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U-max-statistics

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Abstract

In 1948, W. Hoeffding [W. Hoeffding, A class of statistics with asymptotically normal distribution, Ann. Math. Statist. 19 (1948) 293–325] introduced a large class of unbiased estimators called U-statistics, defined as the average value of a real-valued k-variate function h calculated at all possible sets of k points from a random sample. In the present paper, we investigate the corresponding extreme value analogue which we shall call U-max-statistics. We are concerned with the behavior of the largest value of such a function h instead of its average. Examples of U-max-statistics are the diameter or the largest scalar product within a random sample. U-max-statistics of higher degrees are given by triameters and other metric invariants.

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1. Introduction

U-statistics form a very important class of unbiased estimators for distributional properties such as moments or Spearman's rank correlation. A U-statistic of degree k with symmetric kernel h is a function of the form

$$U(\xi_1, \dots, \xi_n) = \binom{n}{k}^{-1} \sum_J h(\xi_{i_1}, \cdots, \xi_{i_k}),$$

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where the sum is over $J = \{(i_1, \ldots, i_k): 1 \le i_1 < \cdots < i_k \le n\}$, $\xi_1, \ldots, \xi_n$ are random elements in a measurable space $\mathcal{S}$, and h is a real-valued Borel function on $\mathcal{S}^k$, symmetric in its k arguments. In his seminal paper, Hoeffding [9] defined U-statistics for not necessarily symmetric kernels and for random points in d-dimensional Euclidean space $\mathbb{R}^d$. Later the concept was extended to arbitrary measurable spaces. Since 1948, most of the classical asymptotic results for sums of i.i.d. random variables have been formulated in the setting of U-statistics, such as central limit laws, strong laws of large numbers, Berry–Esséen type bounds, and laws of the iterated logarithm.

The purpose of this article is to investigate the extreme value analogue of U-statistics, i.e.

$$H_n = \max_J h(\xi_{i_1}, \ldots, \xi_{i_k}).$$

A typical example of such U-max-statistic is the diameter of a sample of points in a metric space, obtained by using the metric as kernel. Grove and Markvorsen [7] introduced an infinite sequence of metric invariants generalizing the notion of diameter to "triameter", "quadrameter", etc. on compact metric spaces. Their k-extent is the maximal average distance between k points, which is an example of a U-max-statistic of arbitrary degree k. Further examples are given by the largest surface area or perimeter of a triangle formed by point triplets as well as the largest scalar product within a sample of points in $\mathbb{R}^d$.

The key to our results is the observation that for all $z \in \mathbb{R}$ the U-max-statistic H_n does not exceed z if and only if U_z vanishes, where

$$U_z = \sum_J \mathbf{1}\{h(\xi_{i_1}, \ldots, \xi_{i_k}) > z\}.$$

The random variable U_z counts the number of exceedances of the threshold z and is a normalized U-statistic in the usual sense. We approximate its distribution by means of a Poisson approximation result for the sum of dissociated random indicator kernel functions by Barbour et al. [3], which determines the distribution of H_n up to some known error. In order to deduce the corresponding limit law for H_n, the behavior of the upper tail of the distribution of h must be known. This often requires complicated geometric computations. The general results are used to derive limit theorems with rates of convergence for the following settings: largest interpoint distance and scalar product of a sample of points in the d-dimensional closed unit ball $\mathbb{B}^d = \{x \in \mathbb{R}^d: \|x\| \le 1\}$, where $\|\cdot\|$ denotes the Euclidean norm, the directions of the points have a density on the surface $\mathbb{S}^{d-1}$ of $\mathbb{B}^d$ and are independent of the norms; smallest spherical distance of a sample of points with density on $\mathbb{S}^{d-1}$; largest perimeter of all triangles formed by point triplets in a sample of uniformly distributed points on the unit circle $\mathbb{S}$.

2. Poisson approximation for U-max-statistics

The following result is easily derived from Consequence (3.2) of Theorem 2.N by Barbour et al. [3]. We use the convention that improper sums for $k = 1$ equal to zero.

Theorem 2.1. *Let $\xi_1, \ldots, \xi_n$ be i.i.d. $\mathcal{S}$-valued random elements and $h: \mathcal{S}^k \to \mathbb{R}$ a symmetric Borel function. Putting*

$$p_{n,z} = \mathbf{P}\{h(\xi_1, \ldots, \xi_k) > z\},$$
$$\lambda_{n,z} = \binom{n}{k} p_{n,z},$$

$$\tau_{n,z}(r) = p_{n,z}^{-1}\mathbf{P}\{h(\xi_1,\ldots,\xi_k) > z, h(\xi_{1+k-r},\xi_{2+k-r},\ldots,\xi_{2k-r}) > z\},$$
$$r = 1,\ldots,k-1,$$

we have, for any $n \geq k$ and any $z \in \mathbb{R}$,

$$|\mathbf{P}\{H_n \leq z\} - \exp\{-\lambda_{n,z}\}|$$
$$\leq (1-\exp\{-\lambda_{n,z}\})\left\{p_{n,z}\left[\binom{n}{k} - \binom{n-k}{k}\right] + \sum_{r=1}^{k-1}\binom{k}{r}\binom{n-k}{k-r}\tau_{n,z}(r)\right\}. \quad (2.1)$$

Clearly the result can be reformulated as well for the minimum value of the kernel by replacing h with $-h$. One of the main applications of this theorem consists in determining a suitable sequence of transformations $z_n: T \to \mathbb{R}$ with $T \subset \mathbb{R}$, such that the right-hand side of (2.1) converges to zero as $n \to \infty$ for all $z = z_n(t)$, $t \in T$, and the limits of $\exp\{-\lambda_{n,z_n(t)}\}$ are nontrivial for each $t \in T$. The usual choice is $T = [0,\infty)$. One way to achieve this goal is based on the following two remarks and will eventually lead to the well-known Poisson limit theorem by Silverman and Brown [14], originally proved by a suitable coupling.

