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Semismall perturbations, semi-intrinsic ultracontractivity, and integral representations of nonnegative solutions for parabolic equations

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Abstract

We consider nonnegative solutions of a parabolic equation in a cylinder $D \times I$, where D is a noncompact domain of a Riemannian manifold and $I = (0, T)$ with $0 < T \leqslant \infty$ or $I = (-\infty, 0)$. Under the assumption [SSP] (i.e., the constant function 1 is a semismall perturbation of the associated elliptic operator on D), we establish an integral representation theorem of nonnegative solutions: In the case $I = (0, T)$, any nonnegative solution is represented uniquely by an integral on $(D \times \{0\}) \cup (\partial_M D \times [0, T))$, where $\partial_M D$ is the Martin boundary of D for the elliptic operator; and in the case $I = (-\infty, 0)$, any nonnegative solution is represented uniquely by the sum of an integral on $\partial_M D \times (-\infty, 0)$ and a constant multiple of a particular solution. We also show that [SSP] implies the condition [SIU] (i.e., the associated heat kernel is semi-intrinsically ultracontractive).

Keywords: Semismall perturbation; Semi-intrinsic ultracontractivity; Parabolic equation; Nonnegative solution; Integral representation; Martin boundary; Elliptic equation

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1. Introduction

This paper is a continuation of [34]. It is concerned with integral representations of nonnegative solutions to parabolic equations and perturbation theory for elliptic operators.

We consider nonnegative solutions of a parabolic equation

$$(\partial_t + L)u = 0 \quad \text{in } D \times I, \tag{1.1}$$

where $\partial_t = \partial/\partial t$, L is a second order elliptic operator on a noncompact domain D of a Riemannian manifold M, and I is a time interval: $I = (0, T)$ with $0 < T \leqslant \infty$ or $I = (-\infty, 0)$.

During the last few decades, much attention has been paid to the structure of all nonnegative solutions to a parabolic equation, perturbation theory for elliptic operators, and their relations. (See [1,2,4–6,11,14,17,19,20,22,25–34,36–38,40–42].) Among others, Murata [34] has established integral representation theorems of nonnegative solutions to Eq. (1.1) under the condition [IU] (i.e., intrinsic ultracontractivity) on the minimal fundamental solution $p(x, y, t)$ for (1.1). Furthermore, he has shown that [IU] implies [SP] (i.e., the constant function 1 is a small perturbation of L on D). It is known [30] that [SP] implies [SSP] (i.e., 1 is a semismall perturbation of L on D).

In this paper, we show that [SSP] implies [SIU] (i.e., semi-intrinsic ultracontractivity) and give integral representation theorems of nonnegative solutions to (1.1) under the condition [SSP]. We consider that [SSP] is one of the weakest possible condition for getting "explicit" integral representation theorems.

Now, in order to state our main results, we fix notations and recall several notions and facts. Let M be a connected separable n-dimensional smooth manifold with Riemannian metric of class C^0. Denote by ν the Riemannian measure on M. $T_x M$ and TM denote the tangent space to M at $x \in M$ and the tangent bundle, respectively. We denote by $\mathrm{End}(T_x M)$ and $\mathrm{End}(TM)$ the set of endomorphisms in $T_x M$ and the corresponding bundle, respectively. The inner product on TM is denoted by $\langle X, Y\rangle$, where $X, Y \in TM$; and $|X| = \langle X, X\rangle^{1/2}$. The divergence and gradient with respect to the metric on M are denoted by div and ∇, respectively. Let D be a noncompact domain of M. Let L be an elliptic differential operator on D of the form

$$Lu = -m^{-1} \operatorname{div}(mA\nabla u) + Vu, \tag{1.2}$$

where m is a positive measurable function on D such that m and m^{-1} are bounded on any compact subset of D, A is a symmetric measurable section on D of $\mathrm{End}(TM)$, and V is a real-valued measurable function on D such that

$$V \in L^p_{\mathrm{loc}}(D, m\,d\nu) \quad \text{for some } p > \max\left(\frac{n}{2}, 1\right).$$

Here $L^p_{\mathrm{loc}}(D, m\,d\nu)$ is the set of real-valued functions on D locally pth integrable with respect to $m\,d\nu$. We assume that L is locally uniformly elliptic on D, i.e., for any compact set K in D there exists a positive constant λ such that

$$\lambda|\xi|^2 \leqslant \langle A_x\xi, \xi\rangle \leqslant \lambda^{-1}|\xi|^2, \quad x \in K,\ (x, \xi) \in TM.$$

We assume that the quadratic form Q on $C_0^\infty(D)$ defined by

$$Q[u] = \int_D \big(\langle A\nabla u, \nabla u\rangle + Vu^2\big) m\, d\nu$$

is bounded from below, and put

$$\lambda_0 = \inf\left\{ Q[u];\ u \in C_0^\infty(D),\ \int_D u^2 m\, d\nu = 1 \right\}.$$

Then, for any $a < \lambda_0$, $(L - a, D)$ is subcritical, i.e., there exists the (minimal positive) Green function of $L - a$ on D. We denote by L_D the selfadjoint operator in $L^2(D; m\, d\nu)$ associated with the closure of Q. The minimal fundamental solution for (1.1) is denoted by $p(x, y, t)$, which is equal to the integral kernel of the semigroup e^{-tL_D} on $L^2(D, m\, d\nu)$.

Let us recall several notions related to [SSP].

[IU] λ_0 is an eigenvalue of L_D; and there exists, for any $t > 0$, a constant $C_t > 0$ such that

$$p(x, y, t) \leqslant C_t \phi_0(x)\phi_0(y), \quad x, y \in D,$$

where ϕ_0 is the normalized positive eigenfunction for λ_0.

This notion was introduced by Davies and Simon [13], and investigated extensively because of its important consequences (see [7–10,12,23,24,31,34,42], and references therein). It looks, on the surface, not related to perturbation theory. But it has turned out [34] that [IU] implies the following condition [SP] for any $a < \lambda_0$.

[SP] The constant function 1 is a small perturbation of $L - a$ on D, i.e., for any $\varepsilon > 0$ there exists a compact subset K of D such that

$$\int_{D\setminus K} G(x, z)G(z, y)m(z)\, d\nu(z) \leqslant \varepsilon G(x, y), \quad x, y \in D \setminus K,$$

where G is the Green function of $L - a$ on D.

This condition is a special case of the notion introduced by Pinchover [37]. Recall that [SP] implies the following condition [SSP] (see [30]).

