Supremum principles for the higher-order derivatives of solutions to the Dirichlet problem for parabolic equations without compatibility conditions between the data

Denis R. Akhmetov \(^a\), Renato Spigler \(^b\),\(^*\)

\(^a\) Sobolev Institute of Mathematics, 4 Acad. Koptyug prosp., 630090 Novosibirsk, Russia
\(^b\) Dipartimento di Matematica, Università “Roma Tre”, I Largo San Leonardo Murialdo, 00146 Rome, Italy

Received 25 August 2005
Available online 22 August 2006
To Louis Nirenberg on his 80th birthday

Abstract

The Dirichlet problem is considered for the heat equation \(u_t = au_{xx}, a > 0\) a constant, for \((x, t) \in [0, 1] \times [0, T]\), without assuming any compatibility condition between initial and boundary data at the corner points \((0, 0)\) and \((1, 0)\). Under some smoothness restrictions on the data (stricter than those required by the classical maximum principle), weak and strong supremum and infimum principles are established for the higher-order derivatives, \(u_t\) and \(u_{xx}\), of the bounded classical solutions. When compatibility conditions of zero order are satisfied (i.e., initial and boundary data coincide at the corner points), these principles allow to estimate the higher-order derivatives of classical solutions uniformly from below and above on the entire domain, except that at the two corner points. When compatibility conditions of the second order are satisfied (i.e., classical solutions belong to \(C^{2,1}_{x,t}\) on the closed domain), the results of the paper are a direct consequence of the classical maximum and minimum principles applied to the higher-order derivatives. The classical principles for the solutions to the Dirichlet problem with compatibility conditions are generalized to the case of the same problem without any compatibility condition. The Dirichlet problem without compatibility conditions is then considered for general linear one-dimensional parabolic equations. The previous results as well as some new properties of the corresponding Green functions derived here allow
to establish uniform $L^1$-estimates for the higher-order derivatives of the bounded classical solutions to the general problem.

© 2006 Elsevier Inc. All rights reserved.

MSC: 35A05; 35B05; 35B50; 35K05; 35K20; 35R05

Keywords: Heat equation; Parabolic equations; Dirichlet problem with and without compatibility conditions between initial and boundary data; Green functions; Classical solutions; Maximum principles; Strong maximum principle; Weak maximum principle; Uniform $L^1$-estimates for higher-order derivatives

0. Introduction

In the qualitative theory of partial differential equations of the parabolic type, the maximum principle plays a special role concerning classical solutions of several initial–boundary value problems. Exploiting such a principle, for instance, uniqueness of classical solutions and their continuous dependence on initial and boundary data can be established, see [6,10,13,19–21,25,26,29], e.g. Such fundamental results, well known for a long time for boundary value problems with compatibility conditions between initial and boundary data, can be found in almost every textbook on partial differential equations, see [27], e.g. On the other hand, at the present time, boundary value problems without any compatibility condition between initial and boundary data, have been playing an important role. However, unboundedness of all derivatives of classical solutions to the Dirichlet problem in the neighborhoods of the points where such conditions are not satisfied, represents a serious difficulty. For this reason, a general theory for this type of initial–boundary value problems has not yet been developed, at least to a satisfactory level, even for linear second-order parabolic equations with constant coefficients.

In [27, Section 3], it was pointed out that, confining to the case of solutions of the heat equation, continuous in the closed domain $0 \leq x \leq l$, $0 \leq t \leq T$ is too restrictive. In fact, even in the simplest problem of the cooling process of a uniformly heated rod, whose endpoints are kept at a zero temperature and subject to an initial temperature distribution, $u(x,0) = \text{const} \neq 0$, the solution will be discontinuous at the corner points, $(0,0)$ and $(l,0)$. It seems therefore necessary to include the possibility of boundary value problems with piecewise continuous initial data, not assuming that the initial data match the boundary conditions.

On the other hand, it was observed in [25, Introduction], that the aforementioned boundary value problems have not been fully investigated even in the simplest case of the heat equation. This question is discussed in detail in [17].

In this paper, we investigate the Dirichlet problem for general linear one-dimensional parabolic equations of the second order, with variable coefficients, with and without compatibility conditions between initial and boundary data. In Section 1, we state the problem to be investigated. The purposes of the paper are first put forth (Section 1.1), then we formulate the Dirichlet problem with compatibility conditions and make some remarks (Section 1.2). In Section 1.3, the delicate issue of the compatibility conditions for the Dirichlet problem for parabolic equations is discussed in detail for the first time. Section 1.4 is devoted to the main problem under investigation: The Dirichlet problem, in general without any compatibility condition. Section 1.5 is devoted to the special (and delicate) case of this kind, occurring when the compatibility conditions of zero order (Section 1.3.2) are satisfied. A very important example of a special bounded classical solution to such a delicate problem could perhaps be constructed, which does not solve the problem as a function continuous up to the corner points laying on the initial line, $t = 0$, even
though the compatibility conditions of zero order are fulfilled at these points. This conjecture is
formulated as Problem 1.15. In Section 1.6, we recall the definition of “continuity modulus” of a
given uniformly continuous function, give additional definitions and remarks, and introduce cer-
tain function spaces, depending on an arbitrary continuity modulus. The “weak and strong supre-
mum and infimum principles” for the higher-order derivatives of bounded classical solutions to
the Dirichlet problem for the heat equation are presented in Section 2. Also, two-side estimates
of classical solutions are established there, and a hypothesis about an extension of the aforemen-
tioned weak supremum principle to general parabolic equations of the second order is formulated.
Section 3 is devoted to the Green functions for the homogeneous Dirichlet problem. The def-
inition of fundamental solution to a linear second-order one-dimensional parabolic equation and
that of Green function for the homogeneous Dirichlet problem for the aforementioned equation
are recalled in Section 3.1. Some remarks on uniqueness of such functions are made. In Sec-
tion 3.2, the known properties of fundamental solutions and Green functions are recalled. Some
necessary notation and some properties of certain fundamental solutions and Green functions are
given in Section 3.3. Furthermore, in Section 3.4, we confer the definition of the Green function
for the Cauchy problem with the definition of the fundamental solution, and realize that these are,
in general, quasi-equal. The main achievements of Section 3 are presented in Sections 3.5 and
3.6 which deal with some new properties of the Green functions. In Section 4, finally, resting on
the results of the previous Sections 0, 2, and 3, uniform \( L^1 \)-estimates for higher-order derivatives
of bounded classical solutions to the Dirichlet problem are obtained. A considerably stronger
result than the previous one can be established in the case of homogeneous Dirichlet problems.
The paper ends with a short summary, where the main results are discussed.

The subject of this paper has an independent interest from the point of view of the general the-
ory of linear partial differential equations of parabolic type (in particular, estimating \( L^1 \)-norms
of higher-order derivatives of classical solutions have a special interest in itself [3]). It is also con-
ected to the problem of existence of global (in time) solutions to nonlinear parabolic equations.
In connection with this issue, we emphasize Vaĭgant’s example [28], which shows nonexistence
of global (in time) classical solutions to the Navier–Stokes system, despite the fact that a theo-
rem for the existence of a local (in time) classical solution is available. We stress that a similar
phenomenon can be observed even in considerably simpler cases. For instance, on the basis of
[1, Lemma 5], one can construct an example of blow-up of the higher-order derivatives of the
classical solution, \( u(x, t) \), to the Cauchy problem

\[
u_t = u_{xx} + f(x, t) \quad \text{for } x \in \mathbb{R}, \ t > 0, \quad u(x, 0) \equiv 0 \quad \text{for } x \in \mathbb{R},
\]

with a uniformly continuous bounded source-term, \( f(x, t) \in C(\mathbb{R}^2) \). Namely, for given arbitrary
\( x_0 \in \mathbb{R} \) and \( T > 0 \), the function \( f(x, t) \) can be chosen in such a way that: (1) a local classical
solution to the Cauchy problem exists up to time \( T \); but (2) the highest-order derivatives of
the solution \( u_t \) and \( u_{xx} \) at \( x = x_0 \) will diverge (blow-up) to infinity when \( t \) approaches \( T \), i.e.,
\( \lim_{t \to T^-} u_t(x_0, t) = +\infty \), and \( \lim_{t \to T^-} u_{xx}(x_0, t) = +\infty \). In particular, see [9, Section 113]:

\[
u_t(x_0, T) := \lim_{t \to T^-} \frac{u(x_0, T) - u(x_0, t)}{T - t} = +\infty.
\]

Recall that, for the classical solutions of the Dirichlet problem

\[
u_t = au_{xx} \quad \text{in } QT,
\]

(0.1)
where $a > 0$ is a constant and $Q_T := \{(x,t) \in (0,1) \times (0,T)\}$, $T > 0$, the following theorems hold (see [10,13,24] and Remark 1.7):

**Theorem 0.1** (Weak maximum and minimum principles). If $u(x,t)$ is a classical solution of problem (0.1)–(0.3), then

$$
\max_{\overline{Q}_T} u(x,t) = \max_{\Gamma} u(x,t), \quad \min_{\overline{Q}_T} u(x,t) = \min_{\Gamma} u(x,t),
$$

where $\Gamma := \overline{Q}_T \setminus Q_T$ is the parabolic boundary of $\overline{Q}_T$.

**Theorem 0.2** (Strong maximum and minimum principles). Let $u(x,t)$ be a classical solution to the problem (0.1)–(0.3). If

$$u(x_0,t_0) = \max_{\Gamma} u(x,t) \quad \text{or} \quad u(x_0,t_0) = \min_{\Gamma} u(x,t)
$$
at some point $(x_0,t_0) \in Q_t$, then

$$u(x,t) \equiv u(x_0,t_0) \quad \text{in} \ \overline{Q}_{t_0}.$$

One of the main results of this paper is the following generalization of Theorem 0.1.

**Theorem 0.3.** If $u(x,t)$ is a bounded classical solution of problem (2.1)–(2.3) (see Remark 1.7), then

$$
\sup_{\overline{Q}_T} u(x,t) = \sup_{\tilde{\Gamma}} u(x,t), \quad \inf_{\overline{Q}_T} u(x,t) = \inf_{\tilde{\Gamma}} u(x,t),
$$

where $\tilde{Q}_T := \overline{Q}_T \setminus ((0,0) \cup (1,0))$ and $\tilde{\Gamma} := \Gamma \setminus ((0,0) \cup (1,0))$.

We omit the proof.

**Remark 0.4.** Let the initial data $\phi(x)$ belong to the space $C[0,1]$, while the boundary data $\psi_1(t)$ and $\psi_2(t)$ belong to $C[0,T]$. If $u(x,t)$ is the unique bounded classical solution of the problem (2.1)–(2.3) (see Remark 1.7) and, in addition,

$$\varphi(0) > \psi_1(0) \quad \text{and} \quad \varphi(1) < \psi_2(0)
$$
or, alternatively,

$$\varphi(0) < \psi_1(0) \quad \text{and} \quad \varphi(1) > \psi_2(0),
$$

then

$$
\sup_{\overline{Q}_T} u_t(x,t) = +\infty, \quad \inf_{\overline{Q}_T} u_t(x,t) = -\infty.
$$
One of the main purposes of this paper is to establish that, for the unique bounded classical solution, \( u(x, t) \), to problem (2.1)–(2.3) the following properties hold true, whenever the derivatives \( u_t(x, t) \) and \( u_{xx}(x, t) \) are continuous in \( \bar{Q}_T \):

1. if \( \varphi(0) \geq \psi_1(0) \) and \( \varphi(1) \geq \psi_2(0) \), then
\[
\sup_{\bar{Q}_T} u_t(x, t) = \sup_I u_t(x, t);
\]

2. if \( \varphi(0) \leq \psi_1(0) \) and \( \varphi(1) \leq \psi_2(0) \), then
\[
\inf_{\bar{Q}_T} u_t(x, t) = \inf_I u_t(x, t).
\]

The initial–boundary data of the problem (2.1)–(2.3) need to enjoy the following smoothness properties:

\[
\varphi(x) \in C^2(0, 1), \quad \psi_1(t), \psi_2(t) \in C^1(0, t].
\]

1. Statement of the problem

1.1. The main goal and other purposes of the paper

The main goal of the paper is to establish uniform \( L^1 \)-estimates for the higher-order derivatives of bounded classical solutions of the Dirichlet problem for linear second-order one-dimensional parabolic equations of general form, without assuming any compatibility condition between initial and boundary data. To this purpose, a method, based on a number of auxiliary results is proposed. Such results have an independent interest. More precisely, in order to exploit this method, it is necessary:

(a) to establish a supremum principle for higher-order derivatives (Theorem 2.3);
(b) to generalize the well-known (classical) maximum principle (Theorem 0.3);
(c) to obtain certain new, rather special, estimates for the Green functions associated to the present problems (Theorems 3.18 and 3.20).

1.2. The Dirichlet problem with compatibility conditions

Let \( T > 0 \) be an arbitrary fixed constant, and \( L \) the operator
\[
Lu := u_t - a(x, t)u_{xx} - b(x, t)u_x - c(x, t)u,
\]
where the coefficients \( a(x, t), b(x, t), \) and \( c(x, t) \) are real-valued functions defined in \( Q_T := \{ (x, t) \in (0, 1) \times (0, T) \} \). The operator \( L \) is called parabolic in \( Q_T \) if \( a(x, t) > 0 \) in \( Q_T \), and uniformly parabolic in \( Q_T \) if
\[
\inf_{Q_T} a(x, t) > 0.
\]
Sometimes we consider the operator \( L \) in the slab \( H_T := \{(x,t) \in \mathbb{R} \times (0, T]\}, assuming that it is parabolic or uniformly parabolic in \( H_T \). With the symbols \( \overline{Q}_T \) and \( \overline{H}_T \) we denote the closures of the sets \( Q_T \) and \( H_T \), respectively. \( \Gamma := \overline{Q}_T \setminus Q_T \) is the parabolic boundary of \( \overline{Q}_T \), and 
\[
\overline{Q}_T := \overline{Q}_T \setminus \{(0,0) \cup (1,0)\}, \quad \tilde{\Gamma} := \Gamma \setminus \{(0,0) \cup (1,0)\}.
\]

Consider the Dirichlet problem

\[
Lu = f(x,t) \quad \text{in} \quad Q_T, \tag{1.2}
\]
\[
u(0,t) = \psi_1(t), \quad u(1,t) = \psi_2(t) \quad \text{for} \ t \in (0, T], \tag{1.3}
\]
\[
u(x,0) = \phi(x) \quad \text{for} \ x \in [0, 1]. \tag{1.4}
\]

The classical statement of the Dirichlet problem (1.2)–(1.4) is to find a classical solution of the problem in the sense of:

**Definition 1.1.** A function \( u(x,t) \) is called a classical solution to problem (1.2)–(1.4), if it

1. is defined and continuous on \( \overline{Q}_T \), and possesses continuous derivatives \( u_t, u_x, \) and \( u_{xx} \) in \( Q_T \);
2. satisfies Eq. (1.2) and the initial–boundary conditions (1.3), (1.4) at each point.

**Remark 1.2.** Throughout the paper, partial derivatives of functions at internal points of \( \overline{Q}_T \) are intended as limits of the corresponding differential quotients. For the derivative \( u_t(x,t) \) at the upper and lower boundaries of \( \overline{Q}_T \), we mean the one-side limits

\[
u_t(x,T) := \lim_{t \to T^-} \frac{u(x,T) - u(x,t)}{T - t}, \quad u_t(x,0) := \lim_{t \to 0^+} \frac{u(x,t) - u(x,0)}{t}
\]

for \( x \in [0, 1] \). For \( u_x(x,t) \) and \( u_{xx}(x,t) \) on the left-hand and on the right-hand boundaries of \( \overline{Q}_T \), we intend the one-side limits

\[
\begin{align*}
u_x(0,t) := & \lim_{x \to 0^+} \frac{u(x,t) - u(0,t)}{x}, \quad u_x(1,t) := \lim_{x \to 1^-} \frac{u(1,t) - u(x,t)}{1 - x} , \\
u_{xx}(0,t) := & \lim_{x \to 0^+} \frac{u_x(x,t) - u_x(0,t)}{x}, \quad u_{xx}(1,t) := \lim_{x \to 1^-} \frac{u_x(1,t) - u_x(x,t)}{1 - x}
\end{align*}
\]

for \( t \in [0, T] \).

