Asymptotic behavior of solutions of nonlinear neutral differential equations with impulses

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Abstract

This paper is concerned with a nonlinear neutral differential equations with impulses of the form

\[ x(t) + C(t)x(t - \tau) + P(t)f(x(t - \delta)) = 0, \quad t \geq t_0, \quad t \neq t_k, \quad (\ast) \]

\[ x(t_k) = b_k x(t_k^-) + (1 - b_k) \int_{t_k - \delta}^{t_k} P(s + \delta)f(x(s)) ds, \quad k = 1, 2, \ldots \quad (\ast\ast) \]

Sufficient conditions are obtained for every solution of (\ast) and (\ast\ast) that tends to a constant as \( t \to \infty \).

Keywords: Asymptotic behavior; Liapunov functional; Neutral differential equation; Impulse

1. Introduction and preliminaries

The theory of impulsive differential equations is now being recognized to be not only richer than the corresponding theory of differential equations without impulses but also represents a more natural framework for mathematical modelling of many real-world phenomena [1]. The
number of publications dedicated to its investigation has grown constantly in the recent years and a well-developed theory has taken in shape. See monographs [1,2] and references therein. However, the theory of impulsive functional differential equations has been less developed due to numerous theoretical and technical difficulties caused by their peculiarities. There are a few publications on qualitative theory. In particular, oscillation and asymptotic behavior of solutions of some impulsive delay differential equations have been studied by several authors (see [3–5,7–10]). Stability of some impulsive functional differential equations in more general form also has been studied by several authors (for example, see [5,6,12]). However, there is little in the way of results for the asymptotic behavior of solutions of impulsive neutral differential equations [10].

On the other hand, it is well known that the asymptotic constancy is widely investigated for delay differential equations. For example, see [15] and references therein.

In this paper, we study the asymptotic behavior of solutions of a nonlinear neutral delay differential equation with impulses of the form

\[ x(t) + C(t)x(t - \tau) + P(t)f(x(t - \delta)) = 0, \quad t \geq t_0, \quad t \neq t_k, \quad (1) \]

\[ x(t_k) = b_kx(t_k^-) + (1 - b_k) \int_{t_k-\delta}^{t_k} P(s + \delta)f(x(s))ds, \quad k = 1, 2, \ldots, \quad (2) \]

where \( \tau > 0, \delta > 0, C(t), P(t) \in PC([t_0, \infty), \mathcal{R}) \) and \( P(t) \geq 0, f \in C(R, R), 0 < t_k < t_{k+1}, \) with \( \lim_{k \to \infty} t_k = \infty, \) and \( b_k, k = 1, 2, \ldots, \) are constants. \( PC([t_0, \infty), \mathcal{R}) \) denotes the set of all functions \( g : [t_0, \infty) \to \mathcal{R} \) such that \( g \) is continuous for \( t_k \leq t < t_{k+1} \) and \( \lim_{r \to t_k^-} g(t) = g(t_k^-) \) exists for all \( k \geq 1. \)

In system (1)–(2) the impulsive term is also delayed, that is, it contains an integral term. A more general form was considered in [11,12], in which the existence and uniqueness of solutions and the stability were studied for the following more general impulsive differential equation

\[
\begin{align*}
\begin{cases}
x'(t) = f(t,x_t), & t \geq t_0, \quad t \neq t_k, \\
\Delta x(t) = I_k(t,x_t), & t = t_k, \quad k = 1, 2, \ldots.
\end{cases}
\end{align*}
\]

We note that though the impulse in (2) is a special form of the impulse term form, the method given in this paper will mark this impulse term form. We also note that when all \( b_k = 1, k = 1, 2, \ldots, \) system (1)–(2) reduces to the following delay differential equation without impulses

\[ x(t) + C(t)x(t - \tau) + P(t)f(x(t - \delta)) = 0, \quad t \geq t_0, \]

whose asymptotic behavior of solutions in some cases (for example, \( C(t) \equiv 0; \ C(t) = c; f(x) = x \) and \( C(t) \) and \( P(t) \) are continuous functions) have been studied by several authors (see [13,14]). Note that we apply our theorems to systems without impulses, and improve the result in [13].

