Absolute Valued Algebras Containing a Central Idempotent

MOHAMED LAMEI EL-MALLAH*

Department of Mathematics, Faculty of Science, Cairo University, Cairo, Giza, Egypt

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1. INTRODUCTION

Let $A$ be an absolute valued real algebra not necessarily associative or finite dimensional but containing a central idempotent, that is, a non-zero idempotent which commutes with all elements of $A$. We show that $A$ is necessarily an inner product space (Theorem 3.6). This result is an important generalization of a result given in [10]. Moreover, using Theorem 3.6, we show that $A$ must be an algebra with an involution (Theorem 3.7). This result and our results given in [3, 4] enable us to give a classification of every finite dimensional absolute valued algebra satisfying the identity $(x, x, x) = 0$ (Theorem 4.1). We also show that Theorem 3.7 would not be true if “central idempotent” was weakened to “central element.” In [3] we have shown that every finite dimensional absolute valued algebra is an inner product space. It is also to be noted that all the known examples of infinite dimensional absolute valued algebras are inner product spaces. It may be conjectured that every absolute valued algebra is an inner product space.

2. PRELIMINARIES AND NOTATIONS

By an absolute valued algebra we shall mean a normed space which is also an algebra such that $\|xy\| = \|x\| \|y\|$. Throughout $A$ will denote an absolute valued algebra over the reals $R$. We assume that $A$ contains a central idempotent $e$. $A$ is an inner product space if on $A$ a positive definite inner product $((x, y))$ is defined, so that $\|x\|^2 = ((x, x))$. $A$ is an algebra with an involution if an operation $*$ defined on $A$ satisfies the following

* Current address: King Saud University, Abha Branch, Box 157. Abha, Saudi Arabia.
conditions:

(i) \((\lambda x + \mu y)^* = \lambda \hat{x} + \mu \hat{y}\);
(ii) \(\hat{x}^* = x\); (iii) \(x\bar{x} = \bar{x}x\);
(iv) \((xy)^* = j\bar{x}\); (v) \(\|\hat{x}\| = \|x\|\);

for any \(\lambda, \mu \in R\) and \(x, y \in A\).

By \(R, C, H, D, \tilde{H}\), and \(\tilde{D}\) we shall, respectively, denote the real field, the complex field, the quaternion algebra, the Cayley–Dickson (or octonion) algebra, the para-quaternion algebra, and the para-octonion algebra \([6, 3]\). \(\mathcal{C}\) will denote the algebra of all complex numbers with the usual addition and scalar multiplication, where the product of \(x\) and \(y\) is equal to \(\hat{x}\hat{y}\). By \(P\) we shall denote the pseudo-octonion algebra \([5]\). The subspace spanned by a set of elements will be denoted by putting brackets around the set of elements, and \((x, y)\) will denote the commutator of \(x, y\). It is obvious that an absolute valued algebra contains no divisors of zero.

3. Results

The two following lemmas are proved in \([9]\) and are useful for our discussion:

**Lemma 3.1.** If all the elements of a subset \(B\) of \(A\) commute with each other, then \([B]\) is an inner product space.

**Lemma 3.2.** Let \(x, y \in A\), \(\|x\| = \|y\| = 1\), and \(\|x - y\| = 2\). If \(x\) commutes with \(y\), then \(x + y = 0\).

Now for any \(x \in A\) linearly independent to the central idempotent \(e\), we have

**Lemma 3.3.** In the subspace \([e, x]\) the following are equivalent:

(i) \(e\) is orthogonal to \(x\),
(ii) \(x^2 = -\|x\|^2 e\).