Remark 1. As already Silverman and Brown [14] stated,

$$p_{n,z} \leq \tau_{n,z}(1) \leq \cdots \leq \tau_{n,z}(k) = 1.$$

Remark 2. If the sample size n tends to infinity, then the right-hand side of (2.1) is asymptotically

$$\mathcal{O}\left(p_{n,z}n^{k-1} + \sum_{r=1}^{k-1}\tau_{n,z}(r)n^{k-r}\right),$$

and for $k > 1$ the sum is dominating, see [3, p. 35].

Remark 3. The symmetry condition on h can be avoided if h is symmetrized by

$$h^*(x_1,\ldots,x_k) = \max_{j_1,\ldots,j_k} h(x_{j_1},\ldots,x_{j_k}),$$

where the maximum is taken over all permutations of $1,\ldots,k$.

The conditions stated in [14] suffice to ensure that Theorem 2.1 provides a nontrivial Weibull limit law.

Corollary 2.2 (*Silverman–Brown limit law [14]*). *In the setting of Theorem 2.1, if for some sequence of transformations $z_n: T \to \mathbb{R}$ with $T \subset \mathbb{R}$, the conditions*

$$\lim_{n\to\infty} \lambda_{n,z_n(t)} = \lambda_t > 0 \quad (2.2)$$

and

$$\lim_{n\to\infty} n^{2k-1} p_{n,z_n(t)}\tau_{n,z_n(t)}(k-1) = 0 \quad (2.3)$$

hold for each $t \in T$, then

$$\lim_{n\to\infty} \mathbf{P}\{H_n \leq z_n(t)\} = \exp\{-\lambda_t\} \quad (2.4)$$

for each $t \in T$.

Remark 4. Condition (2.2) implies $p_{n,z_n(t)} = \mathcal{O}(n^{-k})$, and by Remarks 1 and 2 we obtain for (2.4) the rate of convergence

$$\mathcal{O}\left(n^{-1} + \sum_{r=1}^{k-1} n^{2k-r} p_{n,z_n(t)} \tau_{n,z_n(t)}(r)\right)$$

with upper bound

$$\mathcal{O}(n^{2k-1} p_{n,z_n(t)} \tau_{n,z_n(t)}(k-1)). \tag{2.5}$$

If $k > 2$, it is sometimes useful to replace (2.3) by the weaker condition

$$\lim_{n\to\infty} n^{2k-r} p_{n,z_n(t)} \tau_{n,z_n(t)}(r) = 0 \tag{2.6}$$

for each $r \in \{1, \ldots, k-1\}$, a fact that follows immediately from Theorem 2.1 and Remark 2.

Remark 5. A limit law for a U-max-statistic of n independent random points with distribution κ holds automatically for the Poisson point process with intensity measure $n\kappa$. This follows from [3, Prop. 2.3.5].

Appel and Russo [2] obtained a Weibull limit law similar to Corollary 2.2 for bivariate h. They assume that the upper tail of the distribution of $h(\xi_1, x)$ does not depend on x for almost all $x \in \mathcal{S}$, which implies that (2.2) and (2.3) hold. However, this condition is fulfilled only in very rare settings, e.g. for uniformly distributed points on $\mathbb{S}^{d-1}$.

3. Largest interpoint distance

The asymptotic behavior of the range of a univariate sample can be determined by classical extreme value theory, see e.g. [6, Sec. 2.9]. The largest interpoint distance

$$H_n = \max_{1\le i<j\le n} \|\xi_i - \xi_j\|$$

within a sample of points in $\mathbb{R}^d$ is a natural and consistent generalization of the range to spatial data. Matthews and Rukhin [11] derived its limiting behavior for a normal sample, a work which has been generalized by Henze and Klein [8] to a sample of points with symmetric Kotz distribution. Appel et al. [1] found corresponding limit laws in the setting of uniformly distributed points in two-dimensional compact sets which are not too smooth near the endpoints of their largest axis. They also provided bounds for the limit law of the diameter of uniformly distributed points in ellipses and the unit disk. The exact limit distribution for the disk and in more general settings were found independently by Lao [10] and Mayer and Molchanov [13]. Lao [10] used Theorem A of [14] to obtain the exact limit law for the diameter of a uniform sample in $\mathbb{B}^d$. The results in [13] rely on a combination of geometric considerations and blocking techniques and yield e.g. the special case of Theorem 3.1 for spherically symmetric distributions.

In what follows, we denote by $\langle \cdot, \cdot \rangle$ the scalar product, by μ_d the d-dimensional Lebesgue measure and by $\mathscr{H}_m$ the m-dimensional Hausdorff measure on $\mathbb{R}^d$. The restriction of $\mathscr{H}_{d-1}$ to $\mathbb{S}^{d-1}$ is usually denoted by μ_{d-1}. Furthermore, $b_m = \pi^{m/2}/\Gamma(\frac{m}{2}+1)$ and $\omega_m = m\pi^{m/2}/\Gamma(\frac{m}{2}+1)$ are volume and surface area of the unit m-dimensional ball. Γ and B denote the complete Gamma and Beta functions.