[SSP] The constant function 1 is a semismall perturbation of $L - a$ on D, i.e., with x^0 being a fixed reference point in D, for any $\varepsilon > 0$ there exists a compact subset K of D such that

$$\int_{D\setminus K} G\big(x^0, z\big)G(z, y)m(z)\, d\nu(z) \leqslant \varepsilon G\big(x^0, y\big), \quad y \in D \setminus K.$$

This condition [SSP] implies that L_D admits a complete orthonormal base of eigenfunctions $\{\phi_j\}_{j=0}^{\infty}$ with eigenvalues $\lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots$ repeated according to multiplicity; furthermore, for any $j = 1, 2, \dots,$ the function ϕ_j/ϕ_0 has a continuous extension $[\phi_j/\phi_0]$ up to the Martin boundary $\partial_M D$ of D for $L - a$ (see Theorem 6.3 of [38]).

We show in this paper that [SSP] also implies the following condition [SIU].

[SIU] λ_0 is an eigenvalue of L_D; and there exist, for any $t > 0$ and compact subset K of D, positive constants A and B such that

$$A\phi_0(x)\phi_0(y) \leqslant p(x, y, t) \leqslant B\phi_0(x)\phi_0(y), \quad x \in K,\ y \in D.$$

This notion was introduced by Bañuelos and Davis [9], where they called it one half IU. Here we should recall that [IU] implies that for any $t > 0$ there exists a constant $c_t > 0$ such that

$$c_t\phi_0(x)\phi_0(y) \leqslant p(x, y, t), \quad x, y \in D.$$

We see that the same argument as in the proof of Theorem 3.1 in [25] (or the argument in the proof of Theorem 1.2 below) shows that [SIU] implies the following condition [NUP] (i.e., non-uniqueness for the positive Cauchy problem).

[NUP] The Cauchy problem

$$(\partial_t + L)u = 0 \quad \text{in } D \times (0, T), \qquad u(x, 0) = 0 \quad \text{on } D \tag{1.3}$$

admits a nonnegative solution u which is not identically zero.

We say that [UP] holds for (1.3) when any nonnegative solution of (1.3) is identically zero. We note that [UP] implies that the constant function 1 is a "big" perturbation of $L - a$ on D in some sense (see Theorem 2.1 of [32]).

Fix $a < \lambda_0$, and suppose that [SSP] holds. Let $D^* = D \cup \partial_M D$ be the Martin compactification of D for $L - a$, which is a compact metric space. Denote by $\partial_m D$ the minimal Martin boundary of D for $L - a$, which is a Borel subset of the Martin boundary $\partial_M D$ of D for $L - a$. Here, we note that $\partial_M D$ and $\partial_m D$ are independent of a in the following sense: if [SSP] holds, then for any $b < \lambda_0$ there is a homeomorphism Φ from the Martin compactification of D for $L - a$ onto that for $L - b$ such that $\Phi|_D = identity$, and Φ maps the Martin boundary and minimal Martin boundary of D for $L - a$ onto those for $L - b$, respectively (see Theorem 1.4 of [30]).

Now, we are ready to state our main results. In the following theorems we assume that [SSP] holds for some fixed $a < \lambda_0$.

Theorem 1.1. *The condition* [SSP] *implies* [SIU].

Theorem 1.2. *Assume* [SSP]. *Then, for any* $\xi \in \partial_M D$ *there exists the limit*

$$\lim_{D \ni y \to \xi} \frac{p(x, y, t)}{\phi_0(y)} \equiv q(x, \xi, t), \quad x \in D,\ t \in \mathbf{R}. \tag{1.4}$$

Here, as functions of (x,t), $\{p(x,y,t)/\phi_0(y)\}_y$ *converges to* $q(x,\xi,t)$ *as* $y \to \xi$ *uniformly on any compact subset of* $D \times \mathbf{R}$. *Furthermore,* $q(x,\xi,t)$ *is a continuous function on* $D \times \partial_M D \times \mathbf{R}$ *such that*

$$q > 0 \quad \text{on } D \times \partial_M D \times (0,\infty), \tag{1.5}$$

$$q = 0 \quad \text{on } D \times \partial_M D \times (-\infty,0], \tag{1.6}$$

$$(\partial_t + L)q(\cdot,\xi,\cdot) = 0 \quad \text{on } D \times \mathbf{R}. \tag{1.7}$$

Theorem 1.3. *Assume* [SSP]. *Consider Eq.* (1.1) *for* $I = (0,T)$ *with* $0 < T \leqslant \infty$. *Then, for any nonnegative solution* u *of* (1.1) *there exists a unique pair of Borel measures* μ *on* D *and* λ *on* $\partial_M D \times [0,T)$ *such that* λ *is supported by the set* $\partial_m D \times [0,T)$, *and*

$$u(x,t) = \int_D p(x,y,t)\,d\mu(y) + \int_{\partial_M D \times [0,t)} q(x,\xi,t-s)\,d\lambda(\xi,s) \tag{1.8}$$

for any $(x,t) \in D \times I$.

Conversely, for any Borel measures μ *on* D *and* λ *on* $\partial_M D \times [0,T)$ *such that* λ *is supported by* $\partial_m D \times [0,T)$ *and*

$$\int_D p\big(x^0,y,t\big)\,d\mu(y) < \infty, \quad 0 < t < T, \tag{1.9}$$

$$\int_{\partial_M D \times [0,t)} q\big(x^0,\xi,t-s\big)\,d\lambda(\xi,s) < \infty, \quad 0 < t < T, \tag{1.10}$$

where x^0 *is a fixed point in* D, *the right-hand side of* (1.8) *is a nonnegative solution of* (1.1) *for* $I = (0,T)$ *with* $0 < T \leqslant \infty$.

The proof of this theorem will be given in Sections 4 and 5. It is based upon the abstract integral representation theorem established in [34], without assuming [IU], via a parabolic Martin representation theorem and Choquet's theorem (see [18,21,35]). Its key step is to identify the parabolic Martin boundary.

This theorem is an improvement of Theorem 1.2 of [34]; where the condition [IU], which is more stringent than [SSP], is assumed. It is also an answer to a problem raised in Remark 4.13 of [34]. Note that (1.8) gives explicit integral representations of nonnegative solutions to (1.1) provided that the Martin boundary $\partial_M D$ of D for $L - a$ is determined explicitly. We consider that [SSP] is one of the weakest possible condition for getting such explicit integral representations.

Let us recall that when [UP] holds for (1.3), the structure of all nonnegative solutions to (1.1) for $I = (0,T)$ is extremely simple. Namely, the following theorem holds (see [5]).

Fact AT. *Assume* [UP]. *Then, for any nonnegative solution* u *of* (1.1) *with* $I = (0,T)$, *there exists a unique Borel measure* μ *on* D *such that*

$$u(x,t) = \int_D p(x,y,t)\,d\mu(y), \quad (x,t) \in D \times I. \tag{1.11}$$

Conversely, for any Borel measure μ *on* D *satisfying* (1.9)*, the right-hand side of* (1.11) *is a nonnegative solution of* (1.1) *with* $I = (0, T)$.

It is quite interesting that when [UP] holds, the elliptic Martin boundary disappears in the parabolic representation theorem; while it enters in many cases of [NUP].

