by* (3.1),
(ii) $\mathcal{L}_b^-$ *into* $\mathcal{L}_d^-$, *where* $d = be^{a\omega}(1 + b\omega^{-1}[e^{a\omega} - 1])^{-1}$.

The proof of the above theorems will be based on the properties of $\exp(a\Delta_{\theta,\omega})$, which may be studied on the base of (1.3) and the properties of $\exp(a\omega zD)$ and $\exp(b\omega^{-1} \times [e^{a\omega} - 1]\Delta_\theta)$, given by Propositions 2.4–2.9. Clearly, the operator $\exp(a\Delta_{\theta,\omega})$ defined by (1.2) may be applied to any $f \in \mathcal{P}$, furthermore, $\exp(a\Delta_{\theta,\omega}) : \mathcal{P} \to \mathcal{P}$. Similarly, for any real b and c, $\exp(bzD) : \mathcal{P} \to \mathcal{P}$ and $\exp(c\Delta_\theta) : \mathcal{P} \to \mathcal{P}$.

Lemma 3.1. *For every polynomial* $f \in \mathcal{P}$ *and any* $a \in \mathbb{R}$, *the operator* $\exp(a\Delta_{\theta,\omega})$ *defined by the series expansion* (1.2) *obeys* (1.3).

Proof. Obviously, it is enough to prove the lemma only for $f_m(z) = z^m$, $m \in \mathbb{N}$. Since $a\Delta_\theta$ and $a\omega zD$ do not commute (except for a or ω being zero), one has to apply an appropriate decomposition technique. It will be based on the Trotter–Kato product formula, which, for our aims, may be written as follows (for more details on this item see [7] and references therein):

$$\exp(A+B)f = \lim_{n\to\infty}\left(\exp\left(\frac{1}{n}A\right)\exp\left(\frac{1}{n}B\right)\right)^n f, \tag{3.7}$$

where A and B are linear continuous operators on a topological vector space and the convergence is understood in the topology of the range space.

In what follows, given $m \in \mathbb{N}$, one has

$$\exp(a\Delta_{\theta,\omega})f_m(z) = \lim_{n\to\infty}\left(\exp\left(\frac{a}{n}\Delta_\theta\right)\exp\left(\frac{a\omega}{n}zD\right)\right)^n f_m(z), \tag{3.8}$$

where the convergence is in $\mathcal{T}_{\mathbb{C}}$. By (2.6) and (2.5),

$$\exp(a\Delta_\theta)z^m = \sum_{k=0}^{m}\frac{a^k}{k!}q_\theta^{(m,k)}z^{m-k},$$

which yields, together with (2.6), that for any $n, m \in \mathbb{N}$,

$$\begin{aligned}
\left(\exp\left(\frac{a}{n}\Delta_\theta\right)\exp\left(\frac{a\omega}{n}zD\right)\right)^n z^m &= \sum_{k_1=0}^{m}\sum_{k_2=0}^{m-k_1}\cdots\sum_{k_n=0}^{m-k_1-\cdots-k_{n-1}}\frac{1}{k_1!k_2!\ldots k_n!}\\
&\quad\times\left(\frac{a}{n}\right)^{k_1+k_2+\cdots+k_n} q_\theta^{(m,k_1)}q_\theta^{(m-k_1,k_2)}\ldots q_\theta^{(m-k_1-\cdots-k_{n-1},k_n)}\\
&\quad\times\exp\left\{\frac{a\omega}{n}(m+m-k_1+\cdots+m-k_1-\cdots-k_{n-1})\right\}z^{m-k_1-\cdots-k_n}\\
&= \sum_{k_1=0}^{m}\sum_{k_2=0}^{m-k_1}\cdots\sum_{k_n=0}^{m-k_1-\cdots-k_{n-1}}\frac{1}{k_1!k_2!\ldots k_n!}\left(\frac{a}{n}\right)^{k_1+k_2+\cdots+k_n} z^{m-k_1-\cdots-k_n}\\
&\quad\times\frac{\gamma_\theta(m)\exp\left\{\frac{a\omega}{n}(nm-(n-1)k_1-\cdots-k_{n-1})\right\}}{\gamma_\theta(m-k_1-\cdots-k_n)}\\
&\overset{\text{def}}{=} \sum_{k=0}^{m}a^k\frac{\gamma_\theta(m)}{\gamma_\theta(m-k)}(e^{a\omega}z)^{m-k}\Psi_n(k,a\omega).
\end{aligned} \tag{3.9}$$