Theorem 3.1. *Let $\xi_1, \xi_2, \ldots$ be i.i.d. points in $\mathbb{B}^d$, $d \geq 2$, such that $\xi_i \stackrel{d}{=} \|\xi_i\| U_i$, $i \geq 1$, where U_i and $\|\xi_i\|$ are independent and $U_i \in \mathbb{S}^{d-1}$. Assume that the distribution function F of $1 - \|\xi_1\|$ satisfies*

$$\lim_{s \downarrow 0} s^{-\alpha} F(s) = a \in (0, \infty)$$

for some $\alpha \geq 0$. Further assume that U_1 has a density f with respect to μ_{d-1} and that

$$\int_{\mathbb{S}^{d-1}} f(u) f(-u)^2 \mu_{d-1}(\mathrm{d}u) < \infty. \tag{3.1}$$

Then

$$\lim_{n\to\infty} \mathbf{P}\left\{n^{2/\gamma}(2 - H_n) \leq t\right\} = 1 - \exp\left\{-\frac{\sigma_1}{2} t^\gamma\right\}$$

for $t > 0$, where

$$\gamma = (d-1)/2 + 2\alpha$$

and

$$\sigma_1 = \frac{(4\pi)^{\frac{d-1}{2}} a^2 \Gamma^2(\alpha+1)}{\Gamma\left(\frac{d+1}{2} + 2\alpha\right)} \int_{\mathbb{S}^{d-1}} f(u) f(-u) \mu_{d-1}(\mathrm{d}u).$$

The rate of convergence is $\mathcal{O}\left(n^{-\frac{d-1}{d-1+4\alpha}}\right)$.

Remark 6. If the density f is bounded (or f is centrally symmetric and f^3 integrable over $\mathbb{S}^{d-1}$) then Condition (3.1) is fulfilled. However, if

$$\int_{\mathbb{S}^{d-1}} f(u) f(-u) \mu_{d-1}(\mathrm{d}u) = 0,$$

the limit distribution is trivial.

Remark 7. Spherically symmetric distributed points have independent and uniformly distributed directions and hence [13, Th. 4.2] follows immediately from Theorem 3.1 with

$$\int_{\mathbb{S}^{d-1}} f(u) f(-u) \mu_{d-1}(\mathrm{d}u) = \frac{\Gamma(\frac{d}{2})}{2\pi^{d/2}}.$$

The special case $\alpha = 1$ and $a = d$ yields the limit law for the diameter of a sample of uniformly distributed points in $\mathbb{B}^d$, see [10] or [13].

Remark 8. If $\|\xi_i\| = 1$ almost surely, then $\alpha = 0$ and $a = 1$. For instance, if U_i are uniformly distributed on $\mathbb{S}^{d-1}$, then for $t > 0$

$$\lim_{n\to\infty} \mathbf{P}\left\{n^{4/(d-1)}(2 - H_n) \leq t\right\} = 1 - \exp\left\{-\frac{2^{d-3}\Gamma(\frac{d}{2})}{\pi^{\frac{1}{2}}\Gamma(\frac{d+1}{2})} t^{\frac{d-1}{2}}\right\},$$

see [2] or [13]. Another example appears if U_i has the von Mises–Fisher distribution of dimension $d \geq 2$ [5] with density

$$f_F(u) = C_d(\kappa) \exp\{\kappa \langle \mu, u \rangle\}, \quad u \in \mathbb{S}^{d-1},$$

where $\mu \in \mathbb{S}^{d-1}$ represents the mean direction and $\kappa > 0$ is the concentration parameter. The normalizing constant $C_d(\kappa)$ is given by

$$C_d(\kappa) = \frac{\kappa^{d/2-1}}{(2\pi)^{d/2} I_{d/2-1}(\kappa)},$$

where I_ν denotes the modified Bessel function of the first kind of order ν. Since

$$\int_{\mathbb{S}^{d-1}} f_F(u) f_F(-u) \mu_{d-1}(\mathrm{d}u) = C_d^2(\kappa) \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})},$$

the corresponding limit law follows immediately.

Remark 9. For fixed α, a and d, the limit law in Theorem 3.1 depends only on the value of $\int_{\mathbb{S}^{d-1}} f(u) f(-u) \mu_{d-1}(\mathrm{d}u)$. Among the class $\mathcal{F}$ of centrally symmetric densities satisfying Condition (3.1), this integral is minimized by the uniform density f_U. Hence for uniformly distributed directions, H_n is (asymptotically) stochastically minimal among $\mathcal{F}$. By the Cauchy–Schwarz inequality,

$$\left(\int_{\mathbb{S}^{d-1}} f(u) f_U(u) \mu_{d-1}(\mathrm{d}u) \right)^2 \le \int_{\mathbb{S}^{d-1}} f(u)^2 \mu_{d-1}(\mathrm{d}u) \int_{\mathbb{S}^{d-1}} f_U(u)^2 \mu_{d-1}(\mathrm{d}u),$$

and hence

$$\int_{\mathbb{S}^{d-1}} f_U(u)^2 \mu_{d-1}(\mathrm{d}u) \le \int_{\mathbb{S}^{d-1}} f(u)^2 \mu_{d-1}(\mathrm{d}u)$$

for all $f \in \mathcal{F}$ with equality if and only if $f = f_U \mu_{d-1}$ almost everywhere.

A key part of the proof of Theorem 3.1 is the asymptotic tail behavior of the distribution of the distance between two i.i.d. points.