Obviously, the classical statement of the problem (1.2)–(1.4) requires for the initial data \( \phi(x) \) in (1.4) to be a continuous function on the interval \([0,1]\), that is to belong to the space \( C[0,1] \), and moreover, to be in accordance with the boundary data in (1.3), i.e., the relations

\[
\phi(0) = \lim_{t \to 0^+} \psi_1(t), \quad \phi(1) = \lim_{t \to 0^+} \psi_2(t)
\]

have to be satisfied. These relations are called the compatibility conditions of zero order (see Section 1.3.2).
1.3. On the compatibility conditions of order $2k$

Provided

(1) an assumption on smoothness of the coefficients of the operator $L$ in (1.1) and
(2) a hypothesis about existence of a classical solution to the Dirichlet problem (1.2)–(1.4) belonging to a certain function space,

one can usually realize that the right-hand side $f(x,t)$ in (1.2), the boundary functions $\psi_1(t)$ and $\psi_2(t)$ in (1.3), and the initial data $\varphi(x)$ in (1.4) should belong to some function spaces (i.e., should to possess some smoothness). We will refer to this fact as to the smoothness conditions. After that, using assumption (1), hypothesis (2), and the smoothness conditions, with the help of Eq. (1.2) one can obtain a set of relations among the data, at the corner points $(x,t) = (0,0)$ and $(x,t) = (1,0)$. These relations are usually called as compatibility conditions.

Definition 1.3. Let $k \geq 0$ be integer. When

(1) all the coefficients $a(x,t), b(x,t)$, and $c(x,t)$ of the operator $L$ belong to the space $C^{2(k-1),k-1}_x \left( \overline{Q}_T \right)$, if $k \geq 1$, and to the set $C(Q_T)$, if $k = 0$, while

(2) the hypothetical classical solution $u(x,t)$ to problem (1.2)–(1.4) belongs to the space $C^{2k,k}_x \left( \overline{Q}_T \right),$

then the arising smoothness and compatibility conditions we call as the smoothness conditions of order $2k$ (or the $2k$-order smoothness conditions) and the compatibility conditions of order $2k$ (or the $2k$-order compatibility conditions) (see [4,7]).

Remark 1.4 (On the domains of the data). In the framework of Definitions 1.1 and 1.3, the boundary functions $\psi_1(t)$ and $\psi_2(t)$, originally defined for $t \in (0, T]$, turn out to be continuous on the interval $(0, T]$, and must have the finite limits

$$\lim_{t \to 0^+} \psi_1(t), \quad \lim_{t \to 0^+} \psi_2(t),$$

that is these functions can always (i.e., for all integers $k \geq 0$) be continuously extended to the interval $[0, T]$, setting

$$\psi_1(0) := \lim_{t \to 0^+} \psi_1(t), \quad \psi_2(0) := \lim_{t \to 0^+} \psi_2(t).$$

Similarly, the right-hand side $f(x,t)$, originally defined on $Q_T$, can be extended continuously to $\overline{Q}_T$, provided that $k \geq 1$.

For $k \geq 1$, the smoothness conditions of order $2k$ are nothing but the conditions

$$\varphi(x) \in C^{2k}[0,1], \quad \psi_1(t), \psi_2(t) \in C^k[0, T], \quad f(x,t) \in C^{2(k-1),k-1}_{x,t} \left( \overline{Q}_T \right), \quad (1.5)$$
while the system of relations representing the compatibility conditions of order $2k$, becomes more complicated when $k$ is greater, and it is difficult to present such a system in general form. However, it can be easily written in the case of $k = 1$.

1.3.1. The compatibility conditions of second order

If $k = 1$, that is if the coefficients of the operator $L$ belong to $C(\overline{Q}_T)$, and the hypothetical classical solution $u(x,t)$ to the problem (1.2)–(1.4) belongs to $C^2_{x,t}(\overline{Q}_T)$, then the smoothness conditions (of second order) are nothing but

$$\varphi(x) \in C^2[0,1], \quad \psi_1(t), \psi_2(t) \in C^1[0,T], \quad f(x,t) \in C(\overline{Q}_T), \quad (1.6)$$

and the compatibility conditions (of second order) are nothing but

$$\begin{cases}
\varphi(0) = \psi_1(0), & \varphi(1) = \psi_2(0), \\
 a(0,0)\varphi''(0) + b(0,0)\varphi'(0) + c(0,0)\varphi(0) + f(0,0) = \psi_1'(0), \\
 a(1,0)\varphi''(1) + b(1,0)\varphi'(1) + c(1,0)\varphi(1) + f(1,0) = \psi_2'(0).
\end{cases} \quad (1.7)$$

These compatibility conditions mean that Eq. (1.2) could be satisfied at the corner points $(x,t) = (0,0)$ and $(x,t) = (1,0)$.

1.3.2. The compatibility conditions of zero order

If $k = 0$, that is if the coefficients of the operator $L$ belong to the set $C(Q_T)$, and the hypothetical classical solution $u(x,t)$ to problem (1.2)–(1.4) belongs to the space $C(\overline{Q}_T)$ (cf. Definition 1.1), then the smoothness conditions (of zero order) are nothing but

$$\varphi(x) \in C[0,1], \quad \psi_1(t), \psi_2(t) \in C[0,T], \quad f(x,t) \in C(Q_T), \quad (1.8)$$

and the compatibility conditions (of zero order) are nothing but

$$\varphi(0) = \psi_1(0), \quad \varphi(1) = \psi_2(0). \quad (1.9)$$

Hence, equation in (1.2) does not hold in general, even formally, at the corner points $(x,t) = (0,0)$ and $(x,t) = (1,0)$. Moreover, all derivatives of solutions (as well as the solutions themselves) do not exist, in general, at the corner points, as limits of the corresponding derivatives (solutions) from inside of the domain.

In this paper, we shall only use the compatibility conditions of zero or of the second order.

1.4. The Dirichlet problem without compatibility conditions

If the smoothness conditions of zero order (1.8) hold true, but the compatibility conditions of zero order (1.9) are not satisfied, i.e., $\varphi(0) \neq \psi_1(0)$ and/or $\varphi(1) \neq \psi_2(0)$, then a classical solution (in the sense of Definition 1.1) to problem (1.2)–(1.4) does not exist. However, in this case, it is possible to replace the initial condition (1.4) by the condition

$$u(x,0) = \varphi(x) \quad \text{for } x \in (0,1) \quad (1.10)$$

and to consider the classical statement of the Dirichlet problem (1.2), (1.3), (1.10), i.e., to look for a classical solution of the problem in the sense of the following definition.
Definition 1.5. A function $u(x, t)$ is called a classical solution to problem (1.2), (1.3), (1.10), if:

1. it is defined and continuous in $\hat{Q}_T$, and possesses continuous derivatives $u_t$, $u_x$, and $u_{xx}$ in $\hat{Q}_T$;
2. it satisfies Eq. (1.2) and the initial–boundary conditions (1.3), (1.10) at each point.

In contrast to Definition 1.1, Definition 1.5 does not require for a solution $u(x, t)$ to be defined (prescribed) at the corner points $(x, t) = (0, 0)$ and $(x, t) = (1, 0)$. In other words, the only difference between the classical statement of the Dirichlet problem (1.2), (1.3), (1.10) and the classical statement of the Dirichlet problem (1.2)–(1.4) is that one looks for a solution continuous in $\tilde{Q}_T$, rather than in $\hat{Q}_T$, and requires that the initial condition is satisfied on the open interval $(0, 1)$, rather than on $[0, 1]$.

Remark 1.6 (On the behavior of solutions from Definition 1.5). With the previous definition, the solution $u(x, t)$ may become unbounded as $(x, t)$ tends to $(0, 0)$ or to $(1, 0)$ from inside of the domain. What happens at the vicinities (neighborhoods) of the corner points, $(0, 0)$ and $(1, 0)$, is not prescribed. In particular, it is possible

1. that $\lim_{x \to 0^+} \varphi(x) = \pm \infty$ and $\lim_{x \to 1^-} \varphi(x) = \pm \infty$;
2. or that $\varphi(x) \in C^\infty[0, 1]$ and $\psi_1(t), \psi_2(t) \in C^\infty[0, T]$ (that is, in particular, the functions are defined at the corner points) and also the compatibility conditions of zero order (1.9) hold, but the solution $u(x, t)$ does not have the limits as $(x, t) \to (0, 0)$ and as $(x, t) \to (1, 0)$ (see Example 1.10 below).

Remark 1.7 (On the two kinds of classical solutions). Throughout all the paper, when we write about classical solutions of different Dirichlet problems with an initial condition on the closed interval $[0, 1]$ (cf. condition (1.4)), we mean classical solutions in the sense of Definition 1.1. Alternatively, when we write about classical solutions of different Dirichlet problems with initial condition on the open interval $(0, 1)$ (see condition (1.10)), we mean classical solutions in the sense of Definition 1.5.

Remark 1.8 (On uniqueness of classical solutions). If the coefficient $c(x, t)$ of the operator $L$, defined in (1.1) and parabolic in $Q_T$, is such that $\sup_{Q_T} c(x, t) < +\infty$, then it can exist at most one classical solution to problem (1.2)–(1.4) (see [13, Theorem 5]). In contrast, a classical solution to problem (1.2), (1.3), (1.10) may be not unique (see Example 1.10 below). However, the latter is unique (if it exists) in the class of bounded functions, whenever the operator $L$ in (1.1) is uniformly parabolic in $Q_T$, the coefficient $a(x, t)$ is continuous on $\overline{Q}_T$, and $b(x, t)$ and $c(x, t)$ are bounded in $Q_T$ (see [13, Theorem 11]).

Remark 1.9 (On initial data in (1.10)). The classical statement of the Dirichlet problem (1.2), (1.3), (1.10) requires for the initial data $\varphi(x)$ in (1.10) to be a continuous function on the interval $(0, 1)$ and does not require that it has limits when $x \to 0^+$ and $x \to 1^-$ (in particular, such limits could exist but be infinite, see Remark 1.6). If it happens that the initial data $\varphi(x)$ of the aforementioned problem can be extended continuously from the interval $(0, 1)$ to the interval $[0, 1]$ (unless originally $\varphi(x) \in C[0, 1]$), then no restrictions are required on the limits $\lim_{x \to 0^+} \varphi(x)$ and $\lim_{x \to 1^-} \varphi(x)$ (i.e., $\varphi(0)$ and $\varphi(1)$, if $\varphi(x)$ is continuous in $[0, 1]$).
Example 1.10 (On existence of unbounded classical solutions to problem (1.2), (1.3), (1.10)). Consider the Dirichlet problem for the heat equation which is a model, e.g., for describing the cooling process of a uniform rod heated at the initial time, \( t = 0 \), to the unit temperature, while at the endpoints, \( x = 0 \) and \( x = 1 \), a zero temperature is imposed for all \( t > 0 \):

\[
\begin{align*}
  u_t &= u_{xx} \quad \text{in } Q_T, \\
  u(0, t) &= 0, \quad u(1, t) = 0 \quad \text{for } t \in (0, T], \\
  u(x, 0) &= 1 \quad \text{for } x \in (0, 1).
\end{align*}
\]

It is known [13], that there exists a unique bounded classical solution \( u(x, t) \) to this problem, and it is clear that its derivative \( u_t \not\equiv 0 \) solves, in the classical sense, the problem

\[
\begin{align*}
  v_t &= v_{xx} \quad \text{in } Q_T, \\
  v(0, t) &= 0, \quad v(1, t) = 0 \quad \text{for } t \in (0, T], \\
  v(x, 0) &= 0 \quad \text{for } x \in (0, 1).
\end{align*}
\]

On the other hand, problem (1.11)–(1.13) has the trivial classical solution \( v(x, t) \equiv 0 \). This means that the problem (1.11)–(1.13) has two different classical solutions. By the uniqueness theorem for bounded classical solutions (see Remarks 1.7 and 1.8), the trivial solution \( v(x, t) \equiv 0 \) is the unique bounded classical solution to (1.11)–(1.13). Consequently, the other classical solution \( u_t \not\equiv 0 \) must be unbounded in \( Q_T \). Obviously, it becomes unbounded as \( t \to 0^+ \), in the neighborhoods of the corner points \( (x, t) = (0, 0) \) and \( (x, t) = (1, 0) \).

1.5. Connection between the Dirichlet problems

In this subsection, we consider the case when smoothness and compatibility conditions of zero order are satisfied, see (1.8) and (1.9). In such a case, we can consider both Dirichlet problems at the same time, i.e., problem (1.2)–(1.4) and problem (1.2), (1.3), (1.10). Clearly, a classical solution to the first problem is also a bounded classical solution to the second one. If Problem 1.15 below has a positive answer, then the inverse is not true in general, that is there are bounded classical solutions to (1.2), (1.3), (1.10), with (1.8) and (1.9) satisfied, which cannot be extended continuously to \( \overline{Q}_T \), that is, up to the corner points \( (0, 0) \) and \( (1, 0) \). In such a case, problem (1.2)–(1.4) (with the same data) does not have classical solutions, if problem (1.2), (1.3), (1.10) cannot have more than one bounded classical solution.

A similar phenomenon can be observed when a classical solution to (1.2)–(1.4) exists, namely: Despite that smoothness and compatibility conditions of any order are satisfied, problem (1.2), (1.3), (1.10) can have unbounded classical solutions which (obviously) cannot be extended continuously to \( \overline{Q}_T \) (see Example 1.10). What follows makes the situation clearer.

Remark 1.11. If problem (1.2)–(1.4) has a classical solution, and the problem (1.2), (1.3), (1.10) (with the same data) has a unique bounded classical solution, then these solutions are identically equal in \( \overline{Q}_T \).

Remark 1.12. Under the smoothness and compatibility conditions of zero order only, a classical solution to problem (1.2)–(1.4), or even to problem (1.2), (1.3), (1.10), does not exist in general.
Some additional smoothness properties of the boundary data, $\psi_1(t)$ and $\psi_2(t)$, and smoothness and growth properties of the right-hand side $f(x, t)$ guarantee the existence of classical solutions to problem (1.2)–(1.4). In particular, the following statement holds true:

**Remark 1.13.** If $\varphi(x) \in C[0, 1]$ and $\psi_1(t), \psi_2(t) \in \mathcal{B}_\omega$, with $\omega(\delta) \equiv \delta^\alpha$, $0 < \alpha < 1$, then the problem
\[
\begin{align*}
  u_t &= u_{xx} \quad \text{in } Q_T, \\
  u(0, t) &= \psi_1(t), \\
  u(1, t) &= \psi_2(t) \quad \text{for } t \in (0, T], \\
  u(x, 0) &= \varphi(x) \quad \text{for } x \in (0, 1)
\end{align*}
\]
has a unique bounded classical solution. If, in addition, the compatibility conditions of zero order (1.9) are satisfied, then the problem
\[
\begin{align*}
  u_t &= u_{xx} \quad \text{in } Q_T, \\
  u(0, t) &= \psi_1(t), \\
  u(1, t) &= \psi_2(t) \quad \text{for } t \in (0, T], \\
  u(x, 0) &= \varphi(x) \quad \text{for } x \in [0, 1]
\end{align*}
\]
has a unique classical solution, and this solution coincides (in $\tilde{Q}_T$) with the aforementioned solution.