**Proof.** (i) \(\Rightarrow\) (ii). Let \(e\) be orthogonal to \(x\) and assume \(\|x\| = 1\). We get

\[\|x^2 - e\| = \|x - e\| \|x + e\| = 2.\]

Since \(\|x^2\| = \|x\|^2 = 1\), \(\|e\| = 1\), and \(e\) commutes with \(x^2\), we have, according to Lemma 3.2, \(x^2 + e = 0\). That is, \(x^2 = -e\). If \(\|x\| \neq 1\) we take \(y = x/\|x\|\). Then \(\|y\| = 1\) and, as was shown before, \(y^2 = -e\). So \(x^2 = -\|x\|^2 e\).
(ii) $\rightarrow$ (i). Let $x^2 = -\|x\|^2 e$ and suppose that $e$ is not orthogonal to $x$. Since $[e, x]$ is an inner product space, it follows that

$$[e, x] = [e, x_0],$$

where $\|x_0\| = 1$ and $e$ is orthogonal to $x_0$. Consequently, $x = \alpha e + \beta x_0$ with $\alpha \neq 0, \beta \neq 0$, in $R$. Now since $x^2 = -\|x\|^2 e$ and since $x_0^2 = -e$ from the first half of the proof, we get

$$-\|x\|^2 e = (\alpha^2 - \beta^2) e + 2\alpha\beta e x_0.$$

From this, and from the fact that $A$ has no divisors of zero, it follows that

$$-\|x\|^2 e = (\alpha^2 - \beta^2) e + 2\alpha\beta e x_0.$$

Thus $x_0 = ye$, which is impossible since $e$ is orthogonal to $x_0$. Therefore $e$ must be orthogonal to $x$.

We now consider the non-empty subset $B = \{x \in A: x^2 = -\|x\|^2 e\}$ of $A$ and prove:

**Lemma 3.4.** $B$ is a subspace of $A$.

**Proof:** For any $\alpha \in R$ and $x \in B$, we have

$$(\alpha x)^2 = \alpha^2 x^2 = \alpha^2 (-\|x\|^2 e) = -\|\alpha x\|^2 e,$$

which implies $\alpha x \in B$.

Let $x$ and $y$ be two elements in $B$. To prove that $x + y \in B$, we distinguish two cases:

1. If $x$ commutes with $y$. Then, according to Lemma 3.1, $[e, x, y]$ is an inner product space and, by Lemma 3.1, we have

$$((e, x + y)) = ((e, x)) + ((e, y)) = 0.$$

That is, $e$ is orthogonal to $x + y$. Consequently, applying Lemma 3.3, $(x + y)^2 = -\|x + y\|^2 e$. Hence $x + y \in B$.

2. If $x$ does not commute with $y$. Since $e$ commutes with $x + y$ and $x - y$, $[e, x + y]$ and $[e, x - y]$ are two 2-dimensional inner product spaces. Consequently, we get $[e, x + y] = [e, z]$ with $\|z\| = 1$ and $e$ is orthogonal to $z$ and $[e, x - y] = [e, w]$ with $\|w\| = 1$ and $e$ is orthogonal to $w$. It then follows that

$$x + y = \alpha e + \beta z, \quad \beta \neq 0$$

$$x - y = \lambda e + \mu w, \quad \mu \neq 0.$$
From which

\[(x + y)^2 = (x^2 - \beta^2)e + 2\alpha\beta ez\]
\[(x - y)^2 = (\lambda^2 - \mu^2)e + 2\lambda\mu ew.\]

Adding these two equalities, we have

\[2(-\|x\|^2 - \|y\|^2)e = (x^2 + \lambda^2 - \beta^2 - \mu^2)e + 2\alpha\beta ez + 2\lambda\mu ew.\]

Since \(A\) has no divisors of zero we obtain

\[2\alpha\beta z + 2\lambda\mu w = ye.\]

Now if \(\alpha \neq 0\) or \(\lambda \neq 0\) then \(z\) commutes with \(w\) and so

\[0 = (x + y, x - y) = -(x, y) + (y, x) = 2(y, x),\]

which implies that \(x\) commutes with \(y\), a contradiction. Therefore, \(\alpha = \lambda = 0\) and we get

\[x + y = \beta z \quad \text{with} \quad \|x + y\| = |\beta|\]
\[x - y = \mu w \quad \text{with} \quad \|x - y\| = |\mu|\]

and consequently,

\[(x + y)^2 = -\beta^2 e = -\|x + y\|^2 e.\]

Hence \(x + y \in B\).