Here for $k \in \mathbb{N}_0$ and $b \in \mathbb{R}$, we have set

$$\begin{aligned}
\Psi_n(k,b) &= \frac{1}{n^k}\sum_{k_1,k_2,\ldots,k_n=0}^{k}\delta_{k,k_1+\cdots+k_n}\frac{e^{\frac{b}{n}(k_1+2k_2+\cdots+nk_n)}}{k_1!k_2!\ldots k_n!}\\
&= \frac{1}{k!n^k}\left(e^{b/n}+e^{2(b/n)}+\cdots+e^{n(b/n)}\right)^k = \frac{1}{k!}\left(\frac{e^{b/n}(e^b-1)}{n(e^{b/n}-1)}\right)^k\\
&\to \frac{1}{k!}\left(\frac{e^b-1}{b}\right)^k,
\end{aligned} \tag{3.10}$$

when $n \to +\infty$. By (3.8), (3.9), and (3.10) one obtains

$$\begin{aligned}\exp(a\Delta_{\theta,\omega}) f_m(z) &= \lim_{n\to\infty}\left(\exp\left(\frac{a}{n}\Delta_\theta\right)\exp\left(\frac{a\omega}{n}zD\right)\right)^n z^m\\ &= \sum_{k=0}^{m}\frac{(\omega^{-1}[e^{a\omega}-1])^k}{k!}\cdot\frac{\gamma_\theta(m)}{\gamma_\theta(m-k)}(e^{a\omega}z)^{m-k}\\ &= \big(\exp\big(\omega^{-1}[e^{a\omega}-1]\Delta_\theta\big) f_m\big)(e^{a\omega}z)\\ &= \exp(a\omega z D)\exp\big(\omega^{-1}[e^{a\omega}-1]\Delta_\theta\big) f_m(z). \qquad \Box\end{aligned}$$

Given $k, n, m \in \mathbb{N}$, we set

$$\kappa_n(k, m, \theta, \omega) = \big[D^n(\Delta_{\theta,\omega})^k z^m\big]_{z=0}. \tag{3.11}$$

After some algebra one obtains

$$\kappa_n(k, m, \theta, \omega) = \frac{n!}{(m-n)!} q_\theta^{(m,m-n)} \omega^{k+n-m} \sum_{l=m-n}^{k}\binom{k}{l} n^{k-l}\alpha_{m-n}^{(l)}, \tag{3.12}$$

if $m \geqslant n$, and $\kappa_n(k, m, \theta, \omega) = 0$ if $m < n$. Here, for $p \geqslant s,\ p, s \in \mathbb{N}$,

$$\alpha_p^{(s)} \stackrel{\text{def}}{=} \sum_{k=0}^{p}\binom{p}{k}(-1)^k(p-k)^s > 0.$$

Proof of Theorem 3.1. Given $f \in \mathcal{P}$ and $b > 0$ obeying the condition $b\omega^{-1}[e^{a\omega}-1] < 1$, one has from (1.3), (2.6), and Proposition 2.4

$$\begin{aligned}\big\|\exp(a\Delta_{\theta,\omega}) f\big\|_c &= \big\|\exp(a\omega z D)\exp\big\{\omega^{-1}(e^{a\omega}-1)\big\} f\big\|_c\\ &= \big\|\exp\big\{\omega^{-1}(e^{a\omega}-1)\big\} f\big\|_{ce^{-a\omega}}\\ &\leqslant \big(1 - b\omega^{-1}(e^{a\omega}-1)\big)^{-\theta}\|f\|_b,\end{aligned} \tag{3.13}$$

where $c = c(b)$ is given by (3.1). Then by claim (i) of Proposition 2.3, $\exp(a\Delta_{\theta,\omega})$ may be continuously extended to the whole $\mathcal{A}_b$. The extension will also obey (1.3). This yields in turn that the estimate (3.13) holds for any $f \in \mathcal{E}$, provided $\|f\|_b < \infty$. $\Box$