**Remark 1.14.** If problem (1.14)–(1.16) has a bounded classical solution which cannot be extended continuously to $\overline{Q}_T$, then problem (1.17)–(1.19) (with the same data) does not have classical solutions.

Consider the following problem.

**Problem 1.15.** Take initial data
\[
\varphi(x) \equiv 0 \quad \text{in } [0, 1].
\]
Do boundary data
\[
\psi_1(t), \psi_2(t) \in C[0, T], \quad \psi_1(0) = 0, \quad \psi_2(0) = 0
\]
exist such that:

1. there exists a bounded classical solution to problem (1.14)–(1.16) with the initial–boundary data in (1.20), (1.21), but
2. this solution cannot be extended continuously to $\overline{Q}_T$?

If so, then every extension to $\overline{Q}_T$ (i.e., to the corner points $(0, 0)$ and $(1, 0)$) of such a bounded classical solution to problem (1.14)–(1.16) would not solve problem (1.17)–(1.19) in the classical sense (see Definitions 1.1, 1.5 and Remark 1.7), in spite of the fact that smoothness and compatibility conditions of zero order are satisfied. Note also that, in order to construct such a solution, it is necessary that problem (1.17)–(1.19) does not possess classical solutions (see Remark 1.14).
1.6. Modulus of continuity and some function spaces

In this paper, we use certain function spaces which depend on some given “continuity modulus” \( \omega = \omega(\delta) \). These spaces are not very popular in the current literature, hence we give here their precise definitions.

**Definition 1.16.** A monotone nondecreasing function \( \omega(\delta) \), defined and continuous for \( \delta \geq 0 \), is called a *continuity modulus*, if the following requirements are satisfied: \( \omega(0) = 0 \) and \( \omega(\delta_1 + \delta_2) \leq \omega(\delta_1) + \omega(\delta_2) \) for \( \delta_1, \delta_2 \geq 0 \).

**Remark 1.17.** If \( \varphi(x) \in C(\mathbb{R}) \) is a uniformly continuous function, then the function

\[
\omega_\delta := \sup_{|x-y| \leq \delta} |\varphi(x) - \varphi(y)|
\]

satisfies Definition 1.16, and is called the *continuity modulus of \( \varphi \).*

The following statement holds.

**Lemma 1.18.** Any continuity modulus, \( \omega \), enjoys the properties:

\[
\omega(\alpha \delta) \leq (\alpha + 1)\omega(\delta) \quad \text{for } \alpha, \delta \geq 0; \quad \frac{\omega(t)}{t} \leq 2 \frac{\omega(\delta)}{\delta} \quad \text{for } 0 < \delta \leq 1.
\]

**Definition 1.19.** A continuity modulus \( \omega \) is said to satisfy the *Dini condition* if

\[
\int_0^1 \frac{\omega(\tau)}{\tau} \, d\tau < +\infty. \quad (1.22)
\]

**Definition 1.20.** We say that a continuity modulus \( \omega \) is *double-integrable according to Dini*, if it satisfies the Dini condition, and, in addition,

\[
\int_0^1 \frac{\omega^*(\tau)}{\tau} \, d\tau < +\infty, \quad \text{where } \omega^*(\tau) := \int_0^{\tau} \frac{\omega(\sigma)}{\sigma} \, d\sigma. \quad (1.23)
\]

**Remark 1.21.** The double-integrability condition (1.23) can also be written as

\[
\int_0^1 \int_0^\tau \frac{\omega(\sigma)}{\sigma} \, d\sigma \, d\tau = \int_0^1 \frac{\omega(\sigma)}{\sigma} \log\left( \frac{1}{\sigma} \right) \, d\sigma < +\infty.
\]

**Remark 1.22.** The Dini condition is satisfied, for instance, by any continuity modulus, \( \omega \), such that the inequality

\[
\omega(\delta) \leq \frac{C}{\log^\alpha(1/\delta)}
\]
holds for all $\delta \in (0, \varepsilon)$ for some $\varepsilon > 0$, $C > 0$ and $\alpha > 1$ being some given constants. When $\alpha > 2$, these continuity moduli are double-integrable according to Dini. In particular, the continuity moduli

$$\omega(\delta) \equiv \delta^\alpha, \quad \alpha \in (0, 1],$$

satisfy the Dini condition as well as the condition in (1.23).

For a given arbitrary continuity modulus, $\omega$, we introduce the following spaces of functions. We denote with $C_\omega(Q_T)$ the set of all functions $u(x, t)$, continuous in $Q_T$, having the finite norm

$$\|u\|_{C_\omega(Q_T)} := \|u\|_{C(Q_T)} + \sup_{(x,t), (y,t) \in Q_T, \ x \neq y} \frac{|u(x, t) - u(y, t)|}{\omega(|x - y|)} + \sup_{(x,t), (x,t+\delta) \in Q_T, \ \delta > 0} \frac{|u(x, t) - u(x, t+\delta)|}{\omega(\sqrt{\delta})},$$

where $\|w\|_{C(Q_T)} := \sup_{Q_T} |u(x, t)|$. Similarly, $C_\omega[0, 1]$ denotes the set of all functions $\varphi(x)$, continuous on the interval $[0, 1]$, having the finite norm

$$\|\varphi\|_{C_\omega[0, 1]} := \|\varphi\|_{C[0, 1]} + \sup_{x, y \in [0, 1], \ x \neq y} \frac{|\varphi(x) - \varphi(y)|}{\omega(|x - y|)}.$$

By $C_{\omega}^1[0, T]$ we denote the set of functions $\psi(t)$, continuously differentiable on the closed interval $[0, T]$, with the norm

$$\|\psi\|_{C_{\omega}^1[0, T]} := \sum_{k \leq 1} \|\psi^{(k)}\|_{C[0, T]} + \sup_{t, t+\delta \in [0, T], \ \delta > 0} \frac{|\psi(t) - \psi(t+\delta)|}{\omega(\sqrt{\delta})},$$

while $C_{\omega}^2[0, 1]$ denotes the set of functions $\varphi(x)$, doubly continuously differentiable on the closed interval $[0, 1]$, with the norm

$$\|\varphi\|_{C_{\omega}^2[0, 1]} := \sum_{k \leq 1} \|\varphi^{(k)}\|_{C[0, 1]} + \|\varphi''\|_{C_\omega[0, 1]}.$$

By $R_{\omega} \equiv R_{\omega}(Q_T)$ we denote the set of functions $f(x, t)$, defined and continuous in $Q_T$, equipped with the norm

$$\|f\|_{R_{\omega}} := \sup \left[ \left( \frac{|f(x, t)|}{\omega(\sqrt{t})} + \frac{|f(x, t) - f(y, t)|}{\omega(|x - y|)} + \frac{|f(x, t) - f(x, t+\delta)|}{\omega(\sqrt{\delta})} \right) t \right],$$

where the supremum is taken over all points $(x, t) \neq (y, t), (x, t+\delta) (\delta > 0)$ of the cylinder $Q_T$. Finally, by $B_{\omega} \equiv B_{\omega}[0, T]$ we denote the set of functions $\psi(t)$, defined and continuous on the interval $[0, T]$, with derivatives $\psi'(t)$ continuous in $[0, T]$, which possess the finite norm

$$\|\psi\|_{B_{\omega}} := \|\psi\|_{C[0, T]} + \sup_{t, t+\delta \in [0, T], \ \delta > 0} \left[ \left( \frac{|\psi'(t)|}{\omega(\sqrt{t})} + \frac{|\psi'(t) - \psi'(t+\delta)|}{\omega(\sqrt{\delta})} \right) t \right].$$
It is easy to check that all such spaces, \( C_\omega, C^{1}_\omega, C^{2}_\omega, \mathcal{R}_\omega, \) and \( B_\omega, \) are Banach spaces. Note also that \( C_\omega \subset C, C^{1}_\omega \subset C^{1}, \) and \( C^{2}_\omega \subset C^{2}. \)

2. Supremum and infimum principles for the higher-order derivatives of the heat equation

In this section, we establish weak and strong supremum principles which concern the higher-order derivatives, \( u_t \) and \( u_{xx}, \) of bounded classical solutions to the Dirichlet problem for the heat equation

\[
\begin{align*}
    u_t &= a u_{xx} \quad \text{in } Q_T, \\
    u(0, t) &= \psi_1(t), \quad u(1, t) = \psi_2(t) \quad \text{for } t \in (0, T], \\
    u(x, 0) &= \varphi(x) \quad \text{for } x \in (0, 1),
\end{align*}
\]

where \( a > 0 \) is a constant. Let us first formulate the following theorem.

**Theorem 2.1** (*Smoothness theorem*). Assume that the initial and boundary data of problem (2.1)–(2.3) have the following smoothness properties:

1. \( \varphi(x) \) belongs to \( C^2_\omega[0, 1]; \)
2. \( \psi_1(t) \) and \( \psi_2(t) \) belong to \( C^1_\omega[0, T]; \)
3. the continuity modulus \( \omega \) has the form \( \omega(\delta) = \delta^\alpha, \) \( 0 < \alpha < 1. \)

Then, there exists a unique bounded classical solution, \( u(x, t), \) to problem (2.1)–(2.3). Its derivatives \( u_t(x, t), u_x(x, t), \) and \( u_{xx}(x, t) \) are continuous functions in \( \tilde{Q}_T, \) and moreover,

\[
\begin{align*}
    S := \sup_{\tilde{\Gamma}} u_t(x, t) &= a \sup_{\tilde{\Gamma}} u_{xx}(x, t) < +\infty, \\
    I := \inf_{\tilde{\Gamma}} u_t(x, t) &= a \inf_{\tilde{\Gamma}} u_{xx}(x, t) > -\infty.
\end{align*}
\]

The constants \( S \) and \( I \) can be calculated by formulae \( S = \max\{S_1, S_2, S_3\} \) and \( I = \min\{I_1, I_2, I_3\}, \)

where

\[
\begin{align*}
    S_1 := \max_{[0, T]} \psi_1'(t), & \quad S_2 := \max_{[0, T]} \psi_2'(t), & \quad S_3 := a \max_{[0, 1]} \varphi''(x), \\
    I_1 := \min_{[0, T]} \psi_1'(t), & \quad I_2 := \min_{[0, T]} \psi_2'(t), & \quad I_3 := a \min_{[0, 1]} \varphi''(x).
\end{align*}
\]

The proof of Theorem 2.1 follows immediately from that of Theorem 2.2.

2.1. The weak supremum principle

Before formulating the weak supremum principle for the higher-order derivatives of the heat equation in general, we prove an important special case of it.

**Theorem 2.2** (*An important special case of the weak supremum principle*). Let all assumptions of Theorem 2.1 be satisfied. If, in addition,
\[ \varphi(0) \geq \psi_1(0), \quad \varphi(1) \geq \psi_2(0), \]  
(2.6)

\[ S:= \sup_{\tilde{\Gamma}} u_t(x,t) = a \sup_{\tilde{\Gamma}} u_{xx}(x,t) \leq 0, \]  
(2.7)

then the unique bounded classical solution \( u(x,t) \) to problem (2.1)--(2.3) has the property

\[ u_t \equiv au_{xx} \leq 0 \quad \text{in} \quad \tilde{Q}_T. \]

**Proof.** We will only prove the theorem for the case when

\[ \varphi(0) > \psi_1(0) \quad \text{and} \quad \varphi(1) > \psi_2(0). \]  
(2.8)

To this goal, for each \( n, n = 3, 4, \ldots \), consider the two auxiliary problems

\[ u^n_t = au^n_{xx} \quad \text{in} \quad Q_T, \]  
(2.9)

\[ u^n(0,t) = \psi_1(t), \quad u^n(1,t) = \psi_2(t) \quad \text{for} \quad t \in (0,T], \]  
(2.10)

\[ u^n(x,0) = \varphi_n(x) \quad \text{for} \quad x \in [0,1], \]  
(2.11)

\[ v^n_t = av^n_{xx} \quad \text{in} \quad Q_T, \]  
(2.12)

\[ v^n(0,t) = 0, \quad v^n(1,t) = 0 \quad \text{for} \quad t \in (0,T], \]  
(2.13)

\[ v^n(x,0) = \varphi(x) - \varphi_n(x) \quad \text{for} \quad x \in (0,1), \quad \text{where} \]  
(2.14)

\[ \varphi_n(x) := \begin{cases} 
\varphi_{1,n}(x) & \text{for} \quad x \in [0, \frac{1}{n}], \\
\varphi(x) & \text{for} \quad x \in \left(\frac{1}{n}, 1 - \frac{1}{n}\right), \\
\varphi_{2,n}(x) & \text{for} \quad x \in \left[1 - \frac{1}{n}, 1\right]. 
\end{cases} \]  
(2.15)

We choose the functions \( \varphi_{1,n} \) and \( \varphi_{2,n} \) such that for the problem (2.9)--(2.11) the smoothness and compatibility conditions of the second order are satisfied (cf. conditions (1.6) and (1.7) for the case of problem (1.2)--(1.4)). We can choose these functions in the form of 4th degree polynomials,

\[ \varphi_{1,n}(x) = a_n x^4 + b_n x^3 + c_n x^2 + d_n x + e_n \quad \text{for} \quad x \in [0, 1/n], \]  

\[ \varphi_{2,n}(x) = \tilde{a}(1-x)^4 + \tilde{b}(1-x)^3 + \tilde{c}(1-x)^2 + \tilde{d}(1-x) + \tilde{e} \quad \text{for} \quad x \in [1 - 1/n, 1]. \]

Then, it is necessary and sufficient that the coefficients of \( \varphi_{1,n} \) and \( \varphi_{2,n} \) satisfy the systems of equations

\[ \begin{cases} 
a_n n^{-4} + b_n n^{-3} + c_n n^{-2} + d_n n^{-1} + e_n = \varphi_{1,n}(1/n), \\
4a_n n^{-3} + 3b_n n^{-2} + 2c_n n^{-1} + d_n = \psi'(1/n), \\
12a_n n^{-2} + 6b_n n^{-1} + 2c_n = \varphi_{2,n}'(1/n), \\
e_n = \psi_1(0), \\
2c_n a = \psi_1(0) 
\end{cases} \]

and

\[ \begin{cases} 
a_n n^{-4} + b_n n^{-3} + c_n n^{-2} + d_n n^{-1} + e_n = \varphi(1/n), \\
4a_n n^{-3} + 3b_n n^{-2} + 2c_n n^{-1} + d_n = \psi'(1/n), \\
12a_n n^{-2} + 6b_n n^{-1} + 2c_n = \varphi''(1/n), \\
e_n = \psi_1(0), \\
2c_n a = \psi_1(0) 
\end{cases} \]
It follows that $c_n = \psi'_1(0)/2a$ and $e_n = \psi_1(0)$, while we obtain for the remaining coefficients $a_n$, $b_n$, and $d_n$, in the first system, a system of three linear equations, whose determinant is

$$
\det \begin{pmatrix}
\frac{n^{-4}}{4n^{-3}} & \frac{n^{-3}}{3n^{-2}} & \frac{n^{-1}}{1} \\
12n^{-2} & 6n^{-1} & 0
\end{pmatrix} = \frac{6}{n^5} \neq 0.
$$

Therefore, the first system is uniquely solvable for every $n = 3, 4, \ldots$. Similarly, we can show the unique solvability of the second one.