Finally, we state an integral representation theorem for the case $I = (-\infty, 0)$.

Theorem 1.4. *Assume* [SSP]. *Consider Eq.* (1.1) *for* $I = (-\infty, 0)$. *Then, for any nonnegative solution* u *of* (1.1) *there exists a unique pair of a nonnegative constant* α *and a Borel measure* λ *on* $\partial_M D \times (-\infty, 0)$ *supported by the set* $\partial_m D \times (-\infty, 0)$ *such that*

$$u(x,t) = \alpha e^{-\lambda_0 t}\phi_0(x) + \int\limits_{\partial_M D\times(-\infty,t)} q(x,\xi,t-s)\,d\lambda(\xi,s) \tag{1.12}$$

for any $(x,t) \in D \times (-\infty, 0)$.

Conversely, for any nonnegative constant α *and a Borel measure* λ *on* $\partial_M D \times (-\infty, 0)$ *such that it is supported by* $\partial_m D \times (-\infty, 0)$ *and*

$$\int\limits_{\partial_M D\times(-\infty,t)} q\bigl(x^0,\xi,t-s\bigr)\,d\lambda(\xi,s) < \infty, \quad -\infty < t < 0, \tag{1.13}$$

the right-hand side of (1.12) *is a nonnegative solution of* (1.1).

This theorem is an improvement of Theorem 6.1 of [34], where [IU] is assumed instead of [SSP].

Here, in order to illustrate a scope of Theorems 1.3 and 1.4, we give a simple example. Further examples will be given in Section 7.

Example 1.5. Let D be a domain in $\mathbf{R}^2$ with finite area. Then, by Theorem 6.1 of [33], the constant function 1 is a small perturbation of $L = -\Delta$ on D. Thus Theorems 1.3 and 1.4 hold true for the heat equation

$$(\partial_t - \Delta)u = 0 \quad \text{in } D \times I.$$

Note that there exist many bounded planar domains for which the heat semigroup is not intrinsically ultracontractive (see Example 1 of [13] and Section 4 of [9]). Thus, the last assertion of this example is new for such domains.

The remainder of this paper is organized as follows. In Section 2 we prove Theorem 1.1, and Theorem 1.2 is proved in Section 3. Sections 4 and 5 are devoted to the proof of Theorem 1.3. In Section 4 we show it in the case of $I = (0, \infty)$. In Section 5 we show it in the case of $I = (0, T)$ with $0 < T < \infty$ by making use of results to be given in Section 4. Theorem 1.4 is proved in Section 6. Finally we shall give two more concrete examples in Section 7 with emphasis on sharpness of concrete sufficient conditions of [SSP].

2. [SSP] implies [SIU]

In this section we prove Theorem 1.1.

Proof of Theorem 1.1. We may and shall assume that $a = 0 < \lambda_0$. Let G be the Green function of L on D. For any $t > 0$, put

$$G_t(x, y) = \int_t^\infty p(x, y, s)\, ds,$$

$$G^t(x, y) = \int_0^t p(x, y, s)\, ds.$$

Then $G = G_t + G^t$. Let us show that for any $t > 0$ and any compact subset K of D there exists a constant $A > 0$ such that

$$A\phi_0(x)\phi_0(y) \leqslant p(x, y, t), \quad x \in K,\ y \in D. \tag{2.1}$$

Fix a compact subset K. We may assume that $x^0 \in K$. Let $K_1 \subset D$ be a compact neighborhood of K. Then the same argument as in the proof of Theorem 1.5 of [30] shows that

$$C^{-1} G\big(x^0, z\big) \leqslant \phi_0(z) \leqslant C G\big(x^0, z\big), \quad z \in D \setminus K_1, \tag{2.2}$$

for some constant $C > 0$. Fix $t > 0$, and put

$$\epsilon_t = \frac{1}{2\lambda_0}\big(1 - e^{-t\lambda_0}\big).$$

By [SSP] and (2.2), there exists a compact subset $K_2 \supset K_1$ such that

$$\int_{D\setminus K_2} \phi_0(z)\, G(z, y)\, d\mu(z) \leqslant \epsilon_t \phi_0(y), \quad y \in D \setminus K_2, \tag{2.3}$$

where $d\mu(z) = m(z)\, d\nu(z)$. Since

$$\frac{\phi_0(y)}{\lambda_0} = \int_D G(y, z)\phi_0(z)\, d\mu(z),$$

and $G(y, z) = G(z, y)$, (2.3) yields

$$\frac{\phi_0(y)}{\lambda_0} \leqslant \int_{K_2} G_t(z, y)\, \phi_0(z)\, d\mu(z) + \int_{K_2} G^t(z, y)\phi_0(z)\, d\mu(z) + \epsilon_t \phi_0(y) \tag{2.4}$$

for any $y \in D \setminus K_2$. By Fubini's theorem,

$$\int_D G_t(z,y)\phi_0(z)\,d\mu(z) = \int_t^\infty ds \int_D p(z,y,s)\phi_0(z)\,d\mu(z)$$
$$= \int_t^\infty e^{-\lambda_0 s}\phi_0(y)\,ds$$
$$= \frac{1}{\lambda_0}e^{-\lambda_0 t}\phi_0(y).$$

Thus

$$\int_{K_2} G_t(z,y)\phi_0(z)\,d\mu(z) \leqslant \frac{1}{\lambda_0}e^{-\lambda_0 t}\phi_0(y).$$

This together with (2.4) implies

$$\epsilon_t\phi_0(y) \leqslant \int_{K_2} G^t(z,y)\phi_0(z)\,d\mu(z). \tag{2.5}$$

Choose a compact subset K_3 whose interior includes K_2. By the parabolic Harnack inequality, there exists a constant C_1 depending on t, K_2, K_3 such that

$$p(z,y,s) \leqslant C_1\, p(x,y,2t),$$

for any $x, z \in K_2$, $y \in D \setminus K_3$, and $0 < s \leqslant t$. We have

$$G^t(z,y) = \int_0^t p(z,y,s)\,ds \leqslant C_1 t p\bigl(x^0, y, 2t\bigr), \quad z \in K_2,\ y \in D \setminus K_3. \tag{2.6}$$

Thus

$$\int_{K_2} G^t(z,y)\phi_0(z)\,d\mu(z) \leqslant \Bigl[C_1\, t \int_{K_2} \phi_0(z)\,dz\Bigr] p\bigl(x^0, y, 2t\bigr).$$