With Eqs. (1)–(2), one associates an initial condition of the form

\[ x_{t_0} = \varphi(s), \quad s \in [-\rho, 0], \quad \rho = \max\{\tau, \delta\}, \quad (3) \]

where \( x_{t_0} = x(t_0 + s) \) for \( -\rho \leq s \leq 0 \) and \( \varphi \in PC([-\rho, 0], \mathcal{R}) = \{\varphi : [-\rho, 0] \to \mathcal{R} : \varphi \) is continuous everywhere except at the finite number of points \( \tilde{s} \) and \( \varphi(\tilde{s}^-) = \lim_{s \to \tilde{s}^-} \varphi(s) \) exist with \( \varphi(\tilde{s}^+) = \varphi(\tilde{s}). \)

A function \( x(t) \) is said to be a solution of Eqs. (1)–(2) satisfying the initial value condition (3) if
(i) \( x(t) = \varphi(t - t_0) \) for \( t_0 - \rho \leq t \leq t_0 \), \( x(t) \) is continuous for \( t \geq t_0 \) and \( t \neq t_k \) \((k = 1, 2, \ldots)\); (ii) \( x(t) + C(t)x(t - \tau) \) is continuously differentiable for \( t > t_0 \), \( t \neq t_k \), \( t \neq t_k + \tau \), \( t \neq t_k + \delta \) \((k = 1, 2, \ldots)\) and satisfies (1); (iii) \( x(t^+_k) \) and \( x(t^-_k) \) exist with \( x(t^+_k) = x(t_k) \) and satisfy (2).

As is customary, a solution of (1)–(2) is said to be nonoscillatory if it is eventually positive or eventually negative. Otherwise, it will be called oscillatory.

2. Main results

In connection with the nonlinear function \( f \), the impulsive perturbations \( b_k \) and the impulsive points \( t_k \) in (1)–(2), we assume

\( H_1 \): there is a constant \( M > 0 \) such that
\[
|\varphi(x)| \leq |f(x)| \leq M|x|, \quad x \in \mathbb{R}, \quad xf(x) > 0 \text{ for } x \neq 0;
\]
(4)

\( H_2 \): \( t_k - \tau \) is not an impulsive point, \( 0 < b_k \leq 1 \) \((k = 1, 2, \ldots)\) and \( \sum_{k=1}^{\infty}(1 - b_k) < \infty \).

Theorem 1. Let \((H_1)\) and \((H_2)\) hold. Assume that \( C(t_k) = b_kC(t_k^-) \) and

\[
\lim_{t \to \infty} |C(t)| = \mu < 1,
\]
(5)

\[
\limsup_{t \to \infty} \left[ \mu \left( 1 + \frac{P(t + \tau + \delta)}{P(t + \delta)} \right) + \int_{t-\delta}^{t+\delta} P(s + \delta) ds \right] < \frac{2}{M}. \]
(6)

Then every solution of (1)–(2) tends to a constant as \( t \to \infty \).

Proof. Let \( x(t) \) be any solution of (1)–(2). We shall prove that the limit \( \lim_{t \to \infty} x(t) \) exists and is finite. For this purpose, we rewrite (1)–(2) in the form

\[
\left[ x(t) + C(t)x(t - \tau) - \int_{t-\delta}^{t} P(s + \delta)f(x(s)) ds \right]' + P(t + \delta)f(x(t)) = 0,
\]
\( t \geq t_0, \ t \neq t_k \),
(7)

\[
x(t_k) = b_kx(t_k^-) + (1 - b_k) \int_{t_k^-}^{t_k} P(s + \delta)f(x(s)) ds, \quad k = 1, 2, \ldots.
\]
(8)

From (5) and (6) we can select an \( \epsilon > 0 \) sufficiently small such that \( \mu + \epsilon < 1 \) and

\[
\limsup_{t \to \infty} \left[ \mu + \epsilon \left( 1 + \frac{P(t + \tau + \delta)}{P(t + \delta)} \right) + \int_{t-\delta}^{t+\delta} P(s + \delta) ds \right] < \frac{2}{M}.
\]
(9)

Also, we select \( t^* > t_0 \) sufficiently large such that

\[
|C(t)| \leq \mu + \epsilon, \quad \text{for } t \geq t^*.
\]
(10)
Since
\[ \frac{f^2(x(t-\tau))}{x^2(t-\tau)} \geq 1 \geq \frac{|C(t)|}{\mu + \epsilon}, \quad t \geq t^*, \]
we have
\[ |C(t)|x^2(t-\tau) \leq (\mu + \epsilon)f^2(x(t-\tau)), \quad t \geq t^*. \] (11)

In what follows, for the sake of convenience, when we write a functional inequality without specifying its domain of validity, we mean that it holds for all sufficiently large \( t \).