Lemma 3.2. *If the conditions of Theorem 3.1 hold, then*

$$\lim_{s \downarrow 0} s^{-\gamma} \mathbf{P}\{\|\xi_1 - \xi_2\| \ge 2 - s\} = \sigma_1.$$

Proof. Let η_1 and η_2 be independent random variables with distribution function F and denote by β_u the central angle[1] between U_2 and $u \in \mathbb{S}^{d-1}$. The law of cosines yields

$$\begin{aligned} \mathbf{P}\{\|\xi_1 - \xi_2\| \ge 2 - s\} &= \mathbf{P}\left\{ \|\xi_1\|^2 + \|\xi_2\|^2 + 2\|\xi_1\|\|\xi_2\| \cos\beta_{-U_1} \ge (2-s)^2 \right\} \\ &= \mathbf{P}\left\{ \cos\beta_{-U_1} \ge \frac{(2-s)^2 - (1-\eta_1)^2 - (1-\eta_2)^2}{2(1-\eta_1)(1-\eta_2)} \right\}, \end{aligned}$$

and by expansion of $\cos\beta_{-U_1}$ about 0, we get for sufficiently small s

$$\mathbf{P}\{\|\xi_1 - \xi_2\| \ge 2 - s\} = \mathbf{P}\left\{ |\beta_{-U_1}| \le 2(\tilde{s} - \eta_1 - \eta_2)^{\frac{1}{2}}, \eta_1 + \eta_2 \le \tilde{s} \right\}, \tag{3.2}$$

where $|\tilde{s} - s| \le C_1 s^2$ for some finite C_1, thus $\tilde{s}/s \to 1$ as $s \downarrow 0$. In the next step, we determine the asymptotic behavior of (3.2) for fixed η_1 and η_2 with $\eta_1 + \eta_2 \in (0, \tilde{s})$. Put $s_y = \tilde{s} - y$,

[1] The central angle means here the smaller of the two angles at the center. It does not mean the reflex angle.

$y \in [0, \tilde{s}]$, and denote by $S_u(s_y)$ the set of points on $\mathbb{S}^{d-1}$ whose small central angles to $-u$ are at most $2\sqrt{s_y}$. By Vitali's covering theorem [12, Th. 2.2], the family

$$\{(-u, S_u(s_y)) \colon u \in \mathbb{S}^{d-1}, y \in (0, \tilde{s}), \tilde{s} \in (0, \varepsilon)\}$$

is a μ_{d-1} Vitali relation [4, p. 151] for a sufficiently small ε. As $s_y \downarrow 0$, $S_u(s_y)$ contracts to the singleton $-u$, and hence by Lebesgue's differentiation theorem [4, Th. 2.9.8] we obtain

$$\lim_{s_y \downarrow 0} \frac{\mathbf{P}\left\{|\beta_{-u}| \leq 2s_y^{\frac{1}{2}}\right\}}{\mu_{d-1}(S_u(s_y))} = f(-u) \tag{3.3}$$

for μ_{d-1} almost every $u \in \mathbb{S}^{d-1}$. Asymptotically equivalent lower and upper bounds for the denominator in (3.3) are obtained by the following considerations. For every $u \in \mathbb{S}^{d-1}$, the convex hull $C(u, h)$ of the spherical cap $S_u(s_y)$ is a cap of $\mathbb{B}^d$ of height

$$h = 1 - \cos(2s_y^{\frac{1}{2}}) \tag{3.4}$$

in direction $-u$, i.e. $C(u, h) = \{x \in \mathbb{B}^d \colon \langle x, -u \rangle \geq 1-h\}$. Its base $B = \{x \in C(u, h) \colon \langle x, -u \rangle = 1 - h\}$ is a $(d-1)$-dimensional ball of radius

$$r = \sin(2s_y^{\frac{1}{2}}) \tag{3.5}$$

centered at $-u$. By convexity of $C(u, h)$, the following bounds for the surface area of $S_u(s_y)$ are obvious:

$$\mathscr{H}_{d-1}(B) \leq \mu_{d-1}(S_u(s_y)) \leq \mathscr{H}_{d-1}(B) + \mathscr{H}_{d-1}(M), \tag{3.6}$$

where M is the cylinder mantle of height h over the boundary of B with

$$\mathscr{H}_{d-1}(M) = h\omega_{d-1} r^{d-2}.$$

Furthermore, the surface area of B equals $b_{d-1} r^{d-1}$. By expansion of the trigonometric functions in (3.4) and (3.5),

$$\lim_{s_y \downarrow 0} s_y^{-\frac{1}{2}} r = 2, \tag{3.7}$$

$$\lim_{s_y \downarrow 0} s_y^{-1} h = 2. \tag{3.8}$$

From (3.6)–(3.8), it follows that

$$2^{d-1} b_{d-1} \leq \lim_{s_y \downarrow 0} s_y^{-\frac{d-1}{2}} \mu_{d-1}(S_u(s_y)) \leq 2^{d-1} b_{d-1} + \lim_{s_y \downarrow 0} s_y^{\frac{1}{2}} 2^{d-1} \omega_{d-1},$$

and hence the surface area of $S_u(s_y)$ satisfies

$$\lim_{s_y \downarrow 0} s_y^{-\frac{d-1}{2}} \mu_{d-1}(S_u(s_y)) = 2^{d-1} b_{d-1}. \tag{3.9}$$

Plugging (3.9) in (3.3) implies

$$\lim_{s_y \downarrow 0} s_y^{-\frac{d-1}{2}} \mathbf{P}\left\{|\beta_{-u}| \leq 2s_y^{\frac{1}{2}}\right\} = 2^{d-1} b_{d-1} f(-u)$$

for μ_{d-1} almost every $u \in \mathbb{S}^{d-1}$. Integration with respect to the angular distribution yields

$$\lim_{s_y \downarrow 0} s_y^{-\frac{d-1}{2}} \mathbf{P}\left\{|\beta_{-U_1}| \le 2 s_y^{\frac{1}{2}}\right\} = 2^{d-1} b_{d-1} \int_{\mathbb{S}^{d-1}} f(u) f(-u) \mu_{d-1}(\mathrm{d}u),$$

and by (3.2), we obtain

$$\lim_{s \downarrow 0} \frac{\mathbf{P}\{\|\xi_1 - \xi_2\| \ge 2 - s\}}{\mathbf{E}\left((\tilde{s} - \eta_1 - \eta_2)^{\frac{d-1}{2}} \mathbf{1}\{\eta_1 + \eta_2 \le \tilde{s}\}\right)}$$
$$= 2^{d-1} b_{d-1} \int_{\mathbb{S}^{d-1}} f(u) f(-u) \mu_{d-1}(\mathrm{d}u). \tag{3.10}$$