By construction, the compatibility conditions of second order are satisfied, for problem (2.9)–(2.11), and it is clear that the functions $\varphi_n$ (defined in (2.15)) belong to the space $C^2_w[0,1]$. Therefore, for every $n$, $n = 3, 4, \ldots$, there exists a unique classical solution, $u^n(x,t)$, to problem (2.9)–(2.11), and moreover, its derivatives $D^k_l u^n$ are continuous on $\overline{Q}_T$ for $2k + l \leq 2$ (see [4,19]). Furthermore, $u^n(x,t)$ is infinitely differentiable in $Q_T$. Consequently, the function $w^n := u^n_1$ is continuous in $\overline{Q}_T$ and is a classical solution to the problem

$$
w^n_1 = aw^n_{xx} \quad \text{in} \quad Q_T, \tag{2.16}
$$

$$
w^n(0,t) = \psi'_1(t), \quad w^n(1,t) = \psi'_2(t) \quad \text{for} \quad t \in (0,T], \tag{2.17}
$$

$$
w^n(x,0) = \varphi''_n(x) \quad \text{for} \quad x \in [0,1]. \tag{2.18}
$$

Now we focus on two important properties of the functions $\varphi_n$:

$$
\max_{x \in [0,1]} \varphi''_n(x) \leq 0 \quad \text{for} \quad n \geq N, \tag{2.19}
$$

$$
\|\varphi_n\|_{C[0,1]} \leq \max\{\|\varphi\|_{C[0,1]} + \|\varphi'_1\|_{C[0,1]}, |\psi_1(0)|, |\psi_2(0)|\} \quad \text{for} \quad n \geq N, \tag{2.20}
$$

for some $N$. We first prove inequality (2.19). To this goal, in view of (2.7), it suffices to establish that $\varphi''_{1,n}(x) \leq 0$ for $x \in [0,1/n]$ and $\varphi''_{2,n}(x) \leq 0$ for $x \in [1 - 1/n, 1]$. It suffices to consider only the case of the first function, since the second case can be treated similarly. We have $\varphi''_{1,n}(x) = 12a_n x^2 + 6b_n x + 2c_n$. Note that $\varphi''_{1,n}(0) = \psi'_1(0) \leq 0$ and $\varphi''_{1,n}(1/n) = \psi''_1(1/n) \leq 0$. Moreover, $a_n > 0$ for all $n$ sufficiently large, say, $n \geq N$. Indeed, multiplying both sides of the second equation of the system by $1/n$ and subtracting the first equation from the so-modified second one, we obtain the relation

$$
\frac{2b_n}{n} = n^2 \left[ \frac{\psi'(1/n)}{n} - \varphi'(1/n) + e_n \right] - \frac{3a_n}{n^2} - c_n.
$$

Inserting this expression in the third equation, we derive the equality

$$
\frac{a_n}{n^2} = \frac{\psi''(1/n)}{3} + \frac{\psi'_1(0)}{6a} + n^2 \left[ \varphi(1/n) - \frac{\varphi'(1/n)}{n} - \psi_1(0) \right]. \tag{2.21}
$$
for all \( n = 3, 4, \ldots \). Note that
\[
\varphi(1/n) - \frac{\varphi'(1/n)}{n} \geq \varphi(0) \tag{2.22}
\]
for all such \( n \). In fact, by Lagrange theorem, \( \varphi(1/n) - \varphi(0) = \varphi'(\theta_n)/n \), for some \( \theta_n \in (0, 1/n) \). Furthermore, by (2.7), \( \varphi''(x) \leq 0 \) for \( x \in [0, 1] \), and thus, the function \( \varphi'(x) \) is monotone nonincreasing. Consequently, \( \varphi(1/n) - \varphi(0) \geq \varphi'(1/n)/n \), and inequality (2.22) is proved. In view of (2.21), (2.22) and (2.8), we have
\[
\frac{a_n}{n^2} \geq \frac{\varphi''(1/n)}{3} + \frac{\psi'_1(0)}{6a} + n^2[\varphi(0) - \psi_1(0)] > 0
\]
for \( n \geq N \). Consequently, \( a_n > 0 \) for such values of \( n \). Therefore, \( \varphi''_{1,n}(x) \) is convex (a parabolic arc) for \( x \in [0, 1/n] \), taking nonpositive values at the endpoints of the domain. Thus, \( \varphi_{1,n}(x) \leq 0 \) for all \( x \in [0, 1/n] \), and the property (2.19) is proved. Therefore, by the classical maximum principle (see Theorem 0.1), \( w^n(x, t) \leq 0 \) in \( \bar{Q}_T \) for \( n \geq N \), where \( w^n(x, t) := u^n_{\psi}(x, t) \) is the classical solution to problem (2.16)–(2.18) (see inequality (2.19) and (2.7)), that is, the estimate
\[
u^n_t(x, t) \leq 0 \quad \text{in} \quad \bar{Q}_T \tag{2.23}
\]
holds for all such values of \( n \).

In order to prove property (2.20), note that there exists the point \( x_0 \in [0, 1] \) such that
\[
\|\varphi_n\|_{C[0,1]} = |\varphi_n(x_0)|.
\]
Below, we consider five different possible cases.

**Case 1.** If \( x_0 = 0 \), then \( \|\varphi_n\|_{C[0,1]} = |\psi_1(0)| \).

**Case 2.** If \( x_0 \in (0, 1/n) \), then \( \varphi_n'(x_0) = 0 \). In view of this and the fact that the function \( \varphi_n'(x) \) is monotone nonincreasing for \( n \geq N \), we have \( |\varphi_n'(x)| \leq |\varphi_n'(1/n)| \) for \( x \in [x_0, 1/n] \) and \( n \geq N \). Therefore, by the Lagrange theorem, \( |\varphi_n(x_0) - \varphi_n(1/n)| \leq |\varphi_n'(1/n)|/n \), that is, \( |\varphi_n(x_0)| \leq |\varphi_n(1/n)| + |\varphi_n'(1/n)|/n \), and thus, \( \|\varphi_n\|_{C[0,1]} \leq \|\varphi\|_{C[0,1]} + \|\varphi'_n\|_{C[0,1]} \) for \( n \geq N \).

**Case 3.** If \( x_0 \in [1/n, 1 - 1/n] \), then \( \|\varphi_n\|_{C[0,1]} = |\varphi(x_0)| \leq \|\varphi\|_{C[0,1]} \).

**Case 4.** If \( x_0 \in (1 - 1/n, 1) \), then \( \varphi_n'(x_0) = 0 \). In view of this and the fact that the function \( \varphi_n'(x) \) is monotone nonincreasing for \( n \geq N \) (see already proved property (2.19)), we have \( 0 \leq \varphi_n'(x) \leq \varphi_n'(1 - 1/n) \) for \( x \in [1 - 1/n, x_0] \) and \( n \geq N \). Therefore, by the Lagrange theorem, \( |\varphi_n(x_0) - \varphi_n(1 - 1/n)| \leq \varphi_n'(1 - 1/n)/n \), that is, \( |\varphi_n(x_0)| \leq |\varphi_n(1 - 1/n)| + \varphi_n'(1 - 1/n)/n = |\varphi(1 - 1/n)| + \varphi'(1 - 1/n)/n \), and thus, \( \|\varphi_n\|_{C[0,1]} \leq \|\varphi\|_{C[0,1]} + \|\varphi'_n\|_{C[0,1]} \) for \( n \geq N \).

**Case 5.** If \( x_0 = 1 \), then \( \|\varphi_n\|_{C[0,1]} = |\psi_2(0)| \).

Summing up the cases 1–5, we conclude that the property (2.20) is established.

Consider now problem (2.12)–(2.14). It is known [27] that there exists a unique bounded classical solution, \( v^n(x, t) \), to the problem (2.12)–(2.14). This solution is given by the formula
\[
v^n(x, t) = \sum_{k=1}^{\infty} C_k e^{-k^2 \pi^2 at} \sin(k \pi x), \quad \text{where}
\]
\[ C_k := 2 \int_0^1 (\varphi(\xi) - \varphi_n(\xi)) \sin(k\pi x) \, d\xi \]

are the Fourier coefficients of the initial data \( \varphi(x) - \varphi_n(x) \). Moreover, it is known that \( v^n(x, t) \) is \textit{infinitely} differentiable in \( \overline{Q}_T \cap \{ t > 0 \} \), and every derivative, \( D_{t,x}^{k,l} v^n \), can be evaluated by differentiating the series term by term, cf. [27]. In particular,

\[ v_t^n(x, t) = -a\pi^2 \sum_{k=1}^{\infty} C_k k^2 e^{-k^2\pi^2 a t} \sin(k\pi x) \]

in \( \overline{Q}_T \cap \{ t > 0 \} \). Being (by (2.20)) \( \| \varphi - \varphi_n \|_{C[0,1]} \leq M \), where the constant \( M \) does not depend on \( n = 3, 4, \ldots \), we get (see definition of \( C_k \) and (2.15))

\[ |C_k| \leq 2 \int_0^{1/n} M \, d\xi + 2 \int_{1-1/n}^1 M \, d\xi = \frac{4M}{n} \]

for all \( k = 1, 2, \ldots \). Hence

\[ |v_t^n(x, t)| \leq \frac{4Ma\pi^2}{n} \sum_{k=1}^{\infty} k^2 e^{-k^2\pi^2 a t}, \]

and thus,

\[ \lim_{n \to \infty} v_t^n(x, t) = 0 \quad (2.24) \]

for every fixed point \( (x, t) \in \overline{Q}_T \cap \{ t > 0 \} \).

Therefore, there exists a unique bounded classical solution \( u(x, t) := u^n(x, t) + v^n(x, t) \) to problem (2.1)–(2.3), and \( u_t(x, t) = u_t^n(x, t) + v_t^n(x, t) \) in \( \overline{Q}_T \cap \{ t > 0 \} \) for \( n = 3, 4, \ldots \). Taking the limit for \( n \to \infty \) at each fixed point \( (x, t) \in \overline{Q}_T \cap \{ t > 0 \} \), in such equality, and keeping in mind (2.23) and (2.24), we obtain that

\[ u_t(x, t) \leq 0 \quad \text{in} \quad \overline{Q}_T \cap \{ t > 0 \}. \]

Theorem 2.2 is thus proved. \( \square \)

We can now prove the following more general theorem.

**Theorem 2.3 (Weak supremum principle).** Let all assumptions of Theorem 2.1 be satisfied, and moreover, inequalities (2.6) are true. Then, the unique bounded classical solution \( u(x, t) \) to problem (2.1)–(2.3) has the properties

\[ \sup_{\overline{Q}_T} u_t(x, t) = \sup_{\overline{Q}_T} u_t(x, t), \quad \sup_{\overline{Q}_T} u_{xx}(x, t) = \sup_{\overline{Q}_T} u_{xx}(x, t), \]

\[ \sup_{\Gamma} u_t(x, t) = \sup_{\Gamma} u_t(x, t), \quad \sup_{\Gamma} u_{xx}(x, t) = \sup_{\Gamma} u_{xx}(x, t), \]
and moreover, the function
\[ \tilde{u}(x, t) := u(x, t) - S \left[ t + \left( \frac{x^2 - x}{2} \right) / 2a \right], \] (2.25)
where \( a > 0 \) is the constant coefficient of Eq. (2.1) and \( S \) is the constant defined in (2.4), possesses the property
\[ \tilde{u}_t \equiv a \tilde{u}_{xx} \leq 0 \quad \text{in } \tilde{Q}_T. \]

If one of the inequalities
\[ \varphi(0) > \psi_1(0) \quad \text{or} \quad \varphi(1) > \psi_2(0) \]
is satisfied, then
\[ \inf_{\tilde{Q}_T} u_t(x, t) = -\infty, \quad \inf_{\tilde{Q}_T} u_{xx}(x, t) = -\infty. \]

**Proof.** Let \( u(x, t) \) be the unique bounded classical solution to problem (2.1)–(2.3) (see Theorem 2.1). Then the function \( u(x, t) \) in (2.25) solves, in the classical sense, the problem
\[ \tilde{u}_t = a \tilde{u}_{xx} \quad \text{in } Q_T, \]
\[ \tilde{u}(0, t) = \psi_1(t) - St, \quad \tilde{u}(1, t) = \psi_2(t) - St \quad \text{for } t \in (0, T], \]
\[ \tilde{u}(x, 0) = \varphi(x) - S\left( \frac{x^2 - x}{2a} \right) \quad \text{for } x \in (0, 1), \]
and it remains to use Theorem 2.2. \( \square \)

2.2. The strong supremum principle

Also, under the assumptions of Theorem 2.3, a strong supremum principle holds for the higher-order derivatives of any bounded classical solution to problem (2.1)–(2.3).

**Theorem 2.4** (A basic special case of the strong supremum principle). Let all assumptions of Theorem 2.3 be satisfied, and \( u(x, t) \) denotes the unique bounded classical solution to problem (2.1)–(2.3) (see Theorem 2.1). If, in addition,
\[ S := \sup_{\tilde{Q}_T} u_t(x, t) = a \sup_{\tilde{Q}_T} u_{xx}(x, t) \leq 0 \]
and one of the equalities
\[ u_t(x_0, t_0) = 0 \quad \text{or} \quad u_{xx}(x_0, t_0) = 0 \]
holds at some point \((x_0, t_0) \in Q_T\), then \( S = 0 \) and
\[ u(x, t) \equiv C_1 x + C_2 \quad \text{in } \tilde{Q}_{t_0}, \]
\( C_1 \) and \( C_2 \) being constants.

Before proving Theorem 2.4, let us list some consequences which we state as corollaries.
Corollary 2.5. Let all assumptions of Theorem 2.2 be satisfied, and moreover,
\[ \varphi(0) > \psi_1(0) \quad \text{or} \quad \varphi(1) > \psi_2(0). \]

Then, the strict inequality
\[ u_t(x, t) \equiv au_{xx}(x, t) < 0 \quad \text{in} \ Q_T \]
holds, \( u(x, t) \) being the unique bounded classical solution to problem (2.1)–(2.3).

Corollary 2.6. Let all assumptions of Theorem 2.2 be satisfied, and moreover, for any two constants \( C_1 \) and \( C_2 \),
\[ \varphi(x) \not\equiv C_1x + C_2 \]
on the interval \([0, 1]\). Then
\[ u_t(x, t) \equiv au_{xx}(x, t) < 0 \quad \text{in} \ Q_T, \]
u(x, t) being the unique bounded classical solution to problem (2.1)–(2.3).

Corollary 2.7. Let all assumptions of Theorem 2.2 be satisfied, and moreover,
\[ \psi_1(t) \not\equiv \text{const} \quad \text{or} \quad \psi_2(t) \not\equiv \text{const} \]
on the interval \([0, t_0]\), for some point \( t_0 \in (0, T] \). Then, for all \( x \in (0, 1) \),
\[ u_t(x, t_0) = au_{xx}(x, t_0) < 0, \]
where \( u(x, t) \) is the unique bounded classical solution to problem (2.1)–(2.3).