Let \( V(t) = V_1(t) + V_2(t) \), where

\[ V_1(t) = \left[ x(t) + C(t)x(t-\tau) - \int_{t-\delta}^{t} P(s+\delta)f(x(s))\,ds \right]^2, \]

\[ V_2(t) = \int_{t-\delta}^{t} P(s+2\delta) \int_{s}^{t} P(u+\delta)f^2(x(u))\,du\,ds \]
\[ + (\mu + \epsilon) \int_{t-\delta}^{t} P(s+\tau+\delta)f^2(x(s))\,ds. \]

As \( t \neq t_k \), calculating respectively \( dV_i/dt \) \( (i = 1, 2) \) along the solution of (1)–(2) and using the inequality \( 2ab \leq a^2 + b^2 \), we have

\[ \frac{dV_1}{dt} = -2 \left[ x(t) + C(t)x(t-\tau) - \int_{t-\delta}^{t} P(s+\delta)f(x(s))\,ds \right] P(t+\delta)f(x(t)) \]
\[ = -P(t+\delta) \left[ 2x(t)f(x(t)) + 2C(t)x(t-\tau)f(x(t)) \right. \]
\[ - \int_{t-\delta}^{t} P(s+\delta)(2f(x(s))f(x(t)))\,ds \left. \right] \]
\[ \leq -P(t+\delta) \left[ 2x(t)f(x(t)) - |C(t)|x^2(t-\tau) - |C(t)|f^2(x(t)) \right. \]
\[ - f^2(x(t)) \int_{t-\delta}^{t} P(s+\delta)\,ds - \int_{t-\delta}^{t} P(s+\delta)f^2(x(s))\,ds \]

and

\[ \frac{dV_2}{dt} = -P(t+\delta) \int_{t-\delta}^{t} P(s+\delta)f^2(x(s))\,ds + P(t+\delta)f^2(x(t)) \int_{t-\delta}^{t} P(s+2\delta)\,ds \]
\[ + (\mu + \epsilon)P(t+\tau+\delta)f^2(x(t)) - (\mu + \epsilon)P(t+\delta)f^2(x(t-\tau)). \]
Therefore, from the above two inequalities and (11), we obtain

\[
\frac{dV}{dt} = \frac{dV_1}{dt} + \frac{dV_2}{dt} \\
\leq -P(t+\delta)\left[2x(t)f(x(t)) - |C(t)|f^2(x(t)) - f^2(x(t)) \int_{t-\delta}^{t} P(s+\delta)\,ds \right. \\
- \left. f^2(x(t)) \int_{t-\delta}^{t} P(s+2\delta)\,ds \right] + (\mu + \epsilon) P(t+\tau+\delta)f^2(x(t)) \\
= -P(t+\delta)f^2(x(t)) \left[\frac{2x(t)}{f(x(t))} - |C(t)| - \int_{t-\delta}^{t+\delta} P(s+\delta)\,ds - (\mu + \epsilon) \frac{P(t+\tau+\delta)}{P(t+\delta)} \right] \\
\leq -P(t+\delta)f^2(x(t)) \left[\frac{2}{M} - \int_{t-\delta}^{t+\delta} P(s+\delta)\,ds - (\mu + \epsilon) \left(1 + \frac{P(t+\tau+\delta)}{P(t+\delta)} \right) \right]. \quad (12)
\]