If $\alpha = 0$, then $\mathbf{P}\{\eta_i = 0\} = a$, $i = 1, 2$, thus

$$\lim_{s \downarrow 0} \tilde{s}^{-\gamma} \mathbf{P}\{\|\xi_1 - \xi_2\| \ge 2 - s\} = \sigma_1.$$

If $\alpha > 0$, we use the fact that $s/\tilde{s} \to 1$ as $s \downarrow 0$, and from integration by parts and dominated convergence, it follows that

$$\lim_{s \downarrow 0} s^{-\gamma} \mathbf{E}\left((\tilde{s} - \eta_1 - \eta_2)^{\frac{d-1}{2}} \mathbf{1}\{\eta_1 + \eta_2 \le \tilde{s}\}\right) = \frac{a^2 \Gamma^2(\alpha+1) \Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d+1}{2} + 2\alpha\right)}.$$

With (3.10), the proof is complete. □

Proof of Theorem 3.1. We plug the transformation $z_n(t) = 2 - t n^{-2/\gamma}$ into Corollary 2.2 and use the tail probabilities given in Lemma 3.2 to obtain

$$\lim_{n \to \infty} \binom{n}{2} \mathbf{P}\{\|\xi_1 - \xi_2\| > z_n(t)\} = \frac{\sigma_1}{2} t^{\gamma}.$$

Hence (2.2) holds for each $t > 0$. In the remaining part of the proof, we show that (2.3) holds. Let β_u and β'_u be the central angles between U_2 and $u \in \mathbb{S}^{d-1}$ and between U_3 and $u \in \mathbb{S}^{d-1}$, respectively. Furthermore, let η_1, η_2 and η_3 be independent random variables with distribution function F. Put $s_n = t n^{-2/\gamma}$. Following the proof of Lemma 3.2, we obtain from (3.10)

$$\begin{aligned}
&\lim_{n \to \infty} n^3 \mathbf{P}\{\|\xi_1 - \xi_2\| > z_n(t), \|\xi_1 - \xi_3\| > z_n(t)\} \\
&\quad \le \lim_{n \to \infty} n^3 \mathbf{P}\left\{|\beta_{-U_1}| \le 2 s_n^{\frac{1}{2}}, |\beta'_{-U_1}| \le 2 s_n^{\frac{1}{2}}, \eta_i \le s_n, i = 1, 2, 3\right\} \\
&\quad = \lim_{n \to \infty} n^3 \mathbf{E}\left(\int_{\mathbb{S}^{d-1}} \mathbf{P}\left\{|\beta_{-u}| \le 2 s_n^{\frac{1}{2}}\right\}^2 f(u) \mu_{d-1}(\mathrm{d}u) \mathbf{1}\{\eta_i \le s_n, i = 1, 2, 3\}\right) \\
&\quad \le \lim_{n \to \infty} n^3 C \mathbf{E}(s_n^{d-1} \mathbf{1}\{\eta_i \le s_n, i = 1, 2, 3\}) \\
&\quad = \lim_{n \to \infty} n^3 C s_n^{d-1} F^3(s_n) = a^3 C \lim_{n \to \infty} n^3 s_n^{d-1+3\alpha} \\
&\quad = a^3 C t^{d-1+3\alpha} n^{-\frac{d-1}{d-1+4\alpha}} = 0
\end{aligned}$$

for some finite constant C. The rate of convergence is determined via (2.5). □

4. Largest scalar product

Besides the Euclidean metric, the scalar product is another symmetric kernel on $\mathbb{R}^d \times \mathbb{R}^d$. The behavior of its largest value

$$H_n = \max_{1 \le i < j \le n} \langle \xi_i, \xi_j \rangle$$

within a sample of points in $\mathbb{B}^d$ is determined in the next result.

Theorem 4.1. *Let $\xi_1, \xi_2, \dots$ be i.i.d. points in $\mathbb{B}^d$, $d \ge 2$, such that $\xi_i \stackrel{d}{=} \|\xi_i\| U_i$, $i \ge 1$, where U_i and $\|\xi_i\|$ are independent and $U_i \in \mathbb{S}^{d-1}$. Assume that the distribution function F of $1 - \|\xi_1\|$ satisfies*

$$\lim_{s \downarrow 0} s^{-\alpha} F(s) = a \in (0, \infty)$$

for some $\alpha \ge 0$. Further assume that U_1 has a density f on $\mathbb{S}^{d-1}$ with respect to μ_{d-1} and that

$$\int_{\mathbb{S}^{d-1}} f^3(u) \mu_{d-1}(\mathrm{d}u) < \infty. \tag{4.1}$$

Then

$$\lim_{n \to \infty} \mathbf{P} \left\{ n^{2/\gamma} (1 - H_n) \le t \right\} = 1 - \exp \left\{ -\frac{\sigma_2}{2} t^\gamma \right\}, \quad t > 0,$$

where

$$\gamma = (d-1)/2 + 2\alpha$$

and

$$\sigma_2 = \frac{(2\pi)^{\frac{d-1}{2}} a^2 \Gamma^2(\alpha + 1)}{\Gamma(\frac{d+1}{2} + 2\alpha)} \int_{\mathbb{S}^{d-1}} f^2(u) \mu_{d-1}(\mathrm{d}u).$$

The rate of convergence is $\mathcal{O}(n^{-\frac{d-1}{d-1+4\alpha}})$.