This together with (2.5) implies

$$\phi_0(y) \leqslant C_2 p\bigl(x^0, y, 2t\bigr), \quad y \in D \setminus K_3, \tag{2.7}$$

where

$$C_2 = \frac{1}{\epsilon_t} C_1\, t \int_{K_2} \phi_0(z)\,d\mu(z).$$

By the parabolic Harnack inequality,

$$p(x^0, y, 2t) \leqslant Cp(x, y, 3t), \quad x \in K, \ y \in D,$$

for some constant $C > 0$. This together with (2.7) yields the desired inequality (2.1). It remains to show that for any $t > 0$ and a compact subset K of D there exists a constant B such that

$$p(x, y, t) \leqslant B\,\phi_0(x)\phi_0(y), \quad x \in K, \ y \in D. \tag{2.8}$$

Fix a compact subset K. We may assume that $x^0 \in K$. Let $K_1 \subset D$ be a compact neighborhood of K. By the parabolic Harnack inequality there exists a constant $c > 0$ such that

$$cp(x^0, y, t) \leqslant p(z, y, 2t), \quad z \in K_1, \ y \in D.$$

Thus, for any $y \in D$,

$$\begin{aligned} e^{-2t\lambda_0}\phi_0(y) &= \int_D \phi_0(z)\, p(z, y, 2t)\, d\mu(z) \\ &\geqslant \int_{K_1} \phi_0(z) p(z, y, 2t)\, d\mu(z) \\ &\geqslant c\left[\int_{K_1} \phi_0(z)\, d\mu(z)\right] p(x^0, y, t). \end{aligned}$$

This implies (2.8), since

$$Cp(x^0, y, t) \geqslant p(x, y, t/2), \quad x \in K, \ y \in D,$$

for some constant $C > 0$. (We should note that in proving (2.8) we have only used the consequence of [SSP] that ϕ_0 is a positive eigenfunction.) □

Remark 2.1. It is an open problem whether [SIU] implies [SSP] or not. Furthermore, the problem whether [SSP] implies [SP] or not in the case $n > 1$ is still open.

3. Parabolic Martin kernels

In this section we prove Theorem 1.2. Throughout the present section we assume [SSP]. We may and shall assume that $a = 0 < \lambda_0$. Let G be the Green function of L on D. For any $0 < \delta < t$, put

$$G_\delta^t(x, y) = \int_\delta^t p(x, y, s)\, ds. \tag{3.1}$$

We denote by $\partial_M D$ the Martin boundary of D for L. In order to prove Theorem 1.2, we need two lemmas.

Lemma 3.1. *Let $\xi \in \partial_M D$. Suppose that a sequence $\{y_n\}_{n=1}^{\infty} \subset D$ converges to ξ, and there exists the limit*

$$\lim_{n\to\infty} \frac{G_\delta^t(z, y_n)}{\phi_0(y_n)} = w(z,t), \quad z \in D. \tag{3.2}$$

Then

$$\lim_{n\to\infty} \int_D G(x,z) \frac{G_\delta^t(z, y_n)}{\phi_0(y_n)}\, d\mu(z) = \int_D G(x,z) w(z,t)\, d\mu(z) \tag{3.3}$$

for any $x \in D$, where $d\mu(z) = m(z)\, d\nu(z)$.

Proof. Fix $x \in D$. Let $K_1 \subset D$ be a compact neighborhood of x. By [SSP], there exists a constant $C > 0$ such that

$$C^{-1}\phi_0(y) \leqslant G(x,y) \leqslant C\phi_0(y), \quad y \in D \setminus K_1. \tag{3.4}$$

Let $\epsilon > 0$. Then there exists a compact subset $K \supset K_1$ such that

$$\int_{D\setminus K} G(x,z) \frac{G(z,y)}{G(x,y)}\, d\mu(z) < \frac{\epsilon}{3C}, \quad y \in D \setminus K.$$

Thus, for n sufficiently large,

$$\int_{D\setminus K} G(x,z) \left[\frac{G_\delta^t(z,y_n)}{\phi_0(y_n)}\right] d\mu(z) \leqslant \int_{D\setminus K} G(x,z) \left[\frac{CG(z,y_n)}{G(x,y_n)}\right] d\mu(z) < \frac{\epsilon}{3}.$$

By Fatou's lemma,

$$\int_{D\setminus K} G(x,z) w(z,t)\, d\mu(z) \leqslant \frac{\epsilon}{3}.$$

By Theorem 1.1, there exist constants A_1 and A_2 such that

$$A_1\phi_0(x)\phi_0(y) \leqslant p(x,y,\delta) \leqslant A_2\phi_0(x)\phi_0(y), \quad x \in K,\ y \in D.$$

Then, for any $t > \delta$, the semigroup property yields

$$A_1 e^{-\lambda_0(t-\delta)}\phi_0(x)\phi_0(y) \leqslant p(x,y,t) \leqslant A_2 e^{-\lambda_0(t-\delta)}\phi_0(x)\phi_0(y) \tag{3.5}$$

for any $x \in K,\ y \in D$. Thus there exists a constant $B > 0$ such that for any n

$$\frac{G_\delta^t(z,y_n)}{\phi_0(y_n)} \leqslant B\phi_0(z), \quad z \in K.$$

Then Lebesgue's dominated convergence theorem yields

$$\lim_{n\to\infty}\int_K G(x,z)\left[\frac{G_\delta^t(z,y_n)}{\phi_0(y_n)}\right]d\mu(z)=\int_K G(x,z)w(z,t)\,d\mu(z).$$

Therefore, for n sufficiently large,

$$\left|\int_D G(x,z)\left[\frac{G_\delta^t(z,y_n)}{\phi_0(y_n)}\right]d\mu(z)-\int_D G(x,z)\,w(z,t)\,d\mu(z)\right|<\epsilon.$$

This shows (3.3). □

By Lemma 6.1 of [38], it follows from [SSP] that there exists the limit

$$\lim_{D\ni y\to\xi}\frac{G(y,z)}{\phi_0(y)}=h(\xi,z),\quad (\xi,z)\in\partial_M D\times D,\tag{3.6}$$

and h is a positive continuous function on $\partial_M D\times D$. From this we show the following lemma.

Lemma 3.2. *Under the same assumptions as in Lemma* 3.1*, one has*

$$\begin{aligned}\int_D h(\xi,z)\,G_\delta^t(z,x)\,d\mu(z)&=\lim_{n\to\infty}\int_D\frac{G(y_n,z)}{\phi_0(y_n)}G_\delta^t(z,x)\,d\mu(z)\\&=\int_D G(x,z)\,w(z,t)\,d\mu(z)\end{aligned}\tag{3.7}$$

for any $x\in D$.