Proof of Theorem 3.2. According to (1.2) and (3.11),

$$g^{(n)}(0) \stackrel{\text{def}}{=} \big(\varphi(\Delta_{\theta,\omega}) f\big)^{(n)}(0) = \sum_{k,m=0}^{\infty}\frac{\varphi^{(k)}(0)}{k!}\cdot\frac{f^{(m)}(0)}{m!}\kappa_n(k, m, \theta, \omega), \quad n \in \mathbb{N}_0, \tag{3.14}$$

which, for given positive a and b, yields

$$\big|g^{(n)}(0)\big| \leqslant \|\varphi\|_a\|f\|_b\sum_{k,m=0}^{\infty}\frac{a^k}{k!}\cdot\frac{b^m}{m!}\big|\kappa_n(k, m, \theta, \omega)\big|. \tag{3.15}$$

By (3.12),

$$\left|\kappa_n(k,m,\theta,\omega)\right| \leqslant \kappa_n\big(k,m,\theta,|\omega|\big).$$

Therefore, for a and b obeying the conditions of the theorem and for $c(b)$ given by (3.1), one readily gets from (3.15), (3.14), (3.13), and (2.1)

$$\begin{aligned}\|g\|_c &\leqslant \|\varphi\|_a\|f\|_b \cdot \sup_{n\in\mathbb{N}_0}\left\{c^{-n}\sum_{k,m=0}^{\infty}\frac{a^k}{k!}\cdot\frac{b^m}{m!}\kappa_n\big(k,m,\theta,|\omega|\big)\right\}\\ &= \|\varphi\|_a\|f\|_b \cdot \left\|\exp\{a\Delta_{\theta,|\omega|}\}\exp(bz)\right\|_c\\ &\leqslant \left(1-b\frac{e^{a|\omega|}-1}{|\omega|}\right)^{-\theta}\|\varphi\|_a\|f\|_b = \left(1-b\frac{e^{a\omega}-1}{\omega}\right)^{-\theta}\|\varphi\|_a\|f\|_b. \qquad \square\end{aligned}$$

Lemma 3.2. *For all $\theta \geqslant 0$, $\omega \geqslant 0$ and for arbitrary $\varphi, f \in \mathcal{P}^+$, the polynomial $\varphi(\Delta_{\theta,\omega})f$ also belongs to $\mathcal{P}^+$.*

This lemma will be proven in several steps below. The proof of Theorem 3.3 readily follows from it and from claim (ii) of Proposition 2.3 and Theorem 3.2. Set

$$U = \{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}. \tag{3.16}$$

In [4] we have proven the following statement.

Proposition 3.1. *Let P, Q, and Q_1 be polynomials of a single complex variable. Suppose that P does not vanish on U and*

$$Q(u) + P(u)vQ_1(u) \neq 0, \tag{3.17}$$

whenever $u, v \in U$. Then either

$$S(z) \stackrel{\text{def}}{=} Q(z) + P(z)DQ_1(z) \neq 0, \tag{3.18}$$

whenever $z \in U$, or else $S(z) \equiv 0$.

Lemma 3.3. *For arbitrary $\sigma \geqslant 0$, $\theta \geqslant 0$, and $\omega \geqslant 0$, the second order differential operator $\sigma + \Delta_{\theta,\omega}$ maps $\mathcal{P}^+$ into itself.*

Proof. For an arbitrarily chosen $p \in \mathcal{P}^+$, we have to show that $(\sigma + \Delta_{\theta,\omega})p \in \mathcal{P}^+$. The case of constant p is trivial. For nonconstant $p \in \mathcal{P}^+$, one may write

$$p(z) = \pi_0\prod_{j=1}^{m}(\pi_j + z), \quad m \in \mathbb{N},\ \pi_j \geqslant 0,\ j = 1,2,\ldots,m. \tag{3.19}$$

First we consider the case $\pi_j = 0$, $j = 1,2,\ldots,m$, that is $p(z) = \pi_0 z^m$. Then

$$(\sigma + \Delta_{\theta,\omega})p(z) = z^{m-1}\tilde{p}(z) \in \mathcal{P}^+,$$

since

$$\tilde{p}(z) \stackrel{\text{def}}{=} m(\sigma + m\omega)z + m(\theta + m - 1) \in \mathcal{P}^+.$$