As \( t = t_k \), we have

\[
V(t_k) = \left[ x(t_k) + C(t_k)x(t_k-\tau) - \int_{t_k-\delta}^{t_k} P(s+\delta)f(x(s))\,ds \right]^2 \\
+ \int_{t_k-\delta}^{t_k} P(s+2\delta) \int_{s}^{t_k} P(u+\delta)f^2(x(u))\,du\,ds \\
+ (\mu + \epsilon) \int_{t_k-\tau}^{t_k} P(s+\tau+\delta)f^2(x(s))\,ds \\
= b_k^2 \left[ x(t_k^-) + C(t_k^-)x(t_k^--\tau) - \int_{t_k^-}^{t_k} P(s+\delta)f(x(s))\,ds \right]^2 \\
+ \int_{t_k^-}^{t_k} P(s+2\delta) \int_{s}^{t_k} P(u+\delta)f^2(x(u))\,du\,ds \\
+ (\mu + \epsilon) \int_{t_k^-}^{t_k} P(s+\tau+\delta)f^2(x(s))\,ds \\
\leq V(t_k^-). \quad (13)
\]

From (9), (12) and (13), we can get

\( P(t+\delta)f^2(x(t)) \in L^1(t_0, \infty) \),
and hence for any $\rho > 0$ we have
\[
\lim_{t \to \infty} \int_{t-\rho}^{t} P(s + \delta) f^2(x(s)) \, ds = 0.
\]

Since
\[
\int_{t-\delta}^{t} P(s + 2\delta) \int_{s}^{t} P(u + \delta) f^2(x(u)) \, du \, ds
\leq \int_{t-\delta}^{t} P(s + 2\delta) \, ds \int_{t-\delta}^{t} P(u + \delta) f^2(x(u)) \, du
\leq 2 \int_{t-\delta}^{t} P(u + \delta) f^2(x(u)) \, du \to 0 \quad \text{as } t \to \infty,
\]
and
\[
(\mu + \epsilon) \int_{t-\tau}^{t} P(s + \delta + \tau) f^2(x(s)) \, ds \leq 2 \int_{t-\tau}^{t} P(s + \delta) f^2(x(s)) \, ds \to 0 \quad \text{as } t \to \infty,
\]
it follows that $\lim_{t \to \infty} V_2(t) = 0$. On the other hand, by (9), (12) and (13), we can find that $V(t)$ is eventually decreasing. In view of $V \geq 0$, $\lim_{t \to \infty} V(t) = \beta$ exists and is finite. Thus $\lim_{t \to \infty} V(t) = \lim_{t \to \infty} V_1(t) = \beta$, that is,
\[
\lim_{t \to \infty} \left[ x(t) + C(t)x(t - \tau) - \int_{t-\delta}^{t} P(s + \delta) f(x(s)) \, ds \right]^2 = \beta. \tag{14}
\]

Next we will prove that the limit
\[
\lim_{t \to \infty} \left[ x(t) + C(t)x(t - \tau) - \int_{t-\delta}^{t} P(s + \delta) f(x(s)) \, ds \right]
\]
exists and is finite.

We let $y(t) = x(t) + C(t)x(t - \tau) - \int_{t-\delta}^{t} P(s + \delta) f(x(s)) \, ds$, then
\[
y(t_k) = x(t_k) + C(t_k)x(t_k - \tau) - \int_{t_k-\delta}^{t_k} P(s + \delta) f(x(s)) \, ds
\]
\[
= b_k \left[ x(t_k^-) + C(t_k^-)x(t_k^- - \tau) - \int_{t_k^-}^{t_k} P(s + \delta) f(x(s)) \, ds \right]
\]
\[
= b_k y(t_k^-).
\]
In view of (14), we have
\[
\lim_{t \to \infty} y^2(t) = \beta. \tag{15}
\]
Moreover, system (7)–(8) can be rewritten as

\[
\begin{aligned}
y(t) + P(t + \delta)f(x(t)) &= 0, \quad t \geq t_0, \quad t \neq t_k, \\
y(t_k) &= b_ky(t_k^-), \quad k = 1, 2, \ldots.
\end{aligned}
\] (16)