Proof of Theorem 2.4. By Theorem 2.2, \( u_t(x, t) \leq 0 \) in \( Q_T \). Let be \( u_t(x_0, t_0) = 0 \) at some point \((x_0, t_0) \in Q_T\). Consider the set \( \{(x, t) \in [0, 1] \times [\varepsilon, t_0]\} \), where \( \varepsilon \in (0, t_0) \) is arbitrary. On this closed set, the function \( g(x, t) := u_t(x, t) \) is continuous and satisfies the equation \( g_t = g_{xx} \). Note that the function \( g(x, t) \) is nonpositive on the boundary of this set. By the strong (classical) maximum principle (cf. Theorem 0.2), it follows that \( g(x, t) \equiv 0 \) everywhere on the set \( \{(x, t) \in [0, 1] \times [\varepsilon, t_0]\} \). By the arbitrariness of \( \varepsilon \in (0, t_0) \), we have
\[ u_t(x, t) = 0 \quad \text{in} \ \overline{Q}_t \cap \{t > 0\}. \]
Consequently, \( u(x, t) = \tilde{u}(x) \) in \( \overline{Q}_t \cap \{t > 0\} \), where \( \tilde{u}''(x) = 0 \) for \( x \in (0, 1) \) (see Eq. (2.1)). Thus, \( u(x, t) \equiv C_1x + C_2 \) in \( \overline{Q}_t \), where \( C_1 \) and \( C_2 \) are some constants. The theorem is thus proved. \( \square \)

We are now able to prove the following theorem.
Theorem 2.8 (The strong supremum principle). Let all assumptions of Theorem 2.3 be satisfied. If the unique bounded classical solution \( u(x, t) \) to problem (2.1)–(2.3) has at least one of the properties

\[
    u_t(x_0, t_0) = \sup_{\tilde{\Gamma}} u_t(x, t) \quad \text{or} \quad u_{xx}(x_0, t_0) = \sup_{\tilde{\Gamma}} u_{xx}(x, t),
\]

at some point \((x_0, t_0) \in Q_T\), then there are two constants \( C_1 \) and \( C_2 \) such that

\[
    u(x, t) \equiv u_t(x_0, t_0) \left( t + \frac{x^2}{2a} \right) + C_1 x + C_2 \quad \text{in } \tilde{Q}_{t_0},
\]

where \( a > 0 \) is the constant coefficient in (2.1).

Proof. For the proof of the theorem, it suffices to consider the auxiliary function (2.25) and use Theorem 2.4. \( \Box \)

Corollary 2.9. Let all assumptions of Theorem 2.3 be satisfied, and moreover,

\[
    \varphi(0) > \psi_1(0) \quad \text{or} \quad \varphi(1) > \psi_2(0).
\]

Then, for all \((x, t) \in Q_T\), the strict inequalities

\[
    u_t(x, t) < \sup_{\tilde{\Gamma}} u_t(x, t) \quad \text{and} \quad u_{xx}(x, t) < \sup_{\tilde{\Gamma}} u_{xx}(x, t)
\]

hold, \( u(x, t) \) being the unique bounded classical solution to problem (2.1)–(2.3).

Corollary 2.10. Let all assumptions of Theorem 2.3 be satisfied, and moreover, for any two constants \( C_1 \) and \( C_2 \),

\[
    \varphi(x) \not\equiv \frac{S}{2a} x^2 + C_1 x + C_2
\]

on the interval \([0, 1]\). Here \( a > 0 \) is the constant in (2.1) and \( S \) is the constant defined in (2.4). Then, for all \((x, t) \in Q_T\),

\[
    u_t(x, t) < \sup_{\tilde{\Gamma}} u_t(x, t) \quad \text{and} \quad u_{xx}(x, t) < \sup_{\tilde{\Gamma}} u_{xx}(x, t),
\]

where \( u(x, t) \) is the unique bounded classical solution to problem (2.1)–(2.3).

Corollary 2.11. Let all assumptions of Theorem 2.3 be satisfied, and moreover,

\[
    \psi_1(t) \not\equiv St + \text{const} \quad \text{or} \quad \psi_2(t) \not\equiv St + \text{const}
\]

on the interval \([0, t_0]\), for some point \( t_0 \in (0, T) \), where \( S \) is the constant defined in (2.4). Then, for all \( x \in (0, 1)\),

\[
    u_t(x, t_0) < \sup_{\tilde{\Gamma}} u_t(x, t) \quad \text{and} \quad u_{xx}(x, t_0) < \sup_{\tilde{\Gamma}} u_{xx}(x, t),
\]

where \( u(x, t) \) is the unique bounded classical solution to problem (2.1)–(2.3).
Remark 2.12. If

$$\phi(x) \equiv k(x^2 - x)/2a + C_1x + C_2, \quad \psi_1(t) \equiv kt + C_2, \quad \psi_2(t) \equiv kt + C_1 + C_2,$$

where $k, C_1, C_2$ are arbitrary constants and $a > 0$ is the constant in (2.1), then the unique bounded classical solution of the problem (2.1)–(2.3) is

$$u(x, t) \equiv k\left[t + (x^2 - x)/2a\right] + C_1x + C_2.$$

2.3. Weak and strong infimum principles

Statements similar to those made in Sections 2.1 and 2.2 hold concerning weak and strong infimum principles for the higher-order derivatives of any bounded classical solution to problem (2.1)–(2.3). To formulate such principles, it suffices to apply the statements corresponding to the supremum principles to the function $\tilde{u}(x, t) \equiv -u(x, t)$. Here we state explicitly the theorems corresponding to Theorems 2.3 and 2.8.

Theorem 2.13 (The weak infimum principle). Let all assumptions of Theorem 2.1 be satisfied, and moreover, $\phi(0) \leq \psi_1(0)$ and $\phi(1) \leq \psi_2(0)$. Then, the unique bounded classical solution $u(x, t)$ to problem (2.1)–(2.3) possesses the properties

$$\inf_{\tilde{Q}_T} u_t(x, t) = \inf_{\Gamma_*} u_t(x, t), \quad \inf_{\tilde{Q}_T} u_{xx}(x, t) = \inf_{\Gamma_*} u_{xx}(x, t),$$

and moreover, the function

$$\tilde{u}(x, t) := u(x, t) - I\left[t + (x^2 - x)/2a\right]$$

possesses the property

$$\tilde{u}_t \equiv a\tilde{u}_{xx} \geq 0 \quad \text{in} \quad \tilde{Q}_T,$$

where $a > 0$ is the constant in (2.1) and $I$ is the constant defined in (2.5). If one of the inequalities

$$\phi(0) < \psi_1(0) \quad \text{or} \quad \phi(1) < \psi_2(0)$$

is satisfied, then

$$\sup_{\tilde{Q}_T} u_t(x, t) = +\infty, \quad \sup_{\tilde{Q}_T} u_{xx}(x, t) = +\infty.$$
Theorem 2.14 (The strong infimum principle). Let all assumptions of Theorem 2.13 be satisfied. If the unique bounded classical solution \( u(x, t) \) to problem (2.1)–(2.3) has at least one of the properties
\[
\begin{align*}
&u_t(x_0, t_0) = \inf_{\tilde{\Gamma}} u_t(x, t) \quad \text{or} \quad u_{xx}(x_0, t_0) = \inf_{\tilde{\Gamma}} u_{xx}(x, t), \\
&\text{at some point } (x_0, t_0) \in Q_T,
\end{align*}
\]
then there are two constants \( C_1 \) and \( C_2 \), such that
\[
\begin{align*}
u(x, t) &\equiv u_t(x_0, t_0) \left( t + \frac{x^2}{2a} \right) + C_1x + C_2 \quad \text{in } \tilde{\Gamma}_0,
\end{align*}
\]
where \( a > 0 \) is the constant in (2.1).

2.4. Two-side estimates of solutions

In this subsection, we illustrate the importance of the principles established above. On the basis of such principles, we are able to derive two-side estimates for bounded classical solutions to the Dirichlet problem (2.1)–(2.3). Theorems 2.16 and 2.17 deal with the estimates for solutions in the case when the compatibility conditions of zero order are either satisfied or not. Both such theorems yield two-side estimates. The only difference is that in the assumptions of the first theorem
\[
\varphi(0) \geq \psi_1(0) \quad \text{and} \quad \varphi(1) \geq \psi_2(0),
\]
while in the assumptions of the second theorem
\[
\varphi(0) \leq \psi_1(0) \quad \text{and} \quad \varphi(1) \leq \psi_2(0).
\]

Theorem 2.15 (A particular two-side estimate). Let all assumptions of Theorem 2.2 be satisfied. Then, the unique bounded classical solution \( u(x, t) \) to problem (2.1)–(2.3) obeys the estimate
\[
\min \{\psi_1(t), \psi_2(t)\}_t \leq u(x, t) \leq \varphi(x) \quad \text{in } Q_T,
\]
and thus,
\[
\sup_{\tilde{\Gamma}_T} u(x, t) = \max_{[0, 1]} \varphi(x), \quad \inf_{\tilde{\Gamma}_T} u(x, t) = \min_{Q_T} \{\psi_1(T), \psi_2(T)\}.
\]

Proof. As \( u(x, t) \) is a continuous function in \( \tilde{\Gamma}_T \) and \( u_t \leq 0 \) in \( Q_T \) (by Theorem 2.2), the estimate
\[
u(x, t) \leq u(x, 0)
\]
holds in \( Q_T \). Furthermore, consider the function of \( x, u(x, t_0) \), for \( x \in [0, 1] \), for an arbitrary fixed value of the parameter \( t_0 \in (0, T] \). The following two cases may occur: either
where \( x_0 \) is some point of \((0, 1)\).

Let us show that case in (2.28) implies that in (2.27). In fact, from (2.28) follows that
\[
\min_{x \in [0, 1]} u(x, t_0) = u(x_0, t_0),
\]
(2.28)

This means that (2.27) is always true. Finally, relations (2.26) and (2.27) prove the theorem.

2.5. Weak and strong principles when the compatibility conditions of zero order are satisfied

The following theorems immediately follow from the previous results of the paper and the classical maximum and minimum principles (see Theorems 0.1, 2.3, 2.8, 2.13, and 2.14).

**Theorem 2.18 (Weak principle).** Let the initial–boundary data of problem (0.1)–(0.3) satisfy the assumptions (1)–(3) of Theorem 2.1 and the compatibility conditions of zero order (1.9) be satisfied. Then, there exists a unique classical solution, \( u(x, t) \), to problem (0.1)–(0.3), its
derivatives \( u_t(x,t), u_x(x,t), \) and \( u_{xx}(x,t) \) are continuous and bounded functions in \( \tilde{Q}_T \) and the following relations are true:

\[
\begin{align*}
\max_{\tilde{Q}_T} u(x,t) &= \max_I u(x,t), \\
\min_{\tilde{Q}_T} u(x,t) &= \min_I u(x,t), \\
\sup_{\tilde{Q}_T} u_t(x,t) &= \sup_I u_t(x,t), \\
\inf_{\tilde{Q}_T} u_t(x,t) &= \inf_I u_t(x,t), \\
\sup_{\tilde{Q}_T} u_{xx}(x,t) &= \sup_I u_{xx}(x,t), \\
\inf_{\tilde{Q}_T} u_{xx}(x,t) &= \inf_I u_{xx}(x,t).
\end{align*}
\]

Also, the two-side estimates (2.29) and (2.30) hold true, where \( a > 0 \) is the constant in (0.1), and \( S \) and \( I \) are those defined in (2.4) and (2.5).

Moreover, the functions

\[
\tilde{u}(x,t) := u(x,t) - S\left[ t + \left( x^2 - x \right)/2a \right], \quad \tilde{u}(x,t) := u(x,t) - I\left[ t + \left( x^2 - x \right)/2a \right]
\]

possess the properties

\[
\tilde{u}_t = a\tilde{u}_{xx} \leq 0 \quad \text{in} \quad \tilde{Q}_T, \quad \tilde{u}_t = a\tilde{u}_{xx} \geq 0 \quad \text{in} \quad \tilde{Q}_T.
\]

**Remark 2.19** (On the two-side estimates (2.29) and (2.30)). For the particular special solution \( u(x,t) := at + \left( x^2 - x \right)/2 \) of problem (0.1)–(0.3), we have

\[
S = I = a, \quad \varphi(x) = \left( x^2 - x \right)/2, \quad \psi_1(t) = \psi_2(t) = at,
\]

and thus, the two-side estimates (2.29) and (2.30) are reduced to the equalities

\[
\begin{align*}
\min\{\psi_1(t), \psi_2(t)\} + \frac{S}{2a} \left( x^2 - x \right) &\equiv u(x,t) \equiv \varphi(x) + St \quad \text{in} \quad Q_T, \\
\varphi(x) + It &\equiv u(x,t) \equiv \max\{\psi_1(t), \psi_2(t)\} + \frac{I}{2a} \left( x^2 - x \right) \quad \text{in} \quad Q_T.
\end{align*}
\]

This result means that the two-side estimates (2.29) and (2.30), in general, cannot be improved.

**Remark 2.20** (On the uniform two-side estimates for the higher-order derivatives). Let \( u_n(x,t), \quad n \in \Omega \), be the classical solutions to the problems

\[
D_t u_n = a_n D_{xx}^2 u_n \quad \text{in} \quad Q_T, \\
u_n(0,t) = \psi_{n,1}(t), \quad u_n(1,t) = \psi_{n,2}(t) \quad \text{for} \quad t \in (0,T], \\
u_n(x,0) = \varphi_n(x) \quad \text{for} \quad x \in [0,1],
\]

where \( a_n > 0 \) are constants. If the higher-order derivatives \( D_t u_n \) of the solutions are continuous and bounded in \( \tilde{Q}_T \) for all \( n \in \Omega \), then they can be estimated uniformly from below and from above as

\[
\inf_{n \in \Omega} I_n \leq D_t u_n \leq \sup_{n \in \Omega} S_n,
\]

in \( \tilde{Q}_T \) for all \( n \in \Omega \), where \( S_n := \sup_{\tilde{Q}_T} D_t u_n \) and \( I_n := \inf_{\tilde{Q}_T} D_t u_n \).
Theorem 2.21 (Strong principle). Let all assumptions of Theorem 2.18 be satisfied. If the unique classical solution \( u(x,t) \) to problem (0.1)–(0.3) possesses at least one of the following four properties

\[
\begin{align*}
    u_t(x_0, t_0) &= \sup_{\tilde{\Gamma}} u_t(x,t), \\
    u_{xx}(x_0, t_0) &= \sup_{\tilde{\Gamma}} u_{xx}(x,t), \\
    u_t(x_0, t_0) &= \inf_{\tilde{\Gamma}} u_t(x,t), \\
    u_{xx}(x_0, t_0) &= \inf_{\tilde{\Gamma}} u_{xx}(x,t),
\end{align*}
\]

at some point \((x_0, t_0) \in Q_T\), then two constants \(C_1\) and \(C_2\) exist such that

\[
    u(x,t) \equiv u_t(x_0, t_0) \left( t + \frac{x^2}{2a} \right) + C_1 x + C_2 \text{ in } Q_{t_0},
\]

where \(a > 0\) is the constant in (0, 1).

The following problem is very interesting:

Problem 2.22 (On the boundedness of the higher-order derivatives). Let \( u(x,t) \) be the unique classical solution to problem (0.1)–(0.3). Are the higher-order derivatives, \( u_t(x,t) \) and \( u_{xx}(x,t) \), of the solution bounded in \( \tilde{Q}_T \) provided that they are continuous in \( \tilde{Q}_T \)?

2.6. Hypothesis

So far, we have been concerned only with the heat equations. Attempting to generalize the previous results to general linear one-dimensional parabolic equations, we make here a hypothesis. Let \( L_x \) be the operator

\[
    L_x u := u_t - a(x) u_{xx} - b(x) u_x - c(x) u,
\]

where the coefficients \(a(x), b(x), c(x)\) are real-valued functions, defined on the interval (0, 1). For the Dirichlet problem

\[
    \begin{align*}
    L_x u &= 0 \quad \text{in } Q_T, \\
    u(0, t) &= \psi_1(t), \quad u(1, t) = \psi_2(t) \quad \text{for } t \in (0, T], \\
    u(x, 0) &= \varphi(x) \quad \text{for } x \in (0, 1),
    \end{align*}
\]

we formulate the following hypothesis.