Lemma 4.2. *If the conditions of Theorem 4.1 hold, then*

$$\lim_{s \downarrow 0} s^{-\gamma} \mathbf{P} \{ \langle \xi_1, \xi_2 \rangle \ge 1 - s \} = \sigma_2.$$

Proof. If β_u is the central angle between U_2 and $u \in \mathbb{S}^{d-1}$ and η is distributed as $1 - \|\xi_1\| \|\xi_2\|$, then for $s \in (0, 1)$

$$\begin{aligned} \mathbf{P} \{ \langle \xi_1, \xi_2 \rangle \ge 1 - s \} &= \mathbf{P} \left\{ \|\xi_1\| \|\xi_2\| \cos \beta_{U_1} \ge 1 - s \right\} \\ &= \mathbf{P} \left\{ \cos \beta_{U_1} \ge (1 - s)/(1 - \eta), \eta \le s \right\}. \end{aligned}$$

Expansion of $\cos \beta_{U_1}$ about 0 yields for all sufficiently small s

$$\mathbf{P} \{ \langle \xi_1, \xi_2 \rangle \ge 1 - s \} = \mathbf{P} \left\{ |\beta_{U_1}| \le (2(\tilde{s} - \eta))^{\frac{1}{2}}, \eta \le \tilde{s} \right\}, \tag{4.2}$$

where $|\tilde{s} - s| \le C_1 s^2$ for some finite C_1, and thus $\tilde{s}/s \to 1$ as $s \downarrow 0$. Following the proof of Lemma 3.2, we obtain

$$\lim_{s \downarrow 0} \frac{\mathbf{P} \{ \langle \xi_1, \xi_2 \rangle \ge 1 - s \}}{\mathbf{E} \left((\tilde{s} - \eta)^{\frac{d-1}{2}} \mathbf{1}\{\eta \le \tilde{s}\} \right)} = 2^{\frac{d-1}{2}} b_{d-1} \int_{\mathbb{S}^{d-1}} f^2(u) \mu_{d-1}(\mathrm{d}u). \tag{4.3}$$

If $\alpha = 0$, then $\mathbf{P}\{\eta = 0\} = a^2$ and hence

$$\lim_{s \downarrow 0} s^{-\gamma} \mathbf{P}\{\langle \xi_1, \xi_2 \rangle \geq 1 - s\} = \sigma_2.$$

If $\alpha > 0$, then by $\tilde{s}/s \to 1$ as $s \downarrow 0$ and from integration by parts and dominated convergence, it follows that

$$\lim_{s \downarrow 0} s^{-\gamma} \mathbf{E}\left((\tilde{s} - \eta)^{\frac{d-1}{2}} \mathbf{1}\{\eta \leq \tilde{s}\}\right) = \frac{a^2 \Gamma^2(\alpha + 1) \Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d+1}{2} + 2\alpha\right)}.$$

With (4.3) the proof is complete. □

Proof of Theorem 4.1. By Corollary 2.2, Lemma 4.2 and the transformation $z_n(t) = 1 - tn^{-2/\gamma}$ we obtain

$$\lim_{n \to \infty} \binom{n}{2} \mathbf{P}\{\langle \xi_1, \xi_2 \rangle \geq z_n(t)\} = \frac{\sigma_2}{2} t^{\gamma}.$$

Hence (2.2) holds for any $t > 0$ and it remains to check (2.3). Put $s_n = tn^{-2/\gamma}$ and let β_u and β'_u be the central angles between U_2 and $u \in \mathbb{S}^{d-1}$ and between U_3 and u, respectively. Following the proof of Lemma 4.2, we derive from (4.3)

$$\begin{aligned}
&\lim_{n \to \infty} n^3 \mathbf{P}\{\langle \xi_1, \xi_2 \rangle \geq z_n(t), \langle \xi_1, \xi_3 \rangle \geq z_n(t)\} \\
&\leq \lim_{n \to \infty} n^3 \mathbf{P}\left\{|\beta_{U_1}| \leq (2s_n)^{\frac{1}{2}}, |\beta'_{U_1}| \leq (2s_n)^{\frac{1}{2}}, \|\xi_i\| \geq z_n(t), i = 1, 2, 3\right\} \\
&= \lim_{n \to \infty} n^3 \mathbf{E}\left(\int_{\mathbb{S}^{d-1}} \mathbf{P}\left\{|\beta_u| \leq (2s_n)^{\frac{1}{2}}\right\}^2 \right. \\
&\left. \times\ f(u) \mu_{d-1}(\mathrm{d}u) \mathbf{1}\{\|\xi_i\| \geq z_n(t), i = 1, 2, 3\}\right) \\
&\leq \lim_{n \to \infty} n^3 C \mathbf{E}(s_n^{d-1} \mathbf{1}\{\|\xi_i\| \geq z_n(t), i = 1, 2, 3\}) \\
&= \lim_{n \to \infty} n^3 C s_n^{d-1} \mathbf{P}\{1 - \|\xi_1\| \leq s_n\}^3,
\end{aligned}$$

where C is a finite positive constant. Hence

$$\begin{aligned}
&\lim_{n \to \infty} n^3 \mathbf{P}\{\langle \xi_1, \xi_2 \rangle \geq z_n(t), \langle \xi_1, \xi_3 \rangle \geq z_n(t)\} \leq \lim_{n \to \infty} C n^3 s_n^{d-1} F^3(s_n) \\
&= a^3 C \lim_{n \to \infty} n^3 s_n^{d-1+3\alpha} \\
&= a^3 C t^{d-1+3\alpha} n^{-\frac{d-1}{d-1+4\alpha}} = 0.
\end{aligned}$$

The rate of convergence is determined via (2.5). □

5. Smallest spherical distance

An application of Theorem 4.1 comes from the field of directional statistics. The following theorem determines the limiting behavior of the smallest spherical distance

$$S_n = \min_{1 \leq i < j \leq n} \beta_{i,j}$$

of i.i.d. points $U_1, U_2, \dots$ on $\mathbb{S}^{d-1}$, where $\beta_{i,j}$ denotes the central angle between U_i and U_j. In other words, S_n equals the smallest central angle formed by point pairs within the sample. A similar result for the smallest Euclidean distance within a random sample can be found in [14].