If \( \beta = 0 \), then \( \lim_{t \to \infty} y(t) = 0 \). If \( \beta > 0 \), then there exists an enough large \( T_1 \) such that \( y(t) \neq 0 \) for any \( t > T_1 \). Therefore for \( t_k > T_1, t \in [t_k, t_{k+1}) \) we have \( y(t) > 0 \) or \( y(t) < 0 \) because \( y(t) \) is continuous on \([t_k, t_{k+1})\). Without loss of generality, we assume that \( y(t) > 0 \) on \([t_k, t_{k+1})\), thus \( y(t) > 0 \) on \([t_{k+1}, t_{k+2})\). By induction, we conclude that \( y(t) > 0 \) on \([t_k, \infty)\). From (15), we have that

\[
\lim_{t \to \infty} y(t) = \lim_{t \to \infty} \left[ x(t) + C(t)x(t - \tau) - \int_{t-\delta}^{t} P(s + \delta)f(x(s)) \, ds \right] = \lambda
\] (17)

must exist and be finite. In view of (16), we have

\[
\int_{t-\delta}^{t} P(s + \delta)f(x(s)) \, ds = y(t - \delta) - y(t) + \sum_{t-\delta < t_k < t} [y(t_k) - y(t_k^-)]
\]

\[
= y(t - \delta) - y(t) - \sum_{t-\delta < t_k < t} (1 - b_k)y(t_k^-).
\]

We let \( t \to \infty \) and note that \( \sum_{k=1}^{\infty} (1 - b_k) < \infty \), we have

\[
\lim_{t \to \infty} \int_{t-\delta}^{t} P(s + \delta)f(x(s)) \, ds = 0.
\] (18)

From (17), we have

\[
\lim_{t \to \infty} \left[ x(t) + C(t)x(t - \tau) \right] = \lambda.
\] (19)

Next, we shall prove that

\[
\lim_{t \to \infty} x(t) \text{ exists and is finite.}
\] (20)

To this end, we need to show that \( |x(t)| \) is bounded. In fact, if \( |x(t)| \) is unbounded, then there exists a sequence \( \{s_n\} \) such that \( s_n \to \infty, |x(s_n^-)| \to \infty, \) as \( n \to \infty \) and

\[
|x(s_n^-)| = \sup_{t_0 \leq t \leq s_n} |x(t)|,
\]

where, if \( s_n \) is not an impulsive point then \( x(s_n^-) = x(s_n) \). Thus, we have

\[
|x(s_n^-) + C(s_n^-)x(s_n - \tau)| \geq |x(s_n^-)| - |C(s_n^-)||x(s_n - \tau)|
\]

\[
\geq |x(s_n^-)|(1 - \mu - \epsilon) \to \infty \quad \text{as} \quad n \to \infty
\]

which contradicts (19). So \( |x(t)| \) is bounded.

If \( \mu = 0 \), clearly \( \lim_{t \to \infty} x(t) = \lambda \), which shows that (20) holds.

If \( 0 < \mu < 1 \), it is easily to see that \( C(t) \) is eventually positive or negative. Otherwise, there is a sequence \( \tau_1, \tau_2, \ldots, \tau_k, \ldots \) with \( \tau_k \to \infty \) as \( k \to \infty \) such that \( C(\tau_k) = 0 \), so \( C(\tau_k) \to 0 \). It is a contradiction with \( \mu > 0 \).
By condition (5), one can find a sufficiently large $T_2$ such that for $t > T_2$, $|C(t)| < 1$. Set
$$
\alpha = \liminf_{t \to \infty} x(t), \quad \beta = \limsup_{t \to \infty} x(t),
$$
then we can choose two sequences $\{u_n\}$ and $\{v_n\}$ such that $u_n \to \infty$, $v_n \to \infty$ as $n \to \infty$, and
$$
\lim_{n \to \infty} x(u_n) = \alpha, \quad \lim_{n \to \infty} x(v_n) = \beta.
$$
For $t > T_2$, we consider the following two possible cases.