Proof. Fix $x\in D$. Let $K_1\subset D$ be a compact neighborhood of x. By Theorem 1.1, (3.4) and (3.5), there exists a constant $C_1>0$ such that

$$C_1G(z,x)\leqslant G_\delta^t(z,x)\leqslant G(z,x),\quad z\in D\setminus K_1.$$

Let $\epsilon>0$. By [SSP], there exists a compact subset $K\supset K_1$ such that

$$\int_{D\setminus K}\left[\frac{G(y_n,z)}{\phi_0(y_n)}\right]G_\delta^t(z,x)\,d\mu(z)<\frac{\epsilon}{3},\tag{3.8}$$

for n sufficiently large. By Fatou's lemma,

$$\int_{D\setminus K}h(\xi,z)G_\delta^t(z,x)\,d\mu(z)\leqslant\frac{\epsilon}{3}.\tag{3.9}$$

On the other hand, for any sufficiently large n

$$\left[\frac{G(y_n, z)}{\phi_0(y_n)}\right] G_\delta^t(z, x) \leqslant C_2, \quad z \in K,$$

where C_2 is a positive constant. By Lebesgue's dominated convergence theorem,

$$\lim_{n\to\infty} \int_K \frac{G(y_n, z)}{\phi_0(y_n)} G_\delta^t(z, x)\, d\mu(z) = \int_K h(\xi, z) G_\delta^t(z, x)\, d\mu(z). \tag{3.10}$$

Combining (3.8), (3.9) and (3.10), we get the first equality. It remains to show the second equality of (3.7). By Fubini's theorem and the symmetry

$$p(x, y, t) = p(y, x, t),$$

we have

$$\int_D G(y_n, z) G_\delta^t(z, x)\, d\mu(z) = \int_0^\infty dr \int_\delta^t ds\, p(y_n, x, r+s) = \int_D G(x, z) G_\delta^t(z, y_n)\, d\mu(z).$$

This together with Lemma 3.1 implies the second equality. □

Proof of Theorem 1.2. Let $\{y_j\}_{j=1}^\infty \subset D$ be any sequence converging to $\xi \in \partial_M D$. Put

$$u_j(x, t) = \frac{p(x, y_j, t)}{\phi_0(y_j)} \quad \text{for } t > 0, \qquad u_j(x, t) = 0 \quad \text{for } t \leqslant 0. \tag{3.11}$$

Since [SIU] holds, it follows from the parabolic Harnack inequality and local a priori estimates for nonnegative solutions to parabolic equations (see [6] and [16]) that there exists a subsequence $\{u_{j_k}\}_{k=1}^\infty$ such that u_{j_k} converges, as $k \to \infty$, uniformly on any compact subset of $D \times \mathbf{R}$ to a solution u of the equation

$$(\partial_t + L)u = 0 \quad \text{in } D \times \mathbf{R}$$

satisfying $u > 0$ on $D \times (0, \infty)$ and $u = 0$ on $D \times (-\infty, 0]$. Thus, in order to prove Theorem 1.2, it suffices to show that the limit function u is independent of $\{y_{j_k}\}_{k=1}^\infty$ and uniquely determined by ξ. Let $\{y_j\}_{n=1}^\infty$ and $\{y'_j\}_{n=1}^\infty$ be two sequences in D converging to ξ. Define u_j by (3.11), and u'_j by (3.11) with y_j replaced by y'_j. Suppose that $\{u_j\}_{j=1}^\infty$ and $\{u'_j\}_{j=1}^\infty$ converge to u and u', respectively. For any $t > \delta > 0$, put

$$w(z, t) = \int_\delta^t u(z, s)\, ds, \qquad w'(z, t) = \int_\delta^t u'(z, s)\, ds.$$

Then we have

$$\lim_{n\to\infty}\frac{G_\delta^t(z,y_n)}{\phi_0(y_n)}=w(z,t),\qquad \lim_{n\to\infty}\frac{G_\delta^t(z,y_n')}{\phi_0(y_n')}=w'(z,t).$$

By Lemma 3.2,

$$\int_D G(x,z)w(z,t)\,d\mu(z)=\int_D h(\xi,z)G_\delta^t(z,x)\,d\mu(z)=\int_D G(x,z)w'(z,t)\,d\mu(z).$$

Thus $w(x,t)=w'(x,t)$, which implies $u(x,t)=u'(x,t)$. This completes the proof of Theorem 1.2. □

4. Integral representations; the case $I=(0,\infty)$

In this section we prove Theorem 1.3 in the case $T=\infty$.

We first state an abstract integral representation theorem which holds without [SSP]. For $x\in D$ and $r>0$, we denote by $B(x,r)$ the geodesic ball in the Riemannian manifold M with center x and radius r. Let x^0 be a reference point in D. Choose a nonnegative continuous function a on D such that $a(x)=1$ on $B(x^0,r^0)$ and $a(x)=0$ outside $B(x^0,2r^0)$ for some $r^0>0$ with $B(x^0,3r^0)\Subset D$. Choose a nonnegative continuous function b on $\mathbf{R}$ such that $0<b(t)<e^{\gamma t}$ on $(1,\infty)$ for some $\gamma<\lambda_0$, and $b(t)=0$ on $(-\infty,1]$. Denote by β the measure defined by $d\beta(x,t)=a(x)b(t)m(x)\,d\nu(x)\,dt$. For any nonnegative measurable function u on $Q=D\times(0,\infty)$, we write

$$\beta(u)=\iint_Q u(x,t)\,d\beta(x,t).$$

Denote by $P(Q)$ the set of all nonnegative solutions of (1.1) with $I=(0,\infty)$, and put

$$P_\beta(Q)=\{u\in P(Q);\ \beta(u)<\infty\}.$$

Note that for any $u\in P(Q)$ there exists a function b as above such that $\beta(u)<\infty$; thus $P(Q)=\bigcup_\beta P_\beta(Q)$. Furthermore, the parabolic Harnack inequality shows that if $\beta(u)=0$, then $u=0$. Now, let us define the β-Martin boundary $\partial_M^\beta Q$ of Q with respect to ∂_t+L along the line given in [21] and [18]. Put

$$p(x,t;y,s)=\begin{cases} p(x,y,t-s), & t>s,\ x,y\in D,\\ 0, & t\leqslant s,\ x,y\in D.\end{cases}$$

Define the β-Martin kernel K_β by

$$K_\beta(x,t;y,s)=\frac{p(x,t;y,s)}{\beta(p(\cdot;y,s))},\quad (x,t),(y,s)\in Q,$$

where $\beta(p(\cdot;y,s))=\iint_Q p(z,r;y,s)\,d\beta(z,r)$. Note that $\beta(p(\cdot;y,s))<\infty$ for any $(y,s)\in Q$, since $0<b(t)<e^{\gamma t}$ on $(1,\infty)$ for some $\gamma<\lambda_0$. Let $\{D_j\}_{j=1}^\infty$ be an exhaustion of D such that each D_j is a domain with smooth boundary, $D_j \Subset D_{j+1} \Subset D$, $\bigcup_{j=1}^{\infty} D_j = D$, and $B(x^0, 3r^0) \Subset D_1$. Put $Q_j = D_j \times (1/j, j)$. For $Y = (y, s)$, $Z = (z, r) \in Q$, let

$$\delta_\beta(Y, Z) = \sum_{j=1}^{\infty} 2^{-j} \sup_{X \in Q_j} \frac{|K_\beta(X; Y) - K_\beta(X; Z)|}{1 + |K_\beta(X; Y) - K_\beta(X; Z)|}.$$