Theorem 5.1. *Let $U_1, U_2 \dots$ be i.i.d. points on $\mathbb{S}^{d-1}$, $d \geq 2$, with density f satisfying (4.1). Then*

$$\lim_{n\to\infty} \mathbf{P}\left\{n^{2/(d-1)} S_n \leq t\right\} = 1 - \exp\left\{-\frac{\sigma_3}{2} t^{d-1}\right\}, \quad t > 0,$$

where

$$\sigma_3 = \frac{\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d+1}{2})} \int_{\mathbb{S}^{d-1}} f^2(u)\mu_{d-1}(\mathrm{d}u).$$

The rate of convergence is $\mathcal{O}(n^{-1})$.

If the points are uniformly distributed on $\mathbb{S}^{d-1}$, Theorem 5.1 applies with

$$\int_{\mathbb{S}^{d-1}} f^2(u)\mu_{d-1}(\mathrm{d}u) = \frac{\Gamma(\frac{d}{2})}{2\pi^{d/2}}.$$

If the points on $\mathbb{S}^{d-1}$ follow the von Mises–Fisher distribution as introduced in Section 3,

$$\int_{\mathbb{S}^{d-1}} f_F^2(u)\mu_{d-1}(\mathrm{d}u) = C_d^2(\kappa)/C_d(2\kappa).$$

In dimension 2, S_n equals the minimal spacing, i.e. the smallest arc length between the "order" statistics.

Proof of Theorem 5.1. Clearly, the relation $\cos\beta_{i,j} = \langle U_i, U_j\rangle$ holds for all pairs of i and j between 1 and n. Since the cosine function is continuous and strictly decreasing on $(0, \pi)$ and from

$$\lim_{s\downarrow 0} s^{-\frac{1}{2}} \arccos(1-s) = \sqrt{2},$$

it follows that

$$\begin{aligned}\lim_{n\to\infty} \mathbf{P}\left\{n^{2/(d-1)} S_n \leq t\right\} &= \lim_{n\to\infty} \mathbf{P}\left\{\min_{1\leq i<j\leq n} \beta_{i,j} \leq t n^{-2/(d-1)}\right\} \\ &= \lim_{n\to\infty} \mathbf{P}\left\{\min_{1\leq i<j\leq n} \beta_{i,j} \leq \arccos\left(1 - t^2 n^{-4/(d-1)}/2\right)\right\} \\ &= \lim_{n\to\infty} \mathbf{P}\left\{\max_{1\leq i<j\leq n} \langle U_i, U_j\rangle \geq 1 - t^2 n^{-4/(d-1)}/2\right\}.\end{aligned}$$

Theorem 4.1 yields the proof with $\alpha = 0$ and $a = 1$. □

6. Largest perimeter

Finally, we present a result for a U-max-statistic of degree 3, namely the limit law for the largest value

$$H_n = \max_{1\leq i<j<\ell\leq n} \mathrm{peri}(U_i, U_j, U_\ell)$$

of the perimeter $\text{peri}(U_i, U_j, U_\ell)$ of all triangles formed by triplets of independent and uniformly distributed points $U_1, U_2, \ldots$ on the unit circle $\mathbb{S}$. The random triameter (see [7]) of the sample is the largest perimeter up to a factor 3, hence the limit law for the triameter of $U_1, U_2, \ldots$ can be derived immediately.

Theorem 6.1. *If $U_1, U_2, \ldots$ are independent and uniformly distributed points on $\mathbb{S}$, then*

$$\lim_{n\to\infty} \mathbf{P}\left\{n^3(3\sqrt{3} - H_n) \le t\right\} = 1 - \exp\left\{-\frac{2t}{9\pi}\right\}, \quad t > 0.$$

The rate of convergence is $\mathcal{O}(n^{-\frac{1}{2}})$.

Lemma 6.2. *If U_1, U_2, U_3 are independent and uniformly distributed points on $\mathbb{S}$, then*

$$\lim_{s\downarrow 0} s^{-1}\mathbf{P}\left\{\text{peri}(U_1, U_2, U_3) \ge 3\sqrt{3} - s\right\} = \frac{4}{3\pi}.$$

Proof. Clearly, $\text{peri}(u_1, u_2, u_3)$ is maximal if u_1, u_2, u_3 are the vertices of an equilateral triangle on $\mathbb{S}$, which has perimeter $3\sqrt{3}$. Let β_1 and β_2 be the angles (measured counter-clockwise) between U_1 and U_2 and between U_2 and U_3, respectively. By rotational symmetry, β_1 and β_2 are independent and uniformly distributed on $[0, 2\pi]$. The law of cosines yields for sufficiently small s

$$\begin{aligned}\mathbf{P}\left\{\text{peri}(U_1, U_2, U_3) \ge 3\sqrt{3} - s\right\} &= 2\mathbf{P}\left\{(2 - 2\cos\beta_1)^{\frac{1}{2}} + (2 - 2\cos\beta_2)^{\frac{1}{2}}\right. \\ &\quad \left. + \ (2 - 2\cos(2\pi - \beta_1 - \beta_2))^{\frac{1}{2}} \ge 3\sqrt{3} - s, \beta_1, \beta_2 \in [2\pi/3 \pm c_s]\right\}, \end{aligned} \tag{6.1}$$

where $c_s = C_1\sqrt{s}$ and C_1 is a suitable finite positive constant. The factor 2 represents the symmetric situation where the order of β_1 and β_2 is reversed. If η_1 and η_2 are independent and uniformly distributed on $[-c_s, c_s]$, the last expression equals

$$\begin{aligned}&2\mathbf{P}\left\{(2 - 2\cos(2\pi/3 + \eta_1))^{\frac{1}{2}} + (2 - 2\cos(2\pi/3 + \eta_2))^{\frac{1}{2}}\right. \\ &\quad \left. + \ (2 - 2\cos(2\pi/3 - \eta_1 - \eta_2))^{\frac{1}{2}} \ge 3\sqrt{3} - s\right\}\mathbf{P}\left\{\beta_1 \in [2\pi/3 \pm c_s]\right\}^2.\end{aligned}$$