Now let at least one π_j in (3.19) do not vanish. Set

$$q(z) = p(z^2) = \pi_0 \prod_{j=1}^{m} (\pi_j + z^2). \tag{3.20}$$

Then

$$\big((\sigma + \Delta_{\theta,\omega})p\big)(z^2) = \big((\sigma + \Lambda_{\theta,\omega})q\big)(z), \tag{3.21}$$

where

$$\Lambda_{\theta,\omega} \stackrel{\text{def}}{=} \left(\theta + \frac{z}{2}D\right)\left(\frac{1}{2z}D\right) + \frac{\omega}{2}zD. \tag{3.22}$$

In view of (3.20), the polynomials $q(z)$, $q(-z)$ do not vanish on U (3.16). The proof will be done by showing that

$$\big((\sigma + \Lambda_{\theta,\omega})q\big)(z) \neq 0, \tag{3.23}$$

whenever $z \in U$. Taking into account (3.20) we may write

$$\big((\sigma + \Lambda_{\theta,\omega})q\big)(z) = \big[\sigma + (\theta + \omega z^2)r(z)\big]q(z) + \frac{1}{2}zD\big(q(z)r(z)\big),$$

where

$$r(z) = \sum_{j=1}^{m} \frac{1}{\pi_j + z^2}.$$

Set

$$Q(z) = \big[\sigma + (\theta + \omega z^2)r(z)\big]q(z), \qquad P(z) = \frac{z}{2}, \qquad Q_1(z) = q(z)r(z).$$

By Proposition 3.1, the proof of (3.23) will be done if we show that

$$R(u,v) \stackrel{\text{def}}{=} Q(u) + P(u)vQ_1(u) \neq 0, \tag{3.24}$$

whenever $u, v \in U$. To this end we rewrite the latter as

$$R(u,v) = \frac{1}{2}q(u) \cdot R_1(u,v) \cdot R_2(u,v), \tag{3.25}$$

where

$$R_1(u,v) = 2\omega u^2 + uv + 2\theta$$

and

$$R_2(u,v) = r(u) + \frac{2\sigma}{2\omega u^2 + uv + 2\theta}. \tag{3.26}$$

Since $q(u) \neq 0$ whenever $u \in U$, to prove (3.24) it remains to show that both R_1 and R_2 do not vanish if $u, v \in U$. Given $u \in U$, let us solve the equation $R_1(u,v) = 0$. The result is

$$v = -\frac{2\theta}{|u|^2}\bar{u} - 2\omega u,$$

which yields the following implications:

$$\big(R_1(u,v)=0\big) \quad\Rightarrow\quad (\operatorname{Re} v \leqslant 0) \quad\Rightarrow\quad (v\in\mathbb{C}\setminus U).$$

Recall that both θ and ω are supposed to be real and nonnegative. Then $R_1(u,v)\neq 0$ if $v\in U$. By the same arguments we show that $R_2(u,v)\neq 0$ if $u,v\in U$. To this end we rewrite (3.26) as

$$R_2(u,v)=C(u,v)\bar{u}^2+B(u,v)\bar{v}\bar{u}+A(u,v), \tag{3.27}$$

where for $u,v\in U$, we set

$$C(u,v)=\sum_{j=1}^{m}\frac{1}{|\pi_j+u^2|^2}+\frac{4\sigma\omega}{|2\omega u^2+vu+2\theta|^2}>0,$$

$$B(u,v)=\frac{2\sigma}{|2\omega u^2+vu+2\theta|^2}\geqslant 0,$$

$$A(u,v)=\sum_{j=1}^{m}\frac{\pi_j}{|\pi_j+u^2|^2}+\frac{4\sigma\theta}{|2\omega u^2+vu+2\theta|^2}>0.$$