Then we see that δ_β is a metric on Q, and the topology on Q induced by δ_β is equivalent to the original topology of Q. Denote by $Q^{\beta *}$ the completion of Q with respect to the metric δ_β. Put $\partial_M^\beta Q = Q^{\beta *} \setminus Q$. A sequence $\{Y^k\}_{k=1}^{\infty}$ in Q is called a fundamental sequence if $\{Y^k\}_{k=1}^{\infty}$ has no point of accumulation in Q and $\{K_\beta(\cdot; Y^k)\}_{k=1}^{\infty}$ converges uniformly on any compact subset of Q to a nonnegative solution of (1.1) with $I = (0, \infty)$. By the local a priori estimates for solutions of (1.1), for any $\Xi \in \partial_M^\beta Q$ there exist a unique nonnegative solution $K_\beta(\cdot; \Xi)$ of (1.1) and a fundamental sequence $\{Y^k\}_{k=1}^{\infty}$ in Q such that

$$\lim_{k \to \infty} \sum_{j=1}^{\infty} 2^{-j} \sup_{X \in Q_j} \frac{|K_\beta(X; Y^k) - K_\beta(X; \Xi)|}{1 + |K_\beta(X; Y^k) - K_\beta(X; \Xi)|} = 0.$$

Thus the metric δ_β is canonically extended to $Q^{\beta *}$. Furthermore, $Q^{\beta *}$ becomes a compact metric space, since by the parabolic Harnack inequality, any sequence $\{Y^k\}_{k=1}^{\infty}$ with no point of accumulation in Q has a fundamental subsequence. We call $K_\beta(\cdot; \Xi)$, $\partial_M^\beta Q$ and $Q^{\beta *}$ the β-Martin kernel, β-Martin boundary and β-Martin compactification for $(Q,\ \partial_t + L)$, respectively. Note that $\beta(K_\beta(\cdot; \Xi)) \leqslant 1$ by Fatou's lemma; and so $K_\beta(\cdot; \Xi) \in P_\beta(Q)$. A nonnegative solution $u \in P_\beta(Q)$ is said to be minimal if for any nonnegative solution $v \leqslant u$ there exists a nonnegative constant C such that $v = Cu$. Put

$$\partial_m^\beta Q = \{\Xi \in \partial_M^\beta Q;\ K_\beta(\cdot; \Xi) \text{ is minimal and } \beta(K_\beta(\cdot; \Xi)) = 1\},$$

which we call the minimal β-Martin boundary for $(Q, \partial_t + L)$.

Observe that $D \times [0, \infty)$ is embedded into $Q^{\beta *}$, and $D \times \{0\} \subset \partial_M^\beta Q$. Indeed, with $y \in D$ fixed, for any sequence $\{Y^k\}_{k=1}^{\infty}$ in Q with $\lim_{k \to \infty} Y^k = (y, 0)$ we have $\lim_{k \to \infty} K_\beta(x, t; Y^k) = p(x, t; y, 0)/\beta(p(\cdot; y, 0))$; furthermore, $K_\beta(\cdot; y, 0) \neq K_\beta(\cdot; z, 0)$ if $y \neq z$. We also note that any sequence $\{Y^k = (y^k, s^k)\}_{k=1}^{\infty}$ in Q with $\lim_{k \to \infty} s^k = \infty$ is a fundamental sequence, since $\lim_{k \to \infty} K_\beta(\cdot; Y^k) = 0$. We denote by ϖ the point in $\partial_M^\beta Q$ corresponding to the Martin kernel which is identically zero: $K_\beta(\cdot; \varpi) = 0$. Put

$$\mathcal{L}_m^\beta Q = \partial_m^\beta Q \setminus (D \times \{0\} \cup \{\varpi\}).$$

We obtain the following abstract integral representation theorem in the same way as in the proof of Theorem 2.1 and Lemma 2.2 of [34].

Theorem 4.1. *For any $u \in P_\beta(Q)$, there exists a unique pair of finite Borel measures κ on D and λ on $\partial_M^\beta Q \setminus (D \times \{0\})$ such that λ is supported by the set $\mathcal{L}_m^\beta Q$,*

$$u(x, t) = \int_D \frac{p(x, t; y, 0)}{\beta(p(\cdot; y, 0))}\, d\kappa(y) + \int_{\mathcal{L}_m^\beta Q} K_\beta(x, t; \Xi)\, d\lambda(\Xi) \tag{4.1}$$

for any $(x,t)\in Q$*, and*

$$\beta(u)=\kappa(D)+\lambda\big(\mathcal{L}_m^{\beta}Q\big). \tag{4.2}$$

Furthermore, the function

$$v(x,t)=u(x,t)-\int_D \frac{p(x,t;y,0)}{\beta(p(\cdot;y,0))}\,d\kappa(y)$$

is a nonnegative solution of the equation

$$(\partial_t+L)v=0 \quad \text{in } D\times\mathbf{R}$$

such that $v=0$ *on* $D\times(-\infty,0]$.

Conversely, for any finite Borel measures κ *on* D *and* λ *on* $\partial_M^{\beta}Q\setminus(D\times\{0\})$ *such that* λ *is supported by the set* $\mathcal{L}_m^{\beta}Q$*, the right-hand side of* (4.1) *belongs to* $P_\beta(Q)$.

We put

$$P_\beta^0(Q)=\Big\{v\in P_\beta(Q);\ \lim_{t\downarrow 0}v(x,t)=0 \text{ on } D\Big\}.$$

We show Theorem 1.3 on the basis of Theorem 4.1. To this end it suffices to show (1.8) for $u\in P_\beta^0(Q)$. The key step in the proof is to identify $\mathcal{L}_m^{\beta}Q$. Under the condition [SSP], we shall show that $\mathcal{L}_m^{\beta}Q=\partial_m D\times[0,\infty)$. In the remainder of this section we assume [SSP]. We may and shall assume that $a=0<\lambda_0$.

Lemma 4.2. *For any domains* U *and* W *with* $U\Subset W\Subset D$*, there exist positive constants* C *and* α *such that*

$$p(x,y,t)\leqslant Cf(t)\phi_0(x)\phi_0(y), \quad x\in U,\ y\in D\setminus W,\ t>0, \tag{4.3}$$

where $f(t)=e^{-\alpha/t}$ *for* $0<t<1$*, and* $f(t)=e^{-\lambda_0 t}$ *for* $t\geqslant 1$*. Furthermore,*

$$q(x,\xi,t)\leqslant Cf(t)\phi_0(x), \quad x\in U,\ \xi\in\partial_M D,\ t>0, \tag{4.4}$$

$$G(x,y)\leqslant C\phi_0(x)\phi_0(y), \quad x\in U,\ y\in D\setminus W, \tag{4.5}$$

where G *is the Green function of* L *on* D.