Hypothesis 2.23 (Weak supremum principle). Let all assumptions of Theorem 2.3 be satisfied, and the coefficients \(a(x), b(x), c(x)\) of the operator \( L_x \), uniformly parabolic in \( Q_T \), belong to the space \( C_\omega[0,1] \), where the continuity modulus \(\omega\) satisfies the Dini condition (1.22) and \(c(x) \leq 0\) in (0, 1). Then, the unique bounded classical solution \( u(x,t) \) to problem (2.31)–(2.33) has the property

\[
    \sup_{\tilde{Q}_T} u_t(x,t) = \sup_{\tilde{\Gamma}} u_t(x,t).
\]
3. On the Green functions

In this section, we consider the family of the Green functions which corresponds to the problem

\[ u_t = a(\zeta, \theta)u_{xx} + f(x, t) \quad \text{in} \quad QT, \]

\[ u(0, t) = 0, \quad u(1, t) = 0 \quad \text{for} \quad t \in (0, T], \]

\[ u(x, 0) = 0 \quad \text{for} \quad x \in [0, 1], \]

for all values of the parameters \( \zeta \in [0, 1] \) and \( \theta \in [0, T] \), where the diffusion coefficient \( a(x, t) \), defined and positive in \( \overline{QT} \), is the coefficient of the operator \( L \) in (1.1). The aim of the section is establishing some important properties of such Green functions between them and with regard to the Green function for general problem (3.5)–(3.7). We first recall certain properties of fundamental solutions and of Green functions, and provide some notation.

3.1. Definitions of Green functions and fundamental solutions

In the literature, some variants exist for the definitions of the Green functions and of the fundamental solutions. In this paper, we use the following definitions.

**Definition 3.1.** By “fundamental solution” to equation \( Lu = 0 \) in the slab \( HT \), we mean a function \( Z(x, t, \xi, \tau) \), defined and continuous on the set

\[ D_T := \{(x, t, \xi, \tau): x, \xi \in \mathbb{R}, 0 \leq \tau < t \leq T\}, \]

along with its derivatives \( \frac{\partial Z}{\partial t} \), \( \frac{\partial Z}{\partial x} \), and \( \frac{\partial^2 Z}{\partial x^2} \), enjoying the following properties:

(1) \( LZ = 0 \) in \( D_T \) (note that the operator \( L \) acts on \( x, t \));
(2) \( Z(x, t, \xi, \tau) \) is bounded in \( D_T \) on the set \( |x - \xi| + t - \tau \geq \delta \), for every \( \delta > 0 \);
(3) for every function \( \varphi(x) \), continuous and compactly-supported in \( \mathbb{R} \), and for every \( \tau \in [0, T) \),

\[ \lim_{t \to \tau + 0} \int_{-\infty}^{+\infty} Z(x, t, \xi, \tau)\varphi(\xi) \, d\xi = \varphi(x) \quad \text{(3.4)} \]

for all \( x \in \mathbb{R} \), and the convergence is uniform with respect to \( x \) in every compact subset of \( \mathbb{R} \).

**Remark 3.2.** Let \( C > 0 \) and \( \alpha, \beta \in (0, 1) \) be arbitrary given constants, and \( \gamma(x) \in C[0, +\infty) \) any given nonnegative function such that

\[ \lim_{x \to +\infty} \gamma(x) = 0 \]

and the function \( x\gamma(x) \) is monotone nondecreasing for \( x \geq 0 \). If the coefficients, \( a(x, t) \), \( b(x, t) \), and \( c(x, t) \) of the operator \( L \) defined in (1.1), satisfy all conditions
\[ 0 \leq a(x, t) \leq C \frac{1 + x^2}{t^{1-\alpha}}, \quad |b(x, t)| \leq C \frac{1 + |x|}{t^{1-\alpha}} \left[ 1 + s\gamma(s) \right], \]
\[ c(x, t) \leq C \frac{1 + \log^2(1 + |x|)}{t^{1-\alpha}} \]

in \( H_T \), where (we set, for short) \( s := \log(1 + |x|) \), or, alternatively, they satisfy the conditions

\[ 0 \leq a(x, t) \leq C \frac{1}{t^{1-\alpha}}, \quad |b(x, t)| \leq C \frac{1 + |x||'|(x)}{t^{1-\alpha}}, \quad c(x, t) \leq C \frac{1 + x^2}{t^{1-\alpha}} \]

in \( H_T \), then equation \( Lu = 0 \) may have at most one fundamental solution in the slab \( H_T \).

**Proof.** The proof follows the same lines of that of [13, Section 4, Theorem 1], but with now [16, Theorem 1] should be used as a uniqueness theorem. \( \square \)

**Definition 3.3.** By “Green function” for the problem

\[ Lu = f(x, t) \quad \text{in} \ Q_T, \quad (3.5) \]
\[ u(0, t) = 0, \quad u(1, t) = 0 \quad \text{for} \ t \in (0, T], \quad (3.6) \]
\[ u(x, 0) = 0 \quad \text{for} \ x \in [0, 1], \quad (3.7) \]

we mean a function \( G(x, t, \xi, \tau) \) with the following properties:

(1) it is defined and continuous on the set

\[ \Gamma_T := \{(x, t, \xi, \tau): x, \xi \in [0, 1], \ 0 \leq \tau < t \leq T\}; \]

(2) for any fixed \( (x, t) \in [0, 1] \times (0, T] \), the function \( G(x, t, \xi, \tau) \) is Lebesgue integrable on the set \( \{(\xi, \tau) \in [0, 1] \times [0, t]\}; \)

(3) if \( f(x, t) \in C^{0,0}_{x,t}(\overline{Q_T}) \), \( 0 < \lambda < 1 \), then the function

\[ u(x, t) := \int_0^t \int_0^1 G(x, t, \xi, \tau) f(\xi, \tau) \, d\xi \, d\tau \]

is a classical solution to problem (3.5)–(3.7) (see Definition 1.1 and Remark 1.7). Here the double integral is intended in the sense of Lebesgue.

**Remark 3.4.** If the coefficient \( c(x, t) \) of the operator \( L \) defined in (1.1), parabolic in \( Q_T \), is such that \( \sup_{Q_T} c(x, t) < +\infty \), then it may exist at most one Green function for the problem (3.5)–(3.7).
3.2. Some properties of Green functions and fundamental solutions

As is well known, the following two existence theorems hold [8,10,13–15]:

**Theorem 3.5.** Suppose that the coefficients of the operator $L$, defined in (1.1), uniformly parabolic in $H_T$, belong to $C_{x,t}^{\alpha,0}(\bar{H}_T)$, $0 < \alpha < 1$. Then, there exists a unique fundamental solution, $Z(x,t,\xi,\tau)$, to equation $Lu = 0$ in the slab $H_T$, for which the estimates

$$\left| D_{k,l}^{k,l} Z(x,t,\xi,\tau) \right| \leq C \left( \frac{1}{(t-\tau)^{k+l/2+1/2}} e^{-M(x-\xi)^2/(t-\tau)} \right)$$

hold in $H_T$ for $2k + l \leq 2$, $C$ and $M$ being positive constants.

**Remark 3.6.** The constant $C$ in Theorem 3.5 depends only on the maximum of the $C_{x,t}^{\alpha,0}(\bar{H}_T)$-norms of the coefficients of $L$, on $\inf_{H_T} a(x,t)$, $T$, and $\alpha$, while the constant $M$ depends only on $\inf_{H_T} a(x,t)$.

**Theorem 3.7.** Suppose that the coefficients of the operator $L$, uniformly parabolic in $Q_T$, belong to $C(\bar{Q}_T)$, with the continuity modulus $\omega(\delta) \equiv \delta^\alpha$, where $\alpha \in (0,1)$ is an arbitrary fixed constant. Then, there exists a unique Green function $G(x,t,\xi,\tau)$ for problem (3.5)–(3.7), and moreover:

1. The derivatives $\frac{\partial G}{\partial t}$, $\frac{\partial G}{\partial x}$, and $\frac{\partial^2 G}{\partial x^2}$ are continuous functions in $\Gamma_T$ and satisfy the estimates

$$\left| D_{k,l}^{k,l} G(x,t,\xi,\tau) \right| \leq C \left( \frac{1}{(t-\tau)^{k+l/2+1/2}} e^{-M(x-\xi)^2/(t-\tau)} \right),$$

$$\left| D_{k,l}^{k,l} G(x,t,\xi,\tau) - D_{k,l}^{k,l} \tilde{Z}(x,t,\xi,\tau) \right| \leq C \left( \frac{1}{(t-\tau)^{k+l/2+1/2}} e^{-M(|x-\xi|+\rho(\xi))^2/(t-\tau)} \right),$$

in $\Gamma_T$ for $2k + l \leq 2$, where $C, M > 0$ are constants, and $\rho(x) := \min\{x, 1-x\}$ for $x \in [0,1]$. Here $\tilde{Z}(x,t,\xi,\tau)$ is the fundamental solution to equation $\tilde{L}u = 0$ in the slab $H_T$, where

$$\tilde{L}u := u_t - \tilde{a}(x,t)u_{xx} - \tilde{b}(x,t)u_x - \tilde{c}(x,t)u,$$

the coefficients $\tilde{a}(x,t) \geq a_0 > 0$, $\tilde{b}(x,t)$, and $\tilde{c}(x,t)$ are real-valued functions representing an extension of the coefficients of $L$ from $C(\bar{Q}_T)$ to the space $C(\bar{H}_T)$.

2. $LG = 0$ in $\Gamma_T$.

3. $G(x,t,\xi,\tau) = 0$ in $\Gamma_T$, for $x = 0$ and $x = 1$.

4. If $f(x,t) \in C_{x,t}^{\lambda,0}(\bar{Q}_T)$ ($0 < \lambda < 1$) and $\varphi(x) \in C[0,1]$, then the function

$$u(x,t) := \int_0^1 G(x,t,\xi,0)\varphi(\xi) d\xi + \int_0^t \int_0^1 G(x,t,\xi,\tau) f(\xi,\tau) d\xi d\tau \quad (3.8)$$

is the bounded classical solution to the problem.
(see Definition 1.5 and Remarks 1.7 and 1.8).

**Remark 3.8.** The constants $C$ and $M$ in Theorem 3.7 depend only on the $C_\omega(Q_T)$-norms of the coefficients of $L$, on $\inf_{Q_T} a(x,t)$, $T$, and $\omega(\delta) \equiv \delta^\alpha$ (i.e., on $\alpha$).

**Remark 3.9.** Under the assumptions of Theorem 3.7, its statement (4) can be generalized as follows. If $f(x,t) \in C_\omega(Q_T), 0 < \lambda < 1$, and $\varphi(x) \in C[0,1]$, then, for any given $t_0 \in [0,T)$, the function

$$u(x,t) := \int_0^1 G(x,t,\xi,t_0)\varphi(\xi)\,d\xi + \int_{t_0}^t \int_0^1 G(x,t,\xi,\tau)f(\xi,\tau)\,d\xi\,d\tau$$

is the bounded classical solution to the problem

$$Lu = f(x,t) \quad \text{in} \quad \{(x,t) \in (0,1) \times (t_0,T]\},$$

$$u(0,t) = 0, \quad u(1,t) = 0 \quad \text{for} \ t \in (t_0,T],$$

$$u(x,t_0) = \varphi(x) \quad \text{for} \ x \in (0,1).$$

### 3.3. Some special notations

At this point, we introduce some notation for special fundamental solutions and Green functions, which play an important role in what follows.

**Definition 3.10.** With the symbol $Z_{\xi,\theta}(x,t,\xi,\tau)$, we denote the unique fundamental solution to $u_t = a(\xi,\theta)u_{xx}$ in the slab $H_T$, for fixed values of the parameters $\xi \in \mathbb{R}$ and $\theta \in [0,T]$. Here, the coefficient $a(x,t)$, defined and positive in $Q_T$, is the diffusion coefficient of the operator $L$ in (1.1).

**Definition 3.11.** With the symbol $G_{\xi,\theta}(x,t,\xi,\tau)$, we denote the unique Green function for problem (3.1)--(3.3), for fixed values of the parameters $\xi \in [0,1]$ and $\theta \in [0,T]$. Here, the coefficient $a(x,t)$, defined and positive in $Q_T$, is the diffusion coefficient of the operator $L$ in (1.1).

**Remark 3.12.** For any given constant coefficient $a(\xi,\theta) > 0$, the two functions $Z_{\xi,\theta}$ and $G_{\xi,\theta}$, uniquely defined by Theorems 3.5 and 3.7.

**Remark 3.13.** Under the assumptions of Theorem 3.5, the fundamental solutions $Z$ and $Z_{\xi,\theta}$, defined in Definitions 3.1 and 3.10, satisfy the estimates
\[ |D_{t,x}^{k,l} Z_{\xi,\theta}(x,t,\xi,\tau) - D_{t,x}^{k,l} Z_{\xi,0,\theta,0}(x,t,\xi,\tau)| \leq C \frac{\omega_c(|\xi - \xi_0|) + \omega_c(\sqrt{\theta - \theta_0})}{(t-\tau)^{k+l/2+1/2}} e^{-M(x-\xi)^2/(t-\tau)}, \]

\[ |D_{t,x}^{k,l} Z(x,t,\xi,\tau) - D_{t,x}^{k,l} Z_{\xi,\tau}(x,t,\xi,\tau)| \leq C \frac{1}{(t-\tau)^{k+l/2+1/2}} e^{-M(x-\xi)^2/(t-\tau)}, \]

for \(2k+l \leq 2, x, \xi, \zeta, \zeta_0 \in \mathbb{R}, \theta, \theta_0 \in [0, T],\) and \(0 \leq \tau < t \leq T,\) where \(C, M, \lambda > 0\) are constants and \(\omega_c(\delta) \equiv \delta^\alpha\) is the continuity modulus involved in the definition of the space \(C_{x,t}^{\alpha,0}(\mathcal{H}_T)\) in Theorem 3.5, see [13,15].

### 3.4. Connection between fundamental solutions and Green functions

Consider the Cauchy problem

\[ Lu = f(x,t) \quad \text{in} \quad H_T, \quad (3.12) \]
\[ u(x,0) \equiv 0 \quad \text{for} \quad x \in \mathbb{R}, \quad (3.13) \]

where \(H_T = \{(x,t) \in \mathbb{R} \times (0, T]\) and \(L\) is the operator defined in (1.1).

**Definition 3.14.** A function \(u(x,t)\) is called a classical solution to problem \((3.12), (3.13),\) if:

1. it is defined and continuous on \(\mathcal{H}_T,\) and possesses continuous derivatives \(u_t, u_x,\) and \(u_{xx}\) in \(H_T;\)
2. it satisfies Eq. (3.12) in \(H_T\) and the initial data (3.13) in the classical sense.

**Definition 3.15.** By “Green function” for the Cauchy problem \((3.12), (3.13),\) we intend a function \(Z_0,\) which has the following properties:

1. it is defined and continuous on the set \(D_T = \{(x,t,\xi,\tau): x, \xi \in \mathbb{R}, 0 \leq \tau < t \leq T\};\)
2. for any fixed \((x,t) \in \mathbb{R} \times (0, T],\) the function \(Z_0(x,t,\xi,\tau)\) is Lebesgue integrable on the set \(\{(\xi, \tau) \in \mathbb{R} \times [0,t]\};\)
3. if \(f(x,t) \in C_{x,t}^{k,0}(\mathcal{H}_T),\) \(0 < \lambda < 1,\) then the function

\[ u(x,t) := \int_0^{+\infty} \int_{-\infty}^{t} Z_0(x,t,\xi,\tau) f(\xi,\tau) \, d\xi \, d\tau \]

is a bounded classical solution to the Cauchy problem \((3.12), (3.13)\) (see Definition 3.14). Here the integral is a double Lebesgue integral.