We have just shown that $|R_1(u,v)|=|2\omega u^2+vu+2\theta|>0$ if $u,v\in U$. For $\sigma=0$, one has

$$R_2(u,v)=C(u,v)\left[\bar{u}^2+\frac{A(u,v)}{C(u,v)}\right]\neq 0,$$

whenever $u,v\in U$. For $\sigma\neq 0$, one has $B(u,v)>0$ and by (3.27) one would get from $R_2(u,v)=0$

$$\bar{v}=-\frac{A(u,v)}{B(u,v)}\cdot u-\frac{C(u,v)}{B(u,v)}\cdot\bar{u},$$

which yields in turn

$$(v\in U)\quad\Rightarrow\quad\big(R_2(u,v)\neq 0\big).\qquad\square$$

Proof of Lemma 3.2. Similarly to (3.19) one has for $\varphi\in\mathcal{P}^+$

$$\varphi(\varDelta_{\theta,\omega})=\varphi_0\prod_{j=1}^{m}(\sigma_j+\varDelta_{\theta,\omega}),\quad \sigma_j\geqslant 0,\ j=1,2,\ldots,m,\ m\in\mathbb{N}.$$

By Lemma 3.3 each $(\sigma_j+\varDelta_{\theta,\omega})$ maps $\mathcal{P}^+$ into itself, hence the whole $\varphi(\varDelta_{\theta,\omega})$ does so. $\square$

Proof of Theorem 3.4. By (1.3) the operator $\exp(a\varDelta_{\theta,\omega})$ is a composition of $\exp(a\omega zD)$ and $\exp(\gamma\varDelta_\theta)$ with $\gamma=(e^a\omega-1)/\omega$. The latter operator continuously maps $\mathcal{L}_b^+$ in $\mathcal{L}_\beta^+$ (Proposition 2.5), where $\beta=b[1-(e^a\omega-1)(b/\omega)]$. The former one, also continuously, maps $\mathcal{L}_\beta^+$ into $\mathcal{L}_{c(b)}^+$, which follows from Proposition 2.7. $\square$

In a similar way, the proof of Theorem 3.5 follows from (1.3), (2.6), and Proposition 2.8. The proof of Theorem 3.6 follows from (1.3), (2.6), and Proposition 2.9. The proof of Corollary 3.1 follows from (3.5), (3.6), and Theorem 3.4.

4. Differential equation

Now we may use the operators introduced above to describe the solutions to certain Cauchy problems. First we consider the following one:

$$\frac{\partial f(t,z)}{\partial t} = (\theta + \omega z)\frac{\partial f(t,z)}{\partial z} + z\frac{\partial^2 f(t,z)}{\partial z^2}, \quad \omega \in \mathbb{R},\ z \in \mathbb{C},\ t > 0,$$
$$f(0,z) = g(z). \tag{4.1}$$

Theorem 4.1. *For every $\theta \geqslant 0$, $\omega \in \mathbb{R}$, and $g \in \mathcal{E}$ having the form*

$$g(z) = \exp(-\varepsilon z)h(z), \quad h \in \mathcal{A}_0,\ \varepsilon \geqslant 0, \tag{4.2}$$

(i) *the problem* (4.1) *has a unique solution in $\mathcal{A}_\varepsilon$, which may be written as*

$$f(t,z) = \big(\exp(t\Delta_{\theta,\omega})g\big)(z)$$
$$= \exp\left(-\frac{\omega z}{1-e^{-t\omega}}\right)\int_0^{+\infty} w_\theta\left(\frac{\omega z s}{1-e^{-t\omega}}\right) g\left(s\frac{e^{t\omega}-1}{\omega}\right) s^{\theta-1}e^{-s}\,\mathrm{d}s; \tag{4.3}$$

(ii) *if in* (4.2) $\varepsilon > 0$, *the solution* (4.3) *converges to zero in $\mathcal{A}_\varepsilon$ when $t \to +\infty$;*
(iii) *if in* (4.2) $h \in \mathcal{L}_0 \subset \mathcal{A}_0$, *the solution* (4.3) *belongs either to $\mathcal{L}_0$, for $\varepsilon = 0$, or to $\mathcal{L}^-$, for $\varepsilon > 0$.*

Proof. Let $\varphi_t(z) = \exp(tz)$. The operator valued function $[0, t_0) \ni t \mapsto \varphi_t(\Delta_{\theta,\omega})$ is continuous and differentiable in the norm-topology, and

$$\varphi_t'(\Delta_{\theta,\omega}) = \Delta_{\theta,\omega}\varphi_t(\Delta_{\theta,\omega}).$$