This lemma is shown in the same way as Lemmas 4.2 and 4.4 of [34].

Let $K(x,\xi)$ be the Martin kernel for L on D with reference point $x^0\in D$, i.e., $K(x^0,\xi)=1$, $\xi\in\partial_M D$. The following lemma gives a relation between K and q.

Lemma 4.3. *For any $\xi \in \partial_M D$,*

$$\lim_{D \ni y \to \xi} \frac{G(x,y)}{\phi_0(y)} = \int_0^\infty q(x,\xi,t)\,dt, \quad x \in D, \tag{4.6}$$

$$K(x,\xi) = \frac{\int_0^\infty q(x,\xi,t)\,dt}{\int_0^\infty q(x^0,\xi,t)\,dt}, \quad x \in D. \tag{4.7}$$

This lemma is shown in the same way as Lemma 4.5 of [34].

Lemma 4.4. *Let $\xi, \eta \in \partial_M D$, $0 \leqslant s, r < \infty$ and $C > 0$. If*

$$q(x,\xi,t-s) = Cq(x,\eta,t-r), \quad (x,t) \in Q,$$

then $\xi = \eta$, $s = r$ and $C = 1$.

Proof. Since $q(x,\xi,\tau) > 0$ for $\tau > 0$ and $q(x,\xi,\tau) = 0$ for $\tau \leqslant 0$, we obtain that $s = r$. Thus $q(x,\xi,\tau) = q(x,\eta,\tau)$. This together with (4.7) implies that $K(\cdot,\xi) = K(\cdot,\eta)$ on D. Hence $\xi = \eta$, and so $C = 1$. □

Now, let β be a measure on $Q = D \times (0,\infty)$ as described in the beginning of this section: $d\beta(x,t) = a(x)b(t)m(x)\,d\nu(x)\,dt$. The following proposition determines the β-Martin boundary $\partial_M^\beta Q$, β-Martin compactification $Q^{\beta *}$, and β-Martin kernel K_β for $(\partial_t + L, Q)$. Recall that $p(x,t;y,s) = p(x,y,t-s)$ and $K_\beta(\cdot;y,s) = p(\cdot;y,s)/\beta(p(\cdot;y,s))$. We write

$$q(x,t;\xi,s) = q(x,\xi,t-s)$$

for $\xi \in \partial_M D$ and $0 \leqslant s < \infty$.

Proposition 4.5.

(i) *The β-Martin boundary $\partial_M^\beta Q$ of Q for $\partial_t + L$ is equal to the disjoint union of $D \times \{0\}$, $\partial_M D \times [0,\infty)$ and the one point set $\{\varpi\}$:*

$$\partial_M^\beta Q = D \times \{0\} \cup \partial_M D \times [0,\infty) \cup \{\varpi\}. \tag{4.8}$$

In particular, $\partial_M^\beta Q$ does not depend on β.

(ii) *The β-Martin compactification $Q^{\beta *}$ of Q for $\partial_t + L$ is homeomorphic to the disjoint union of the topological product $D^* \times [0,\infty)$ and the one point set $\{\varpi\}$, where a fundamental neighborhood system of ϖ is given by the family $\{\varpi\} \cup D^* \times (N,\infty)$, $N > 1$. In particular, $Q^{\beta *}$ does not depend on β.*

(iii) *The β-Martin kernel K_β is given as follows. For $(x,t) \in Q$,*

$$K_\beta(x,t;y,0) = \frac{p(x,t;y,0)}{\beta(p(\cdot;y,0))}, \quad (y,0) \in D \times \{0\}, \tag{4.9}$$

$$K_\beta(x,t;\xi,s) = \frac{q(x,t;\xi,s)}{\beta(q(\cdot;\xi,s))}, \quad (\xi,s) \in \partial_M D \times [0,\infty), \tag{4.10}$$

and $K_\beta(x,t;\varpi) = 0$.

This proposition is shown in the same way as Proposition 4.8 of [34].

Lemma 4.6. *Let* $(\xi, s) \in (\partial_M D \setminus \partial_m D) \times [0, \infty)$. *Then there exists a finite Borel measure* γ *on* $\partial_M D$ *supported by* $\partial_m D$ *such that*

$$q(\cdot; \xi, s) = \int_{\partial_m D} q(\cdot; \eta, s)\, d\gamma(\eta). \tag{4.11}$$

Proof. For reader's convenience, we give a sketch of the proof for the case $s = 0$. (For details, see the proof of Lemma 4.10 of [34].) By the elliptic Martin representation theorem, there exists a unique finite Borel measure μ on $\partial_M D$ supported by $\partial_m D$ such that

$$K(x, \xi) = \int_{\partial_m D} K(x, \eta)\, d\mu(\eta).$$

This together with (4.7) implies

$$\int_0^\infty q(x, \xi, t)\, dt = \int_{\partial_m D} \left(\int_0^\infty q(x, \eta, t)\, dt \right) d\gamma(\eta), \tag{4.12}$$

where $d\gamma(\eta) = [H(x^0, \xi)/H(x^0, \eta)]\, d\mu(\eta)$ with

$$H(x, \eta) = \int_0^\infty q(x, \eta, t)\, dt.$$

For $\alpha > 0$, denote by G_α the Green function of $L + \alpha$ on D. By the resolvent equation and [SSP], we then have

$$\int_0^\infty e^{-\alpha t} q(x, \eta, t)\, dt = \int_0^\infty q(x, \eta, t)\, dt - \alpha \int_D G_\alpha(x, z) \left(\int_0^\infty q(z, \eta, t)\, dt \right) m(z)\, d\nu(z), \tag{4.13}$$

for any $\eta \in \partial_M D$. By combining (4.12) and (4.13), we get

$$\int_0^\infty e^{-\alpha t} \left(\int_{\partial_m D} q(x, \eta, t)\, d\gamma(\eta) \right) dt = \int_0^\infty e^{-\alpha t} q(x, \xi, t)\, dt.$$

Thus the Laplace transforms of $q(x, \xi, t)$ and $\int_{\partial_m D} q(x, \eta, t)\, d\gamma(\eta)$ coincide; and so (4.11) holds. □

Lemma 4.7. *Let* $(\xi, s) \in (\partial_M D \setminus \partial_m D) \times [0, \infty)$. *Then* $q(\cdot; \xi, s)$ *is not minimal.*