**Remark 3.16.** If the coefficients of the operator \(L\) satisfy the assumptions in Remark 3.2, then at most one Green function for the Cauchy problem \((3.12), (3.13)\) may exist.
Remark 3.17. If all coefficients of the operator $L$, uniformly parabolic in $H_T$, belong to the space $C^{\alpha,0}_{x,t}(\overline{H_T})$ ($0 < \alpha < 1$), then Definitions 3.1 and 3.15 are equivalent, i.e., there exist just one function $Z(x,t,\xi,\tau)$ and just one function $Z_0(x,t,\xi,\tau)$, and

$$Z(x,t,\xi,\tau) \equiv Z_0(x,t,\xi,\tau) \text{ in } D_T.$$ 

This shows that the fundamental solution to $Lu = 0$ in the slab $H_T$ is nothing but the Green function for the Cauchy problem (3.12), (3.13), under certain assumptions on the coefficients of the operator $L$ (e.g., as in Remark 3.17).

3.5. Some properties of a certain family of Green functions

For a given coefficient $a(x,t)$, positive in $Q_T$, and taking various values of the parameters $\zeta \in [0,1]$ and $\theta \in [0,T]$, we construct a family of the Green functions $G_{\zeta,\theta}(x,t,\xi,\tau)$ (Definition 3.11). In view of Remark 3.17, it is natural to expect that the Green functions $G$ and $G_{\zeta,\theta}$ possess properties similar to those of the fundamental solutions $Z$ and $Z_{\zeta,\theta}$, appearing in Remark 3.13. This is partially confirmed by the following theorem.

Theorem 3.18. If the coefficient $a(x,t)$ of the operator $L$, uniformly parabolic in $Q_T$, belongs to $C_\omega(Q_T)$, then the estimate

$$|D^{k,l}_{x,t}G_{\zeta,\theta}(x,t,\xi,\tau) - D^{k,l}_{x,t}G_{\zeta_0,\theta_0}(x,t,\xi,\tau)| \leq C_{k,l} \frac{\omega(|\zeta - \zeta_0|) + \omega_c(\sqrt{|\theta - \theta_0|})}{(t-\tau)^{k+l/2+1/2}} e^{-M(x-\xi)^2/(t-\tau)},$$

(3.14) holds for all $k,l = 0, 1, 2, \ldots$, $(x,t,\xi,\tau) \in \Gamma_T$, $\zeta,\zeta_0 \in [0,1]$, and $\theta,\theta_0 \in [0,T]$, for some positive constants $C_{k,l}$ and $M$.

Remark 3.19. We stress that the continuity modulus $\omega$ (defined in Definition 1.16) is in such theorem arbitrary.

Proof of Theorem 3.18. The function $G_{\zeta,\theta}(x,t,\xi,\tau)$ can be written explicitly as

$$G_{\zeta,\theta}(x,t,\xi,\tau) = \frac{1}{2\sqrt{\pi a(t-\tau)}} \sum_{k=-\infty}^{+\infty} \left( e^{-(2k+\xi-x)^2/4a(t-\tau)} - e^{-(2k+\xi+x)^2/4a(t-\tau)} \right),$$

(3.15)

where $a = a(\zeta,\theta)$, see [10,18,27]. This function is infinitely differentiable in $\Gamma_T$ with respect to the variables $(x,t,\xi,\tau)$, and its every derivative can be evaluated by differentiating the series term by term. Moreover, for every $k = 0, 1, 2, \ldots$ and for every $l = 0, 1, 2, \ldots$, the derivative $D^{k,l}_{x,t}G_{\zeta,\theta}$ can be estimated as

$$|D^{k,l}_{x,t}G_{\zeta,\theta}(x,t,\xi,\tau)| \leq \frac{C_{k,l}}{(t-\tau)^{k+l/2+1/2}} e^{-M(x-\xi)^2/(t-\tau)},$$

(3.16)

for $(x,t,\xi,\tau) \in \Gamma_T$, $\zeta \in [0,1]$, and $\theta \in [0,T]$, where $C_{k,l} > 0$ and $M > 0$ are constants. For $2k+l \leq 2$, estimate (3.16) follows from Theorem 3.7 and Remark 3.8. Let us prove the estimate
for $2k + l > 2$. To this purpose, consider the function $v(x, t) \equiv D_t G_{\xi, \theta}(x, t, \xi, \tau)$ on the domain $Q_{\tau, T} = \{(x, t) \in [0, 1] \times (\tau, T]\}$, for arbitrary but fixed parameters $\xi, \zeta \in [0, 1]$, $\theta \in [0, T]$, and $\tau \in [0, T)$. This function is the classical solution to the Dirichlet problem

\[ v_t = a(\zeta, \theta) v_{xx} \quad \text{in} \quad Q_{\tau + \epsilon, T}, \]

\[ v(0, t) = 0, \quad v(1, t) = 0 \quad \text{for} \quad t \in (\tau + \epsilon, T], \]

\[ v(x, \tau + \epsilon) = D_t G_{\xi, \theta}(x, \tau + \epsilon, \xi, \tau) \quad \text{for} \quad x \in [0, 1], \]

where $\epsilon \in (0, T - \tau)$ is arbitrary. Therefore, $v(x, t)$ can be represented as

\[ v(x, t) = \frac{1}{2} \int_0^1 G_{\xi, \theta}(x, t, \eta, \tau + \epsilon) D_t G_{\xi, \theta}(\eta, \tau + \epsilon, \xi, \tau) d\eta, \]

in the domain $Q_{\tau + \epsilon, T}$ (see Remark 3.9), and thus,

\[ D_{k,l}^{t,x} v(x, t) = \frac{1}{2} \int_0^1 D_{k,l}^{t,x} G_{\xi, \theta}(x, t, \eta, \tau + \epsilon) D_t G_{\xi, \theta}(n, \tau + \epsilon, \xi, \tau) d\eta \quad (3.17) \]

in $Q_{\tau + \epsilon, T}$, for all $k, l = 0, 1, 2, \ldots$. Estimate (3.16) (already established for $2k + l \leq 2$) and (3.17) yield

\[ |D_{k,l}^{t,x} v(x, t)| \leq \frac{1}{2} C_{k,l}^{t,x} e^{-M(x-\eta)^2/(t-\tau-\epsilon)} \int_0^1 e^{-(\eta-\xi)^2/\epsilon} d\eta \]

for $2k + l \leq 2$. Setting here $\epsilon = (t - \tau)/2$ and changing variable of integration, setting $\eta = y + x$, we obtain

\[ |D_{k,l}^{t,x} v(x, t)| \leq \frac{2^{k+l/2+2} C_{k,l}^{t,x} C_{1,0}^{t,x}}{(t - \tau)^{k+l/2+2}} \int_{-\infty}^{\infty} e^{-2M(y^2 + (y + x - \xi)^2)/(t-\tau)} dy \quad (3.18) \]

for $2k + l \leq 2$. Furthermore, using the well-known result $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$, we obtain

\[ \int_{-\infty}^{\infty} e^{-k(y^2 + (y + z)^2)} dy = \sqrt{\frac{\pi}{2k}} e^{-kz^2/2} \quad (3.19) \]

valid for every $k > 0$ and for every $z \in \mathbb{R}$. We, finally, obtain from (3.18) and (3.19)

\[ |D_{k,l}^{t,x} v(x, t)| \leq \frac{2^{k+l/2+1} C_{k,l}^{t,x} C_{1,0}^{t,x}}{(t - \tau)^{k+l/2+3/2}} \sqrt{\frac{\pi}{M}} e^{-M(x-\xi)^2/(t-\tau)} \quad (3.20) \]
for \(2k + l \leq 2\). This means that the estimate in (3.16) is proved for \(2k + l \leq 4\), as the estimates for \(D^3_x G_{\xi, \theta}\) and \(D^4_x G_{\xi, \theta}\) follow from (3.20) and the relations \(D_{t,x}^{1,1} G_{\xi, \theta} = a(\xi, \theta) D_x^3 G_{\xi, \theta}\) and \(D_{t,x}^{1,2} G_{\xi, \theta} = a(\xi, \theta) D_x^4 G_{\xi, \theta}\). Similarly, using the estimate (3.16) (already established for \(2k + l \leq 4\)) and representation in (3.17), we obtain the estimate (3.16) for \(2k + l \leq 6\), and so on: the estimate (3.16) can be obtained step-by-step for all \(k, l = 0, 1, 2, \ldots\).

Furthermore, consider the function in (3.15) and all its derivatives \(D_{t,x}^{k,l} G_{\xi, \theta}(x, t, \xi, \tau)\) as functions of the single variable \(a := a(\xi, \theta)\), for arbitrary but fixed parameters \((x, t, \xi, \tau) \in \Gamma_T\). Then, the relation

\[
\frac{d}{da} \left( D_{t,x}^{k,l} G_{\xi, \theta} \right) = ka^{k-1} D_x^{2k+l} G_{\xi, \theta} + a^{k-1} (t - \tau) D_{t,x}^{1,2k+l} G_{\xi, \theta}
\]

(3.21)

holds for all \(k, l = 0, 1, 2, \ldots\). In fact, the function in (3.15) can be considered as a function of \(x, \xi\), and the quantity \(a(t - \tau)\), i.e., as \(\tilde{G}(x, \xi, a(t - \tau))\). Then, \(D_{t,x}^{1} G_{\xi, \theta}(x, t, \xi, \tau) = D_x^{1} \tilde{G}(x, \xi, a(t - \tau))\), and hence,

\[
\frac{d}{da} \left( D_{t,x}^{1} G_{\xi, \theta} \right) = \frac{t - \tau}{a} D_{t,x}^{1,1} G_{\xi, \theta}
\]

(3.22)

for all \(l = 0, 1, 2, \ldots\). This relation coincides with that in (3.21) for \(k = 0\). For \(k > 0\),

\[
D_{t,x}^{k,l} G_{\xi, \theta} = a D_{t,x}^{k-1,l+2} G_{\xi, \theta} = \cdots = a^k D_x^{2k+l} G_{\xi, \theta},
\]

as \(D_{t,x}^{1} G_{\xi, \theta} = a D_x^{2} G_{\xi, \theta}\). It follows that

\[
\frac{d}{da} \left( D_{t,x}^{k,l} G_{\xi, \theta} \right) = a^{k-1} D_x^{2k+l} G_{\xi, \theta} + a^{k-1} (t - \tau) D_{t,x}^{1,2k+l} G_{\xi, \theta}
\]

for all \(k, l = 0, 1, 2, \ldots\), and it only remains to use (3.22). Relation (3.21) is thus proved. By (3.16) and (3.21), we have

\[
\left| \frac{d}{da} \left( D_{t,x}^{k,l} G_{\xi, \theta} \right) \right| \leq \frac{a^{k-1}(kC_{0,2k+l} + C_{1,2k+l})}{(t - \tau)^{k+l+1/2+1/2}} e^{-M(x - \xi)^2/(t - \tau)}
\]

\[
\leq \frac{\tilde{C}_{k,l}}{(t - \tau)^{k+l+1/2+1/2}} e^{-M(x - \xi)^2/(t - \tau)}
\]

for all \(k, l = 0, 1, 2, \ldots\) and for every \(a \in [a_0, A]\), where

\[
a_0 := \inf_{\Omega_T} a(x, t) > 0, \quad A := \sup_{\Omega_T} a(x, t) < +\infty, \quad \tilde{C}_{k,l} := \frac{A(kC_{0,2k+l} + C_{1,2k+l})}{a_0}.
\]

It follows by Lagrange theorem that

\[
\left| D_{t,x}^{k,l} G_{\xi, \theta}(x, t, \xi, \tau) - D_{t,x}^{k,l} G_{\xi, \theta}(x, t, \xi, \tau_0) \right| \leq \tilde{C}_{k,l} \frac{|a(\xi, \theta) - a(\xi_0, \theta_0)|}{(t - \tau)^{k+l+1/2+1/2}} e^{-M(x - \xi)^2/(t - \tau)}
\]

for all \(k, l = 0, 1, 2, \ldots, (x, t, \xi, \tau) \in \Gamma_T, \xi, \xi_0 \in [0, 1]\), and \(\theta, \theta_0 \in [0, T]\). Since the coefficient \(a(x, t)\) of the operator \(L\) is in \(C_0(\bar{Q}_T)\), we derive from this estimate and from the properties of the continuity moduli the thesis of the theorem, which is thus proved. \(\square\)
3.6. An important estimate

If the coefficients of the uniformly parabolic operator \( L \) belong to \( C_\omega(Q_T) \), with a continuity modulus \( \omega \), double-integrable according to Dini (Definition 1.20), then there exists a unique Green function \( G(x,t,\xi,\tau) \) for problem (3.5)–(3.7) (see [23]). In this subsection, we formulate the following theorem.

**Theorem 3.20.** If all coefficients of the operator \( L \), uniformly parabolic in \( Q_T \), belong to \( C_\omega(Q_T) \), with a continuity modulus \( \omega \), double-integrable according to Dini, then the Green functions \( G(x,t,\xi,\tau) \) and \( G_{\xi,\theta}(x,t,\xi,\tau) \) (defined in Definitions 3.3 and 3.11) satisfy the estimates

\[
|D_{t,x}^{k,l}G(x,t,\xi,\tau) - D_{t,x}^{k,l}G_{\xi,\theta}(x,t,\xi,\tau)| \leq C \frac{\omega^*(\sqrt{t-\tau})}{(t-\tau)^{k+l/2+1/2}} e^{-M(x-\xi)^2/(t-\tau)} \tag{3.23}
\]

for \( 2k + l \leq 2 \), \((x,t,\xi,\tau) \in \Gamma_T \), \( \xi, \xi_0 \in [0,1] \), and \( \theta, \theta_0 \in [0,T] \), where \( C, M > 0 \) are constants and \( \omega^*(t) := \int_0^T \omega(\sigma)/\sigma \, d\sigma \) (see Definition 1.20).

The proof of this theorem, only formulated here, is left to a future paper.

4. Uniform \( L^1 \)-estimates for the higher-order derivatives

We turn now our attention to the main goal of the paper. Establishing *uniform* \( L^1 \)-estimates for the higher-order derivatives of bounded classical solutions of the Dirichlet problem for linear second-order one-dimensional parabolic equations of general form, *without* assuming any compatibility condition between initial and boundary data. Such estimates can be derived from the results established in Sections 0, 2, and 3. Consider the Dirichlet problem (1.2), (1.3), (1.10). To date, a satisfactory theory does exist for a number of initial–boundary value problems *with* compatibility conditions, for parabolic equations in Hölder spaces as well as in weighted Hölder spaces, see [4,5,10,13–15,19]. A few generalizations also exist for the case of function spaces depending of some given continuity modulus, see [1–3,11,12,22,23]. For parabolic boundary–value problems *without* compatibility conditions between initial and boundary data, only few special results have been established so far, see [7,13,27], for instance.

In this section, we are concerned with *uniform* \( L^1 \)-estimates for the higher-order derivatives of bounded classical solutions to problem (1.2), (1.3), (1.10). When the coefficients \( a(x,t), b(x,t), \) and \( c(x,t) \) of the uniformly parabolic operator \( L \) and the right-hand side \( f(x,t) \) of Eq. (1.2), belong to \( C_{x,t}^{\alpha,\alpha/2}(\overline{Q}_T) \), while the initial data \( \varphi(x) \in C^{2+\alpha}[0,1], \) and \( \psi_1(t) \equiv \psi_2(t) \equiv 0 \), it is well known that, under the compatibility conditions of second order, there exists a unique bounded classical solution \( u(x,t) \) to problem (1.2), (1.3), (1.10). Moreover, this solution belongs to \( C_{x,t}^{2+\alpha,1+\alpha/2}(\overline{Q}_T) \), see [4,19]. That is, the higher-order derivatives of the solution \( u_t \) and \( u_{xx} \) are continuous and bounded on \( \overline{Q}_T \).