The functions (4.2), with a given ε and all $h \in \mathcal{A}_0$, form a subspace of $\mathcal{B}_b \supset \mathcal{A}_\varepsilon$, $b > \varepsilon$. The restrictions of $\varphi_t(\Delta_\theta)$, $t \in [0, t_0)$ to this subspace is a differentiable semigroup. Then the problem (4.1) has a unique solution in the mentioned subspace (see, e.g., Theorem 1.4 [8, p. 109]) having the form

$$f(t,z) = \big(\exp(t\Delta_{\theta,\omega})g\big)(z).$$

This proves uniqueness. The representation (4.3) follows from Theorem 3.5. Further, we substitute in (4.3) the initial condition (4.2) and apply Theorem 3.6 with $u = -\varepsilon$. This yields

$$f(t,z) = \left(1 + \varepsilon\frac{e^{t\omega}-1}{\omega}\right)^{-\theta} \exp\left(-\frac{\varepsilon\omega z}{\omega e^{-t\omega} + \varepsilon(1-e^{-t\omega})}\right)$$
$$\times \left[\exp\left(\frac{e^{t\omega}-1}{\omega + \varepsilon(e^{t\omega}-1)}\Delta_\theta\right)h\right]\left(\frac{\omega^2 z e^{t\omega}}{(\omega + \varepsilon[e^{t\omega}-1])^2}\right)$$
$$\stackrel{\text{def}}{=} \left(1 + \varepsilon\frac{e^{t\omega}-1}{\omega}\right)^{-\theta} \exp\left(-\frac{\varepsilon\omega z}{\omega e^{-t\omega} + \varepsilon(1-e^{-t\omega})}\right) h_t(z).$$

Since $h \in \mathcal{A}_0$, by Theorem 3.6, $h_t \in \mathcal{A}_0$, and by Corollary 3.1, $h_t \in \mathcal{L}_0$ if $h \in \mathcal{L}_0$. The former yields that the solution belongs to $\mathcal{A}_\varepsilon$ and the latter does claim (iii). It remains to prove the convergence stated in (ii). The continuity of the operator $\exp(t\Delta_{\theta,\omega})$ yields that in $\mathcal{A}_0$

$$\begin{aligned} h_t(z) &= \left[\exp\left(\frac{e^{t\omega}-1}{b+\varepsilon(e^{t\omega}-1)}\Delta_\theta\right)h\right]\left(\frac{ze^{t\omega}}{(1+\varepsilon\omega^{-1}[e^{t\omega}-1])^2}\right) \\ &\to \left\{\exp\left(\frac{1}{\varepsilon}\Delta_\theta\right)h\right\}(0), \end{aligned}$$

if $\omega \neq 0$. The case $\omega = 0$ may be handled similarly. Therefore, the product in (4.4) tends to zero in $\mathcal{A}_\varepsilon$ when $t \to +\infty$. □

Given $N \in \mathbb{N}$, let $\mathcal{E}^{(N)}$ stand for the set of analytic functions $F:\mathbb{R}^N \to \mathbb{C}$. For $b > 0$, we set

$$\|F\|_{b,N} \stackrel{\text{def}}{=} \sup_{x\in\mathbb{R}^N}\left\{\left|F(x)\right|\exp\left(-b|x|^2\right)\right\}, \tag{4.4}$$

where $|x|$ is the Euclidean norm. Set

$$\mathcal{A}_a^{(N)} \stackrel{\text{def}}{=} \left\{F \in \mathcal{E}^{(N)} \mid \|F\|_{b,N} < \infty,\ \forall b > a\right\}, \quad a \geqslant 0. \tag{4.5}$$