**Lemma 3.5.** \(B\) is an inner product space.

**Proof.** Let \(x, y\) be two elements in \(B\) with \(\|x\| = \|y\| = 1\). According to Schoenberg's theorem [8], it is sufficient to show that the inequality \(\|x + y\|^2 + \|x - y\|^2 \geq 4\) holds. Since \(x, y \in B\) and \(\|x\| = \|y\| = 1\), \(x^2 = y^2 = -e\). Therefore

\[
\|x + y\|^2 + \|x - y\|^2 = \|(x + y)^2\| + \|(x - y)^2\|
\]
\[
= \|-2e + xy + yx\| + \| -2e - xy - yx\|
\]
\[
\geq \|-4e\|
\]
\[
= 4 \|e\| = 4
\]

and the lemma is proved.

Now for any \(x \in A\), since \([e, x]\) is an inner product space, we can write
$x$ as
\[ x = \alpha e + \beta x_0, \]
where $\|x_0\| = 1$ and $e$ is orthogonal to $x_0$, i.e., $x_0^2 = -e$, which implies $x_0 \in B$ and, consequently, $\beta x_0 \in B$. Since $[e] \cap B = \{0\}$, $A$ can be written as
\[ A = [e] + B, \]
the direct sum of subspaces.

We are now able to prove

**Theorem 3.6.** Any absolute valued algebra containing a central idempotent is an inner product space.

**Proof.** Let $x, y$ be two elements in $A$. Since
\[ x = \lambda e + a, \quad a \in B \]
\[ y = \mu e + b, \quad b \in B \]
we define an inner product $((x, y))$ on $A$ as
\[ ((x, y)) = \lambda \mu + ((a, b)), \]
where $((a, b))$ denotes the inner product defined on $B$. We now show:

(i) $((x, x)) = \lambda^2 + ((a, a)) = \lambda^2 + \|a\|^2 \geq 0$ and $((x, x)) = 0$ if and only if $\lambda = 0$ and $a = 0$, i.e., $x = 0$.

(ii) It is clear that $((x, y)) = ((y, x))$.

(iii) Also
\[ ((y, x)) = (\gamma \lambda) \mu + ((\gamma a, b)) \]
\[ = \gamma [\lambda \mu + ((a, b))] = \gamma ((x, y)). \]

(iv) Let $z = \alpha e + c$, $c \in B$. Then we have
\[ ((x + y, z)) = (\lambda + \mu) \alpha + ((a + b, c)) \]
\[ = \lambda \alpha + ((a, c)) + \mu \alpha + ((b, c)) \]
\[ = ((x, z)) + ((y, z)). \]

Hence $A$ is an inner product space.

Using Theorem 3.6 we can prove:
THEOREM 3.7. Any absolute valued algebra $A$ containing a central idempotent must admit an involution.

Proof. We have shown that

$$A = [e] + B,$$

the direct sum of spaces, where the subspace $B$ contains all elements of $A$ which are orthogonal to $e$.

Now for any element $x = xe + a$ in $A$, where $a \in B$, define $\dot{x} = xe - a$. Let $y = \beta e + b$ be an element in $A$. Then we can show:

(i) Since $\dot{x} + py = (\lambda x + \mu y)(\dot{e} + (\lambda a + \mu b))$ and $\lambda a + \mu b \in B$, we get

$$(\lambda x + \mu y)^* = (\lambda x + \mu b)(\dot{e} + (\lambda a + \mu b)) = \lambda \dot{x} + \mu \dot{y}.$$  

(ii) Since $x$ is central in $A$, it follows that $x\dot{x} = \dot{x}x$.

