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Root Groups

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Let $G(k)$ be a Chevalley group defined over a field k and constructed as in [6] from a faithful representation of a semi-simple Lie algebra whose root system is R.

Let H and U_a $(a \in R)$ be subgroups of $G(k)$ as defined in [6, p. 31]. Let R^+ be a positive system of roots, let $U^+ = \langle U_a : a \in R^+ \rangle$ and let $B = HU^+$. When k is algebraically closed, the subgroup H is a maximal torus and B is a Borel subgroup of G. In this case, it is well known that the groups U_a are the minimal subgroups of G which are normalized but not centralized by H. A question is: What are the minimal subgroups of $G(k)$ which are normalized but not centralized by H, when k is an arbitrary field?

There are several interesting results when k is a finite field [2, 4, 5]. Here we will prove an analogue of one of the results of [4] when the field k has at least five elements.

PROPOSITION. *Let $G(k)$, H and U be as above. Assume that k has at least five elements and if k is of characteristic 2, assume that it is perfect. If X is a subgroup of U which is normalized by H then $X = \langle U_a : U_a \subseteq X \rangle$.*

The proposition follows from a lemma which is of independent interest.

Let V be a Z-module, R^+ a finite subset of $V - \{0\}$, and S a subset of R^+ such that every element x of R^+ has a unique expression $x = \sum_{s \in S} a_s s$ with $a_s \geqslant 0$. Define the height of x to be $\sum_{s \in S} a_s$ and denote it by $\mathrm{ht}(x)$.

LEMMA. *Let G be a group generated by subgroups U_a $(a \in R^+)$ and H such that*

(i) $(U_a, U_b) \subseteq \langle U_{ia+jb} : ia + jb \in R^+, i, j > 0 \rangle$,

(ii) *for all $a \in R^+$, if $x \in U_a - \{e\}$, then $\langle x^H \rangle = U_a$,*

(iii) *for all $a \neq b \in R^+$, if $x \in U_a$ and $y \in U_b - \{e\}$, then there exists an element h in H such that*

$$hxh^{-1} = x, \qquad hyh^{-1} \neq y.$$

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Then any H-invariant subgroup X of $U^+ = \langle U_a : a \in R^+ \rangle$ *is generated by the subgroups* U_a *which it contains.*

Proof. Consider all triples $(G, H, \{U_a\}_{a \in R^+})$ as above. Let N_G be the number of subgroups $U_a \neq e$. If N_G is 0 or 1, then, in view of condition (ii), there is nothing to prove.

So let $N_G > 1$ and let γ be an element of greatest height such that $U_\gamma \neq e$. Now $(U_\gamma, U_a) = e$, for all $a \in R^+$ (by (i)), so U_γ is normal in G. Consider the triple $(\bar{G} = G/U_\gamma, \bar{H}, \{\bar{U}_a\}_{a \in R^+})$. Conditions (i) and (ii) clearly hold for $\bar{G}$. As for (iii), suppose that $\bar{x} \in \bar{U}_a$ and $\bar{y} \in \bar{U}_b - \{\bar{e}\}$, with $x \in U_a$, $y \in U_b$. By hypothesis, there exists an element $h \in H$ such that $hxh^{-1} = x$ and $hyh^{-1} \neq y$. Suppose $(hyh^{-1})^- = \bar{y}$. Then $(h, y) \in U_b \cap U_\gamma$ and $(h, y) \neq e$, so (ii) implies that $U_b = U_\gamma$ and therefore $\bar{y} = \bar{e}$, which contradicts the choice of y. So $\bar{G}$ satisfies (i), (ii) and (iii) whence $\bar{X} = \langle \bar{U}_a : \bar{U}_a \subseteq \bar{X},\ a \neq \gamma \rangle$ by induction.

Suppose $\bar{U}_a \subseteq \bar{X}$ and $u_a \in U_a - \{e\}$. Then $u_a u_\gamma \in X$ for some $u_\gamma \in U_\gamma$. If $u_\gamma \neq e$, then by (iii), $e \neq h u_\gamma h^{-1} u_\gamma^{-1} \in X \cap U_\gamma$ whence by (ii), we have $U_\gamma \subseteq X$. So $X = \langle U_a, U_\gamma : U_a \subseteq X \rangle$ in this case. And if $U_\gamma \nsubseteq X$, then $X = \langle U_a : \bar{U}_a \subseteq \bar{X} \rangle$. This is what we wanted to prove.

Proof of the proposition. $G = G(k)$ has generators $x_a(\xi) \in U_a$, $h_a(t) \in H$ $(a \in R,\ \xi \in k,\ t \in k^*)$ such that

$$(x_a(\xi), x_b(\eta)) = \prod_{\substack{i,j>0 \\ ia+jb \in R}} x_{ia+jb}(N_{a,b,i,j}\xi^i \eta^j),$$

$$h_a(t)\, x_b(\xi)\, h_a(t)^{-1} = x_b(t^{\langle b,a \rangle}\xi).$$

See, for example, [6, p. 30].

Let u and v be positive roots, and let $R_{u,v}$ be the integral closure of u, v in R. Then $u, v \in R_{u,v} \cap R^+$, so we need only check that (iii) holds in positive systems of rank 2.

If α, β is a basis of $R_{u,v} \cap R^+$, then one can check that (iii) holds with $h = h_\alpha(\xi)\, h_\beta(\eta)$ and ξ, η suitable elements of k^*. Hence conditions (i), (ii) and (iii) of the lemma hold. This completes the proof of the proposition.

COROLLARY. *Let k be algebraically closed. View G(k) as an algebraic group over k. If X is a connected, nilpotent subgroup normalized by H, then for some subset* S_X *of* R^+ *and some* $n \in N(H)$, *we have*

$$X = (X \cap H) \times \langle nU_a n^{-1} : a \in S_X \rangle.$$

This is an immediate consequence which the reader can prove himself, of the proposition, definitions and conjugacy of maximal tori.

Another corollary, which uses the argument in [1, p. 12–07] and which describes, for examples, the normalizers of elements of finite order in semi-simple groups over $\mathbb{C}$ is:

COROLLARY. *Let k be algebraically closed. View* $G(k)$ *as an algebraic group over k. Assume that k is not characteristic* 2 *or* 3. *If X is a closed subgroup normalized by H, then either X centralizes H, or else for some non-empty closed set of roots, we have* $X^0 = \langle T \cap X^0, U_a : a \in S_X \rangle$, *where* X^0 *is the connected component of X. Moreover,* $S_X = S_1 \cup S_2$, *where* $S_1 = \{a \in S_X : -a \in S_X\}$, $S_2 = \{b \in S_X : -b \notin S_X\}$, *and* S_1, S_2 *are both closed set of roots.*

The group $L_X = \langle U_a : a \in S_1 \rangle$ *is semi-simple, and the group* $U_X = \langle U_a : a \in S_2 \rangle$ *is a unipotent normal subgroup of X. Moreover,* X/X^0 *has representatives in* $N(H)$, *which induce graph automorphisms on the Dynkin diagram of* L_X *relative to the subtorus* $L_X \cap T$.

One uses the commutator formula and the calculation in [3, p. 295] to show that S_X is a closed set of roots, and therefore that both S_1 and S_2 are also closed. Since $\langle T, X^0 \rangle$ is generated by those of its Borel subgroups which contain T, the previous corollary is applicable to these subgroups; moreover, X normalizes $\langle T, X^0 \rangle$. These facts imply all the assertions.

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