**Remark 4.1.** In [7], it has been shown that, when *at least one* of the compatibility conditions of second order are *perturbed*, the higher-order derivatives of any bounded classical solution \( u(x,t) \) to the problem become, in general, *unbounded* in the neighborhoods of those points of the boundary where the aforementioned compatibility conditions are *not* satisfied. Besides, the behavior of such derivatives is *worse* (i.e., such derivatives tend to infinity, in general, *faster*) whenever the *lower-order* compatibility conditions are perturbed.
If, for instance, the compatibility conditions of zero order are not satisfied, the estimate
\[ |u_t(x, t)| \leq \frac{C}{\min^2\{x, 1-x\} + t} \]  
(4.1)
can be established for \( x \in (0, 1) \) and \( t \in (0, T] \), where \( C > 0 \) is a constant [7]. It follows that
\[ \int_0^T |u_t(x, \tau)| d\tau \leq C \log \left(1 + \frac{T}{\min^2\{x, 1-x\}}\right) \]  
for \( x \in (0, 1) \),
(4.2)
with the same constant \( C \). Therefore, the estimate in (4.1) implies that in (4.2), which concerns the \( L^1 \)-norm of \( u_t(x, \cdot) \). The latter, however, is nonuniform with respect to \( x \), as \( x \to 0^+ \) and \( x \to 1^- \).

The aim of this section is to establish that, in fact (at least under the same assumptions made above on the data), the uniform estimate
\[ \int_0^T |u_t(x, \tau)| d\tau \leq C \]
holds, where the constant \( C \) does not depend on \( x \in [0, 1] \). Such an estimate will be proved under considerably weaker restrictions on the data of problem (1.2), (1.3), (1.10). For example, the initial data \( \varphi(x) \) can be only required to belong to \( C^\alpha[0, 1] \) for an arbitrary \( \alpha > 0 \).

**Remark 4.2.** If \( \varphi(0) \neq \psi_1(0) \) or \( \varphi(1) \neq \psi_2(0) \), then, for a classical solution \( u(x, t) \) to problem (1.2), (1.3), (1.10), an estimate like
\[ \int_0^t |u_t(x, \tau)| d\tau \leq \gamma(t), \]
where the function \( \gamma(t) \) is independent of \( x \in (0, 1) \) and goes to zero as \( t \to 0^+ \), cannot exist.

**Proof.** Note that the estimate
\[ \left| \int_0^t u_t(x, \tau) d\tau \right| \leq \gamma(t), \]
assume to hold for \( x \in (0, 1) \), for a classical solution \( u(x, t) \) to problem (1.2), (1.3), (1.10), implies that
\[ \lim_{t \to 0^+} \left| \psi_1(t) - \varphi(0) \right| = \lim_{t \to 0^+} \lim_{x \to 0^+} \left| u(x, t) - u(x, 0) \right| \leq \lim_{t \to 0^+} \gamma(t), \]
\[ \lim_{t \to 0^+} \left| \psi_2(t) - \varphi(1) \right| = \lim_{t \to 0^+} \lim_{x \to 1^-} \left| u(x, t) - u(x, 0) \right| \leq \lim_{t \to 0^+} \gamma(t). \]
(see Definition 1.5 and Remark 1.7). It follows that $\phi(0) = \psi_1(0)$ and $\phi(1) = \psi_2(0)$, whenever $\lim_{t \to 0^+} \gamma(t) = 0$. These relations are nothing but the compatibility conditions of zero order, and the remark is thus proved. □

4.1. The general Dirichlet problem (1.2), (1.3), (1.10)

First of all, note that, for the Dirichlet problem (2.1)–(2.3) for the heat equation, the following theorem holds.

**Theorem 4.3.** Under assumptions of Theorem 2.2, the relations
\[ \int_0^t |u_t(x, \tau)| d\tau = \varphi(x) - u(x, t) \leq \varphi(x) - \min\{\psi_1(t), \psi_2(t)\} \]
hold in $Q_T$.

The proof of Theorem 4.3 follows from Theorems 2.2 and 2.15. Using Theorems 4.3, 3.18, and 3.20, we can get uniform $L^1$-estimates for the higher-order derivatives of bounded classical solutions to problem (1.2), (1.3), (1.10).

**Theorem 4.4.** Suppose that all the coefficients of the operator $L$, uniformly parabolic in $Q_T$, belong to $C_\omega(Q_T)$ as well as the initial data $\varphi(x) \in C_\omega[0, 1]$, while the right-hand side $f(x, t) \in \mathcal{R}_\omega$ and the boundary functions $\psi_1(t), \psi_2(t) \in B_\omega$, where $\omega(\delta) = \delta^\alpha$ with $\alpha \in (0, 1)$ an arbitrary fixed constant. Then, there exists a unique bounded classical solution $u(x, t)$ to problem (1.2), (1.3), (1.10), and
\[ \int_0^T |D^k,l_{t,x} u(x, \tau)| d\tau \leq C \]
for $x \in [0, 1]$ and $2k + l = 2$, where $C$ is a constant.

**Proof.** Let be, without loss of generality, $\varphi(0) > \psi_1(0)$ and $\varphi(1) > \psi_2(0)$. We can represent the solution $u(x, t)$ to problem (1.2)–(1.4) as $u(x, t) = v(x, t) + w(x, t)$, where $v(x, t)$ is the classical solution to the problem
\[
Lv = f(x, t) \quad \text{in } Q_T, \tag{4.3}
\]
\[
v(0, t) = \psi_1(t), \quad v(1, t) = \psi_2(t) \quad \text{for } t \in (0, T], \tag{4.4}
\]
\[
v(x, 0) = \varphi(x) + [\psi_1(0) - \varphi(0)](1 - x) + [\psi_2(0) - \varphi(1)]x \quad \text{for } x \in [0, 1] \tag{4.5}
\]
and $w(x, t)$ is the bounded classical solution to the problem
\[
Lw = 0 \quad \text{in } Q_T, \tag{4.6}
\]
\[
w(0, t) = 0, \quad w(1, t) = 0 \quad \text{for } t \in (0, T], \tag{4.7}
\]
\[
w(x, 0) = \left[\varphi(0) - \psi_1(0)\right](1 - x) + \left[\varphi(1) - \psi_2(0)\right]x \quad \text{for } x \in (0, 1). \tag{4.8}
\]
There exists a unique classical solution \( v(x,t) \) to problem (4.3)–(4.5), and, in addition,

\[
|D^{k,l}_{t,x} v(x,t)| \leq C \frac{\omega(\sqrt{t})}{t}
\]

in \( Q_T \), for \( 2k + l = 2 \) (see [4]). Recall that, by the assumptions made in the theorem, \( \omega(\delta) \equiv \delta^\alpha \), \( \alpha \in (0, 1) \). It follows that

\[
\int_0^t |D^{k,l}_{t,x} v(x,\tau)| \, d\tau \leq C \omega(\sqrt{t})
\]

(4.9)
in \( \overline{Q}_T \), for \( 2k + l = 2 \), where the constant \( C \) does not depend on \( (x,t) \in \overline{Q}_T \). Consider now problem (4.6)–(4.8). For its solution \( w(x,t) \) we can write, by Theorem 3.7 (see (3.8) with \( f \equiv 0 \)) and Theorem 3.18,

\[
D^{k,l}_{t,x} w(x,t) = I^{k,l}_1(x,t) + I^{k,l}_2(x,t) + I^{k,l}_3(x,t)
\]

\[
:= \int_0^1 D^{k,l}_{t,x} G_{\xi,0}(x,t,\xi,0)|_{\xi=x} w(\xi,0) \, d\xi
\]

\[
+ \int_0^1 \left( D^{k,l}_{t,x} G_{\xi,0}(x,t,\xi,0) - D^{k,l}_{t,x} G_{\xi,0}(x,t,\xi,0)|_{\xi=x} \right) w(\xi,0) \, d\xi
\]

\[
+ \int_0^1 \left( D^{k,l}_{t,x} G(x,t,\xi,0) - D^{k,l}_{t,x} G_{\xi,0}(x,t,\xi,0) \right) w(\xi,0) \, d\xi
\]
in \( Q_T \), for \( 2k + l \leq 2 \). In order to estimate the integral \( I^{k,l}_2 \), it suffices to use (3.14), change the integration variable, \( \xi = x + \sqrt{t}\eta \), and apply Lemma 1.18:

\[
|I^{k,l}_2| \leq C \int_0^1 \frac{\omega(|\xi - x|)}{t^{k+l/2+1/2}} e^{-M(x-\xi)^2/t} \, d\xi = C \int_{-x/\sqrt{t}}^{(1-x)/\sqrt{t}} \frac{\omega(\sqrt{t}|\eta|)}{t^{k+l/2}} e^{-M\eta^2} \, d\eta
\]

\[
\leq C \frac{\omega(\sqrt{t})}{t^{k+l/2}} \int_{-\infty}^{+\infty} (|\eta| + 1) e^{-M\eta^2} \, d\eta \leq C \frac{\omega(\sqrt{t})}{t^{k+l/2}}
\]

(4.10)
in \( Q_T \) for \( 2k + l \leq 2 \), where \( C \) is a constant. Similarly, we conclude from (3.23) that

\[
|I^{k,l}_3| \leq C \int_0^1 \frac{\omega^*(\sqrt{t})}{t^{k+l/2+1/2}} e^{-M(x-\xi)^2/t} \, d\xi \leq C \frac{\omega^*(\sqrt{t})}{t^{k+l/2}} \int_{-\infty}^{+\infty} e^{-M\eta^2} \, d\eta \leq C \frac{\omega^*(\sqrt{t})}{t^{k+l/2}}
\]

(4.11)
in $Q_T$ for $2k + l \leq 2$, where $\omega^*(\tau) := \int_0^\tau \omega(\sigma)/\sigma \, d\sigma$ and $C$ is a constant. It only remains to estimate the integral $I_1^{k,l}$. Note that, for any fixed $x_0 \in [0, 1]$, the integral

$$I_1^{k,l}(x_0, t) := \int_0^1 D_{t,x}^{k,l} G_{\xi,0}(x_0, t, \xi, 0) \bigg|_{\xi = x_0} w(\xi, 0) \, d\xi$$

coincides with $D_{t,x}^{k,l} u^{x_0}(x_0, t)$, the function $u^{x_0}(x, t)$ being the bounded classical solution to the problem

$$u_t^{x_0}(x, t) = a(x, 0) u_{x,x}^{x_0}(x, t) \quad \text{in } Q_T,$$

$$u^{x_0}(0, t) = 0, \quad u^{x_0}(1, t) = 0 \quad \text{for } t \in (0, T],$$

$$u^{x_0}(x, 0) = [\varphi(0) - \psi_1(0)](1 - x) + [\varphi(1) - \psi_2(0)]x \quad \text{for } x \in (0, 1).$$

By Theorem 4.3,

$$\int_0^T |u_t^{x_0}(x_0, \tau)| \, d\tau \leq u^{x_0}(x_0, 0) \leq M := \max\{\varphi(0) - \psi_1(0), \varphi(1) - \psi_2(0)\},$$

where the constant $M$ does not depend on $x_0 \in [0, 1]$. Here, the initial and boundary data for $u^{x_0}(x, t)$ have been exploited. It follows that

$$\int_0^T |I_1^{k,l}(x_0, \tau)| \, d\tau \leq \frac{M}{\min_{x \in [0, 1]} a(x, 0)}$$

(4.12)

for $x_0 \in [0, 1]$ and $2k + l = 2$, where $M$ is the same constant above. On the basis of the estimates (4.9)–(4.12), we conclude that the required $L^1$-estimate holds, and the theorem is thus proved. \(\square\)

4.2. The homogeneous Dirichlet problem

For the Dirichlet problem (3.9)–(3.11) with zero boundary data (3.10), we can establish a stronger result. The following theorem is valid:

**Theorem 4.5.** Let the continuity modulus $\omega$ satisfy the Dini condition, while the continuity modulus $\tilde{\omega}$ is double-integrable according to Dini (see Definitions 1.19 and 1.20). Suppose that all coefficients of the operator $L$, uniformly parabolic in $Q_T$, belong to $C_{\tilde{\omega}}(\bar{Q}_T)$, while the right-hand side $f(x, t) \in R_{\tilde{\omega}}$, and the initial data $\varphi(x) \in C_{\tilde{\omega}}[0, 1]$. Then, there exists a unique bounded classical solution $u(x, t)$ to problem (3.9)–(3.11), and

$$\int_0^T |D_{t,x}^{k,l} u(x, \tau)| \, d\tau \leq C$$

(4.13)

for $x \in [0, 1]$ and $2k + l = 2$, where $C$ is a constant.
Remark 4.6. If the continuity modulus $\tilde{\omega}$ does not satisfy the Dini condition, then the fundamental solution to equation $Lu = 0$ as well as a classical solution to problem (3.9)–(3.11) may not exist, independently of the fulfillment of the compatibility conditions of zero order [1,11,12].

Remark 4.7. If the continuity modulus $\omega$ does not satisfy the Dini condition, then initial data $\varphi(x) \in C[0, 1]$ with $\varphi(0) = 0$ and $\varphi(1) = 0$ do exist such that there is a unique bounded classical solution, $u(x,t)$, to problem (3.9)–(3.11) with the right-hand side $f(x,t) \equiv 0$, but

$$\int_0^T \left| u_t(x_0, \tau) \right| d\tau = +\infty$$

for a certain point $x_0 \in (0, 1)$ (see [3]).

Proof of Theorem 4.5. The proof is carried out in the same way as the proof of Theorem 4.4. The only difference is that, under assumptions weaker compared to those in Theorem 4.4, it is necessary to establish the uniform estimate (4.13) in the case that the compatibility conditions of zero order are satisfied, i.e., when $\varphi(0) = 0$ and $\varphi(1) = 0$. Such an estimate is provided by the following lemma.

Lemma 4.8. If $\varphi(0) = 0$ and $\varphi(1) = 0$ and all the assumptions of Theorem 4.5 are satisfied, then there exists a unique bounded classical solution $u(x,t)$ to problem (3.9)–(3.11), and

$$\left| D^{k,l}_{t,x} u(x,t) \right| \leq C \tilde{\omega}^*(\sqrt{t}) + \omega(\sqrt{t})$$

in $Q_T$ for $2k + l = 2$, where $C$ is a constant and $\tilde{\omega}^*(\tau) := \int_0^\tau \tilde{\omega}(\sigma)/\sigma d\sigma$ (see Definition 1.20).

The proof of this lemma follows from the fact that, under the assumptions made, the Green function to problem (3.5)–(3.7) exists and has certain properties [23]. Consequently, Theorem 4.5 is thus proved.

5. Summary

This paper is concerned with linear parabolic partial differential equations of the second order. A theory is developed for the Dirichlet problem without any compatibility conditions between initial and boundary data. Weak and strong supremum and infimum principles are established, concerning the higher-order derivatives $u_t$ and $u_{xx}$ of bounded classical solutions to such a problem. A generalization of the classical maximum principle is also proved for this problem, and uniform $L^1$-estimates are derived for the higher-order derivatives. These estimates have been established using, in particular, certain properties of the Green functions derived in Theorems 3.18 and 3.20. The importance of such properties rests on the fact that they allow to extend certain properties enjoyed by the solutions of the heat equation to solutions of general parabolic equations. Examples of this are provided by Theorems 4.4 and 4.5. We observe that the new properties of the Green functions established in Theorems 3.18 and 3.20 have an independent interest. Also, we do believe that it is possible to generalize the “weak supremum principle” for the higher-order derivatives from the case of the heat equation to the case of linear second-order one-dimensional parabolic equations of general form (see Hypothesis 2.23).
Acknowledgments

This research was supported, in part, by the Russian Foundation for Basic Research under grant no. 03-01-00162, an INTAS Fellowship for Young Scientists (Post Doctoral Fellowship 03-55-778), and the GNAMPA and GNFM of the Italian INdAM. The authors are grateful to Professor Tadei I. Zelenyak for stating the problem investigated in Section 2 and to Professor Vladimir S. Belonosov for his continuous interest and encouragement in this work.

References