**Case 1.** $0 < C(t) < 1$ for $t > T_2$, we have
$$
\lambda = \lim_{n \to \infty} \left[ x(u_n) + C(u_n)x(u_n - \tau) \right] \leq \alpha + \mu \beta,
$$
and
$$
\lambda = \lim_{n \to \infty} \left[ x(v_n) + C(v_n)x(v_n - \tau) \right] \geq \beta + \mu \alpha.
$$
Thus, we get
$$
\beta + \mu \alpha \leq \alpha + \mu \beta,
$$
that is,
$$
\beta(1 - \mu) \leq \alpha(1 - \mu).
$$
Since $0 < \mu < 1$ and $\beta \geq \alpha$, it follows that $\beta = \alpha$. By (19) we obtain
$$
\beta = \alpha = \frac{\lambda}{1 + \mu},
$$
which shows that (20) holds.

**Case 2.** $-1 < C(t) < 0$ for $t > T_2$, we have
$$
\alpha = \lim_{n \to \infty} x(u_n) = \lim_{n \to \infty} \left[ x(u_n) + C(u_n)x(u_n - \tau) - C(u_n)x(u_n - \tau) \right] = \lambda + \mu \alpha,
$$
it follows that
$$
\alpha = \frac{\lambda}{1 - \mu}.
$$
Similarly,
$$
\beta = \lim_{n \to \infty} x(v_n) = \lim_{n \to \infty} \left[ x(v_n) + C(v_n)x(v_n - \tau) - C(v_n)x(v_n - \tau) \right] = \lambda + \mu \beta.
$$
Therefore $\alpha = \beta = \frac{\lambda}{1 - \mu}$. This shows that (20) holds.

According to the discussion above, we conclude that (20) holds, and so the proof of Theorem 1 is complete. \(\square\)

In the following Theorem 2 we assume that the condition $(H_2)'$ holds which is different from the condition $(H_2)$ in the sense that the condition $\sum_{k=1}^{\infty} (1 - b_k) < \infty$ is being replaced by $t_k - t_{k-1} \geq \eta$ for all $k$.

$(H_2)'$ $t_k - \tau$ is not impulsive point for all $k = 1, 2, \ldots$ and $0 < b_k \leq 1$ and there exists a constant $\eta > 0$ such that $t_k - t_{k-1} \geq \eta$ for all $k$. 
Theorem 2. Let \((H_1)\) and \((H_2)'\) hold. Assume that \(C(t_k) = b_k C(t_k^-)\) and (5), (6) hold. Then every solution of (1)–(2) tends to a constant as \(t \to \infty\).

Proof. From the proof of Theorem 1, we can also prove that (17) holds by using the conditions of Theorem 2. We also note that in the proof of Theorem 1, the condition \((H_2)\) was used only in the proof of (18). Therefore, to prove Theorem 2, we only need to prove that (18) holds by using condition \((H_2)'\). Since \(t_k - t_{k-1} \geq \eta\), it follows that the number of impulsive points in \((t - \delta, t)\) for \(t \geq t_0 + \delta\) is at most \(\left[\frac{\eta}{\eta}\right] = q\). Set \(t - \delta < t_i < t_{i+1} < \cdots < t_{i+q} < t\), \(i = i(t)\). Then, in view of (15),

\[
\lim_{t \to \infty} \sum_{t - \delta < t_k < t} \left[ y(t_k) - y(t_k^-) \right] = \lim_{i \to \infty} \left[ y(t_i) - y(t_i^-) + \cdots + y(t_{i+q}) - y(t_{i+q}^-) \right] = 0.
\]

Since

\[
\int_{t - \delta}^{t} P(s + \delta) f(x(s)) \, ds = y(t - \delta) - y(t) + \sum_{t - \delta < t_k < t} \left[ y(t_k) - y(t_k^-) \right],
\]

it follows that, by passing to the limit as \(t \to \infty\), we conclude that (18) holds. The proof of Theorem 2 is complete. \(\Box\)

By Theorems 1 and 2, we have the following asymptotic behavior result immediately.

Theorem 3. Either the conditions of Theorem 1 or the conditions of Theorem 2 imply that every oscillatory solution of (1)–(2) tends to zero as \(t \to \infty\).

Corollary 1. Assume that

\[
\limsup_{t \to \infty} \int_{t - \tau}^{t + \tau} p(s + \tau) \, ds < 2.
\]

Then every oscillatory solution of

\[
x'(t) + p(t)x(t - \tau) = 0
\]

tends to zero as \(t \to \infty\).