This set equipped with the topology generated by the family of norms $\{\|\cdot\|_{b,N},\ b > a\}$ becomes a Fréchet space. Let $O(N)$ stand for the group of all orthogonal transformations of $\mathbb{R}^N$. A function $F \in \mathcal{E}^{(N)}$ is said to be isotropic if for every $U \in O(N)$ and all $x \in \mathbb{R}^N$, one has $F(Ux) = F(x)$. The subset of $\mathcal{E}^{(N)}$ consisting of isotropic functions is denoted by $\mathcal{E}_{\text{isot}}^{(N)}$. Let also $\mathcal{P}_{\text{isot}}^{(N)} \subset \mathcal{E}_{\text{isot}}^{(N)}$ stand for the set of isotropic polynomials. The classical Study–Weyl theorem (see [6]) implies that there exists a bijection between the set of all polynomials of a single complex variable $\mathcal{P}$ and $\mathcal{P}_{\text{isot}}^{(N)}$ established by

$$\mathcal{P}_{\text{isot}}^{(N)} \ni P(x) = p\big((x,x)\big) \in \mathcal{P},$$

where $(\cdot,\cdot)$ is the scalar product in $\mathbb{R}^N$. Obviously, each a function F having the form

$$F(x) = f\big((x,x)\big), \tag{4.6}$$

with a certain $f \in \mathcal{E}$, belongs to $\mathcal{E}_{\text{isot}}^{(N)}$. Given $\mathcal{X} \subset \mathcal{E}$, we write $\mathcal{X}(\mathbb{R}^N)$ for the subset of $\mathcal{E}_{\text{isot}}^{(N)}$ consisting of the functions obeying (4.6) with $f \in \mathcal{X}$. Consider

$$\mathcal{E}_{\text{isot}}^{(N)} \ni F \mapsto \left(\Delta + \left(\frac{d}{(x,x)} + b\right)(x,\nabla)\right)F \in \mathcal{E}_{\text{isot}}^{(N)},$$

where Δ and ∇ are the Laplacian and gradient in $\mathbb{R}^N$. For F and f satisfying (4.6), one has

$$\left(\Delta + \left(\frac{d}{(x,x)} + b\right)(x,\nabla)\right)F(x) = 4(\Delta_{\theta,\omega}f)\big((x,x)\big), \tag{4.7}$$

where Δ_θ and $\Delta_{\theta,\omega}$ are defined by (1.1) with

$$\theta = \frac{N+d}{2}, \qquad \omega = \frac{b}{4}. \tag{4.8}$$

Consider the following Cauchy problem

$$\frac{\partial F(t,x)}{\partial t} = \left(\Delta + \left(\frac{d}{(x,x)} + b\right)(x,\nabla)\right)F(t,x),$$
$$F(0,x) = G(x) \in \mathcal{E}_{\mathrm{isot}}^{(N)}, \tag{4.9}$$

where $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^N$.

Theorem 4.2. *For every $d \geqslant -N$, $b \in \mathbb{R}$, and G having the form*

$$G(x) = \exp\bigl[-\varepsilon(x,x)\bigr]h\bigl((x,x)\bigr), \quad h \in \mathcal{A}_0,\ \varepsilon \geqslant 0, \tag{4.10}$$

(i) *the problem* (4.9) *has a unique solution in $\mathcal{A}_\varepsilon^{(N)}$, which, for $t > 0$, may be written as follows*:

$$F(t,x) = \exp\left(-\frac{b(x,x)}{4(1-e^{-tb})}\right) \times \int_0^{+\infty} w_\theta\left(\frac{bs(x,x)}{4(1-e^{-tb})}\right) h\left(4s\frac{e^{tb}-1}{b}\right) s^{\theta-1} \times \exp\left(-s\left(1 + 4\varepsilon\frac{e^{tb}-1}{b}\right)\right) \mathrm{d}s, \tag{4.11}$$

where θ is given by (4.8);

(ii) *if in* (4.10) $\varepsilon > 0$, *the solution* (4.11) *converges to zero in $\mathcal{A}_\varepsilon^{(N)}$ when $t \to +\infty$*;

(iii) *if in* (4.10) $h \in \mathcal{L}_0 \subset \mathcal{A}_0$, *the solution* (4.11) *belongs either to $\mathcal{L}_0(\mathbb{R}^N)$, for $\varepsilon = 0$, or to $\mathcal{L}^-(\mathbb{R}^N)$, for $\varepsilon > 0$.*

The proof directly follows from Theorem 4.1 on the base of the correspondence formulas (4.6) and (4.7).

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