(iii) To prove $(xy)^* = \dot{y}\dot{x}$, we know that

$$xy = \alpha \beta e + \alpha eb + \beta ea + ab.$$  

But since $((e, eb)) = ((e, b)) = 0$, $eb \in B$. Similarly $ea \in B$. Therefore, in view of (i), we have

$$(xy)^* = \alpha \beta e - \alpha eb - \beta ea + (ab)^*.$$  

Since $\dot{y}\dot{x} = \alpha \beta e - \beta ea - \alpha eb + ba$, it is sufficient to show that $(ab)^* = ba$ for any two elements $a, b \in B$. In fact, according to property (i) we can assume, without loss of generality, that $\|a\| = \|b\| = 1$. Let us write

$$ab = \lambda e + c \quad \text{with} \quad c \in B. \quad (1)$$  

But since $a + b \in B$, we have

$$(a + b)^2 = -\|a + b\|^2 e,$$

from which

$$-2e + ab + ba = -\|a + b\|^2 e.$$  

Whence

$$ab + ba = \mu e. \quad (2)$$  

From (1), (2) we get

$$ba = (\mu - \lambda)e - c.$$
Now from

\[ ((e, ab)) = ((-a^2, ab)) = -((a, b)) \]
\[ ((e, ba)) = ((-a^2, ba)) = -((a, b)) \]

we deduce

\[ ((e, ab)) = ((e, ba)), \]

that is,

\[ \lambda = \mu - \lambda. \]

Hence \((ab)^* = ba\) as was required.

(v) From \[\|\hat{x}\|^2 = \|xe - a\|^2 = a^2 + \|a\|^2 = \|x\|^2,\] we get \[\|\hat{x}\| = \|x\|.\]

4. **Classification of Finite Dimensional Absolute Valued Algebras Satisfying \((x, x, x) = 0\)**

It should be noted that the dimension of a finite dimensional real division algebra can only be 1, 2, 4, or 8. Throughout \(A\) will be a finite dimensional absolute valued algebra satisfying the identity \((x, x, x) = 0\). If \(\dim A \leq 2\), we know that \(A\) is isomorphic to \(R, \mathbb{C}, \) or \(\mathcal{C}\). In [3] we have shown that if \(\dim A = 4\) then \(A\) is isomorphic to \(H\) or \(\hat{H}\), and if \(\dim A = 8\) and \(A\) contains no central idempotent, than \(A\) is isomorphic to \(P\). Now, to complete the classification, let \(\dim A = 8\) and assume \(A\) contains a central idempotent. Then, according to Theorem 3.7, \(A\) is an algebra with an involution. Applying Theorem 6 of [4], \(A\) is isomorphic to \(D\) or \(\hat{D}\). Thus we have proved:

**Theorem 4.1.** Any finite dimensional absolute valued algebra satisfying the identity \((x, x, x) = 0\) is isomorphic to \(R, \mathbb{C}, \hat{C}, H, \hat{H}, D, \hat{D}, \) or \(P\).

**Corollary 4.2.** Any finite dimensional absolute valued algebra satisfying the identity \((x, x, x) = 0\) is flexible.

**Corollary 4.3.** The pseudo-octonion algebra \(P\) is the only (up to an isomorphism) finite dimensional absolute valued algebra which satisfies the identity \((x, x, x) = 0\) and does not contain a central idempotent.
5. Example

We show that Theorem 3.7 would not be true if "central idempotent" was weakened to "central element." In fact in [4] we proved that if an absolute valued algebra $A$ with an involution contains a central element $a$ which is linearly independent from $a^2$, then $A$ must be isomorphic to $\mathbb{C}$ or $\mathbb{C}$, i.e., dim $A = 2$. But in [2] we gave an example $B$ of a 4-dimensional absolute valued algebra in which there exists a central element $a$ which is linearly independent from $a^2$. We conclude that $B$ does not admit an involution. Therefore the condition "central idempotent" is essential in Theorem 3.7.

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References