Remark. Corollary 1 improves Theorem 2 in [13] by relaxing the following condition in [13]

\[
\limsup_{t \to \infty} \int_{t - \tau}^{t} p(s + \tau) \, ds < 1.
\]

Theorem 4. The conditions in Theorem 1 together with

\[
\int_{t_0}^{\infty} P(t) \, dt = \infty
\]

imply that every solution of (1)–(2) tends to zero as \(t \to \infty\).
Proof. By Theorem 3, we only have to prove that every nonoscillatory solution of (1)–(2) tends to zero as \( t \to \infty \). Let \( x(t) \) be an eventually positive solution of (1)–(2), we shall prove \( \lim_{t \to \infty} x(t) = 0 \). As in the proof of Theorem 1, we can rewrite (1)–(2) in the form (16). Integrating from \( t_0 \) to \( t \) both sides of (16) produces
\[
\int_{t_0}^{t} P(s + \delta) f(x(s)) \, ds = y(t_0) - y(t) - \sum_{t_0 < t_k < t} (1 - b_k) y(t_k^-).
\]
By using (17) and \( \sum_{k=1}^{\infty} (1 - b_k) < \infty \), we have
\[
\int_{t_0}^{\infty} P(s + \delta) f(x(s)) \, ds < \infty,
\]
which, together with (21) yields \( \lim \inf_{t \to \infty} f(x(t)) = 0 \). We claim that
\[
\lim \inf_{t \to \infty} x(t) = 0.
\] (22)
Let \( \{s_m\} \) be such that \( s_m \to \infty \) as \( m \to \infty \) and \( \lim_{m \to \infty} f(x(s_m)) = 0 \). We must have \( \lim \inf_{t \to \infty} x(s_m) = c = 0 \). In fact, if \( c > 0 \), then there is a subsequence \( \{s_{m_k}\} \) such that \( x(s_{m_k}) \geq c/2 \) for \( k \) sufficiently large. By (H1) we have \( f(x(s_{m_k})) \geq \xi \) for some \( \xi > 0 \) and sufficiently large \( k \), which yields a contradiction because of \( \lim_{k \to \infty} f(x(s_{m_k})) = 0 \). Therefore, (22) holds. On the other hand, by Theorem 1, we have \( \lim_{t \to \infty} x(t) \) exists. Therefore \( \lim_{t \to \infty} x(t) = 0 \). Thus the proof of Theorem 4 is complete. \( \square \)

3. Example

Consider the following impulsive differential equation
\[
\begin{cases}
[x(t) + C(t)x(t - \frac{1}{2})]' + \frac{2 + \sin t}{30} [1 + \sin^2 x(t - \pi)] x(t - \pi) = 0, & t \geq 0, \ t \neq k, \\
x(k) = \frac{k}{k + 1} x(k^-) + \frac{1}{k + 1} \int_{k-\pi}^{k} \frac{2 + \sin(s + \pi)}{30} [1 + \sin^2 x(s)] x(s) \, ds, & k = 1, 2, \ldots,
\end{cases}
\]
where \( P(t) = \frac{2 + \sin t}{30}, \ f(x) = (1 + \sin^2 x)x, \ b_k = k/(k + 1), \ C(t) = \frac{1}{8k}, \ t \in [k - 1, k), \ k = 1, 2, \ldots \). Since \( t_k - t_{k-1} = 1 \) for all \( k \),
\[
\lim_{t \to \infty} |C(t)| = \frac{1}{8} < 1,
\]
\[
C(k) = \frac{k}{k + 1} C(k^-),
\]
\[
|x| \leq |(2 + \sin^2 x)x| \leq 2|x|, \quad x^2(1 + \sin^2 x) > 0 \quad (x \neq 0)
\]
and
\[
\lim \sup_{t \to \infty} \left[ \frac{1}{8} \left( 1 + \frac{2 + \sin(s + \frac{1}{2} + \pi)}{2 + \sin(s + \pi)} \right) + \int_{t-\pi}^{t+\pi} \frac{2 + \sin(s + \pi)}{30} ds \right] \leq \frac{1}{2} + \frac{2\pi}{15} < 1,
\]
so, by Theorem 2, every solution of (23) tends to a constant as \( t \to \infty \).
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References