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THE USE OF EXTRAPOLATION FOR CALCULATING ELEMENTARY TRANSCENDENTAL AND BESSEL FUNCTIONS

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Abstract—This paper investigates the use of extrapolation for calculating elementary transcendental and Bessel functions. Although extrapolation can speed the convergence of series used to calculate these functions, it is found that the improvement is not worth the cost computationally.

EXTRAPOLATION METHODS BASED ON A TRANSIENT MODEL OF PARTIAL SERIES

Shanks [1] has developed a model of partial series as a sum of "transients." If a particular series s_N is given by

$$s_N = \sum_{n=1}^{N} a_n, \tag{1}$$

then it is assumed that the behavior of s_N is accurately represented by a sum of exponentials,

$$s_N = B + \sum_{i=1}^{K} a_i\, e^{(-\alpha_i N)}. \tag{2}$$

The constant term B represents the value of the series to which s_N will ultimately converge. The sum of exponentials represents transients that start at $N = 0$, and decay ultimately to zero or diverge ultimately to infinity as N approaches infinity. The α_k terms are, in general, complex, and they allow for the series to oscillate about its final value. This model does not accurately represent all partial series, but it does cover a broad range.

The purpose of extrapolation is to take the partial sums $s_1, s_2, \ldots, s_N$ and produce a new series $\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_p$ that converges faster to the proper limit B in Equation (2). In some sense, extrapolation "filters out" the transients of the partial series to produce a smoother and faster-converging series. A by-product of this model is that the divergent nature of the partial series is also filtered out; extrapolation forces convergence to an "antilimit" of the series [1].

Wynn [2] has devised a simple algorithm for calculating the $e_k(s_N)$ extrapolant. Consider the transformation $\epsilon_{2k}(s_N)$ in which

$$\epsilon_{-1}(s_N) = 0, \tag{3}$$

$$\epsilon_o(s_N) = s_N, \tag{4}$$

$$\epsilon_{s+1}(s_N) = \epsilon_{s-1}(s_{N+1}) + \frac{1}{\epsilon_s(s_{N+1}) - \epsilon_s(s_N)}, \tag{5}$$

$$\epsilon_{2k+1}(s_N) = \frac{1}{e_k(\Delta s_N)}, \tag{6}$$

$$\Delta s_N = s_N - s_{N-1}. \tag{7}$$

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It is true that

$$\epsilon_{2k}(s_N) = e_k(s_N). \tag{8}$$

For example, consider $k = 1$. Equations (4) and (5) produce

$$e_1(s_N) = \epsilon_2(s_N) = \epsilon_o(s_{N+1}) + \frac{1}{\epsilon_1(\Delta s_N)}. \tag{9}$$

From Equation (5) $\epsilon_1(s_N) = 1/e_o(\Delta s_N) = (\Delta s_N)^{-1}$, so that

$$e_1(s_N) = s_{N+1} + \frac{(\Delta s_{N-1})(\Delta s_N)}{\Delta s_N - \Delta s_{N+1}}. \tag{10}$$

Using Equation (7),

$$e_1(s_N) = \frac{s_{N+1}\, s_{N-1} - s_N^2}{s_{N+1} + s_{N-1} - 2s_N}. \tag{11}$$

This form will be recognized as simple Aitken extrapolation [3]. Higher orders of transformations are more difficult to show [2].

As an example of the power of this extrapolation method, Shanks [1] applies it to the slowly convergent Leibnitz series, $\pi = \sum 4(-1)^n/(2n+1)$. Using ten terms of the partial series produces six figure accuracy compared to 4×10^7 terms necessary for comparable accuracy in the simple sum.

Considering the dramatic acceleration in series convergence the Shanks transform can produce [4], the authors thought it worthwhile to investigate the use of this transform in calculating elementary transcendental functions and Bessel functions. The following series forms were used with the Shanks transform along with the intervals over which they were calculated:

$$\sin(2\pi x) = \sum_{n=0}^{\infty} (-1)^n \frac{(2\pi x)^{2n+1}}{(2n+1)!} \qquad [0,\ 1] \tag{12}$$

$$\cos(2\pi x) = \sum_{n=0}^{\infty} \frac{(-2\pi x)^{2n}}{(2n)!} \qquad [0,\ 1] \tag{13}$$

$$e^x = \sum_{n=o}^{\infty} \frac{x^n}{n!} \qquad [0,\ 1] \tag{14}$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2n+1} \qquad [0,\ 10] \tag{15}$$

$$J_o(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} \left(\frac{x}{2}\right)^{2m} \qquad [0, 10] \tag{16}$$

$$1 = J_o(x) + \sum_{n=0}^{\infty} J_{2n}(x). \tag{17}$$

The last series, Equation (17), is used in recursive algorithms to calculate the Bessel function [5].

Table 1 summarizes the maximum error of the partial summations for 5, 10, 15, and 20 series terms, and the corresponding extrapolant error. The results are disappointing for all but the arc tangent series; however, the improvement in the error, even in this case, is not great enough to justify using extrapolation. Using a straight polynomial approximation gives better results [6]. The failure for the Bessel series is especially disappointing, as the current methods for calculating general Bessel series are rather complex [5].

CONCLUSIONS

The use of extrapolation was investigated for calculating elementary transcendental functions. It was concluded that extrapolation does not accelerate the covergence of the series used to calculate these significantly enough to justify its use.

Table 1. Maximum error limits of the Wynn extrapolant and of the partial series for several functions.

	Sine Series				Cosine Series			
n	5	10	15	20	5	10	15	20
Partial sum error	12	1×10^{-3}	1×10^{-9}	1×10^{-13}	20	5×10^{-3}	3×10^{-9}	1×10^{-14}
Extrapolated sum error	2	1×10^{-3}	1×10^{-10}	1×10^{-13}	3	5×10^{-3}	5×10^{-10}	1×10^{-14}

	Exponential Series			
n	5	10	15	20
Partial sum error	0.1	3×10^{-7}	5×10^{-13}	5×10^{-13}
Extrapolated sum error	4×10^{-13}	6×10^{-8}	3×10^{-13}	5×10^{-13}

	Arc Tangent Series			
n	5	10	15	20
Partial sum error	0.05	0.03	0.015	0.011
Extrapolated sum error	2×10^{-4}	6×10^{-8}	4×10^{-12}	5×10^{-15}

	Bessel Series				Bessel Recursion Series			
n	5	10	15	20	5	10	15	20
Partial sum error	400	6	5×10^{-4}	2×10^{-9}	0.6	2.5×10^{-5}	2×10^{-10}	2×10^{-10}
Extrapolated sum error	5	0.2	2×10^{-5}	3×10^{-10}	1.1	2.5×10^{-3}	2×10^{-10}	2×10^{-10}

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