By series expansion, (6.1) equals

$$\begin{aligned}&2(c_s/\pi)^2\mathbf{P}\left\{\eta_1^2 + \eta_2^2 + (\eta_1 + \eta_2)^2 \le 8\tilde{s}/\sqrt{3}\right\} \\ &\quad = 2(c_s/\pi)^2\mathbf{P}\left\{\eta_2 \in \left[-\eta_1/2 \pm (4\tilde{s}/\sqrt{3} - 3\eta_1^2/4)^{\frac{1}{2}}\right]\right\} \\ &\quad = \pi^{-2}\int_{-4\sqrt{\tilde{s}}/3^{3/4}}^{4\sqrt{\tilde{s}}/3^{3/4}} (4\tilde{s}/\sqrt{3} - 3y^2/4)^{\frac{1}{2}}\mathrm{d}y = \frac{4\tilde{s}}{3\pi},\end{aligned} \tag{6.2}$$

where $|\tilde{s} - s| \le C_2 s^{3/2}$ for some finite C_2, and the proof follows from the fact that $\tilde{s}/s \to 1$ as $s \downarrow 0$. □

Proof of Theorem 6.1. We plug into Corollary 2.2 the transformation $z_n(t) = 3\sqrt{3} - tn^{-3}$ and use Lemma 6.2 to determine

$$\lim_{n\to\infty} \binom{n}{3}\mathbf{P}\left\{\text{peri}(U_1, U_2, U_3) > z_n(t)\right\} = \frac{2t}{9\pi}.$$

Hence (2.2) is satisfied for all $t > 0$. Condition (2.3) does not hold, so we use the weaker (2.6) to replace (2.3), i.e. we need to show that

$$\lim_{n\to\infty} n^5 \mathbf{P}\{\mathrm{peri}(U_1, U_2, U_3) > z_n(t), \mathrm{peri}(U_1, U_4, U_5) > z_n(t)\} = 0 \tag{6.3}$$

and

$$\lim_{n\to\infty} n^4 \mathbf{P}\{\mathrm{peri}(U_1, U_2, U_3) > z_n(t), \mathrm{peri}(U_1, U_2, U_4) > z_n(t)\} = 0. \tag{6.4}$$

For (6.3), we follow the proof of Lemma 6.2. In addition, denote by β_1' and β_2' the random angles between U_1 and U_4 and between U_4 and U_5, respectively. It follows immediately from rotational symmetry that $\beta_1, \beta_2, \beta_1'$ and β_2' are independent and uniformly distributed on $[0, 2\pi]$. With Lemma 6.2, we check (6.3) by

$$\begin{aligned}&\lim_{n\to\infty} n^5 \mathbf{P}\{\mathrm{peri}(U_1, U_2, U_3) > z_n(t), \mathrm{peri}(U_1, U_2, U_4) > z_n(t)\}\\ &\quad\le C_1 \lim_{n\to\infty} n^5 \mathbf{P}\{\mathrm{peri}(U_1, U_2, U_3) > z_n(t)\}^2 = C_2 t^2 \lim_{n\to\infty} n^{-1} = 0,\end{aligned}$$

where C_1 and C_2 are suitable finite positive constants. To show (6.4), we follow the proof of Lemma 6.2 and introduce the random variable η_3, independent of η_1 and η_2 and uniformly distributed on $[-c_s, c_s]$. For suitable finite positive constants C_3, C_4 and C_5, we have

$$\begin{aligned}&\mathbf{P}\{\mathrm{peri}(U_1, U_2, U_3) > z_n(t), \mathrm{peri}(U_1, U_2, U_4) > z_n(t)\}\\ &\quad\le C_3 c_s^3 \mathbf{P}\left\{\eta_2, \eta_3 \in \left[-\eta_1/2 \pm (4\tilde{s}/\sqrt{3} - 3\eta_1^2/4)^{\frac{1}{2}}\right]\right\}\\ &\quad= C_4 c_s^2 \int_{-4\sqrt{\tilde{s}}/3^{3/4}}^{4\sqrt{\tilde{s}}/3^{3/4}} \mathbf{P}\left\{\eta_2 \in \left[-y/2 \pm (4\tilde{s}/\sqrt{3} - 3y^2/4)^{\frac{1}{2}}\right]\right\}^2 \mathrm{d}y = C_5 \tilde{s}^{3/2},\end{aligned}$$

and with $s = tn^{-3}$ and $s/\tilde{s} \to 1$ as $s \to 0$

$$\begin{aligned}&\lim_{n\to\infty} n^4 \mathbf{P}\{\mathrm{peri}(U_1, U_2, U_3) > z_n(t), \mathrm{peri}(U_1, U_2, U_4) > z_n(t)\}\\ &\quad\le C_5 t^{3/2} \lim_{n\to\infty} n^{-\frac{1}{2}} = 0.\end{aligned}$$

Hence (6.4) holds, and the rate of convergence is determined by Remark 2. □

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