Proof. For reader's convenience, we give a proof. We have (4.11). Suppose that $q(\cdot;\xi,s)$ is minimal. Then, along the line given in the proof of Lemma 12.12 of [15], we obtain from (4.11) that the support of γ consists of a single point. Thus, for some $\eta\in\partial_m D$ and constant C

$$q(\cdot;\xi,s)=Cq(\cdot;\eta,s).$$

Hence, by Lemma 4.4, $\xi=\eta$; which is a contradiction. □

Lemma 4.8. *Let $(\xi,s)\in\partial_m D\times(0,\infty)$. Then $q(\cdot;\xi,s)$ is minimal if and only if $q(\cdot;\xi,0)$ is minimal.*

Proof. Assume that $q(\cdot;\xi,0)$ is minimal. Suppose that a nonnegative solution u of (1.1) satisfies $u(\cdot)\leqslant q(\cdot;\xi,s)$ on Q. Put $v(x,t)=u(x,t+s)$. Then $v(\cdot)\leqslant q(\cdot;\xi,0)$. Thus $v(\cdot)=Cq(\cdot;\xi,0)$ for some constant C. Hence $u(x,t)=Cq(x,t;\xi,s)$ for $t>s$, and $u(x,t)=0=Cq(x,t;\xi,s)$ for $t\leqslant s$. This shows that $q(\cdot;\xi,s)$ is minimal. Next, assume that $q(\cdot;\xi,s)$ is minimal. Suppose that a nonnegative solution u of (1.1) satisfies $u(\cdot)\leqslant q(\cdot;\xi,0)$ on Q. Put $v(x,t)=u(x,t-s)$ for $t>s$, and $v(x,t)=0$ for $0<t\leqslant s$. Then $v(\cdot)\leqslant q(\cdot;\xi,s)$. Thus $v(\cdot)=Cq(\cdot;\xi,s)$ for some constant C. Hence $u(x,t)=Cq(x,t;\xi,0)$. This shows that $q(\cdot;\xi,0)$ is minimal. □

By Theorem 4.1 and Lemmas 4.7 and 4.8, we have the following proposition.

Proposition 4.9. *There exists a Borel subset R of $\partial_M D$ such that*

$$R\subset\partial_m D,\qquad \mathcal{L}_m^{\beta}Q=R\times[0,\infty),$$

for any $u\in P_\beta^0(Q)$ there exists a unique Borel measure λ on $\partial_M D\times[0,\infty)$ which is supported by $R\times[0,\infty)$ and satisfies

$$u(x,t)=\int\limits_{R\times[0,\infty)} q(x,\xi,t-s)\,d\lambda(\xi,s),\quad (x,t)\in Q. \tag{4.14}$$

Lemma 4.10. *Let $(\xi,s)\in\partial_m D\times[0,\infty)$. Then $q(\cdot;\xi,s)$ is minimal.*

Proof. Suppose that $q(\cdot;\xi,0)$ is not minimal. Then $\xi\notin R$ and

$$q(x,\xi,t)=\int\limits_{R\times[0,\infty)} q(x,\eta,t-s)\,d\lambda(\eta,s)$$

for some Borel measure λ. We have

$$K(x,\xi)\int\limits_0^\infty q\big(x^0,\xi,t\big)\,dt=\int\limits_0^\infty q(x,\xi,t)\,dt=\int\limits_{R\times[0,\infty)} d\lambda(\eta,s)K(x,\eta)\int\limits_0^\infty q\big(x^0,\eta,t\big)\,dt.$$

Thus

$$K(x,\xi)=\int_R K(x,\eta)\,d\Lambda(\eta)$$

for some Borel measure Λ. But $\xi\in\partial_m D\setminus R$ and $R\subset\partial_m D$. This contradicts the uniqueness of a representing measure in the elliptic Martin representation theorem. Hence $q(\cdot;\xi,0)$ is minimal; which together with Lemma 4.8 shows Lemma 4.10. □

Completion of the proof of Theorem 1.3 in the case $I=(0,\infty)$. By Lemma 4.10, $R=\partial_m D$ and

$$\mathcal{L}_m^{\beta} Q=\partial_m D\times[0,\infty).$$

Thus Proposition 4.9 shows Theorem 1.3. □

5. Proof of Theorem 1.3; the case $0<T<\infty$

In this section we prove Theorem 1.3 in the case $0<T<\infty$ by making use of the results in Section 4. To this end, the following proposition plays a crucial role.

Proposition 5.1. *Let $\xi\in\partial_M D$ and $0\leqslant s<r<\infty$. Then*

$$\int_D p(x,y,t-r)q(y,r;\xi,s)\,d\mu(y)=q(x,t;\xi,s),\quad x\in D,\ t>r, \tag{5.1}$$

where $d\mu(y)=m(y)\,d\nu(y)$.

Proof. We first show (5.1) for $\xi\in\partial_m D$. Define $u(x,t)$ by

$$\begin{aligned} u(x,t)&=q(x,t;\xi,s),\quad 0<t\leqslant r,\\ u(x,t)&=\int_D p(x,y,t-r)q(y,r;\xi,s)\,d\mu(y),\quad r<t<\infty. \end{aligned} \tag{5.2}$$

(We call u the minimal extension of q from $t=r$.) Then we see that u is a nonnegative solution of $(\partial_t+L)u=0$ in $D\times(0,\infty)$ such that $u(\cdot)\leqslant q(\cdot;\xi,s)$ on $D\times(0,\infty)$. By Lemma 4.10, $u(\cdot)=Cq(\cdot;\xi,s)$ for some constant C. But $u(x,t)=q(x,t;\xi,s)$ for $0<t\leqslant r$. Thus $C=1$, and so $u(\cdot)=q(\cdot;\xi,s)$.

Next, let $\xi\notin\partial_m D$. By Lemma 4.6, there exists a finite Borel measure γ on $\partial_M D$ supported by $\partial_m D$ such that

$$q(\cdot;\xi,s)=\int_{\partial_m D} q(\cdot;\eta,s)\,d\gamma(\eta). \tag{5.3}$$

Thus

$$\int_D p(x,y,t-r)q(y,r;\xi,s)\,d\mu(y) = \int_{\partial_m D} d\gamma(\eta)\int_D p(x,y,t-r)q(y,r;\eta,s)\,d\mu(y)$$
$$= \int_{\partial_m D} q(x,t;\eta,s)\,d\gamma(\eta)$$
$$= q(x,t;\xi,s).$$

This proves (5.1). □

Lemma 5.2. *Let $\xi,\eta \in \partial_M D$, $0 \leqslant s,r < T$ and $C > 0$. If*

$$q(x,\xi,t-s) = Cq(x,\eta,t-r), \quad x \in D,\ 0 < t < T, \tag{5.4}$$

then $\xi = \eta$, $s = r$ and $C = 1$.

Proof. Choose u such that $\max(r,s) < u < T$, and construct minimal extensions of both sides of (5.4) from $t = u$. Then, by (5.1) we have

$$q(x,\xi,t-s) = Cq(x,\eta,t-r), \quad x \in D,\ 0 < t < \infty.$$

By Lemma 4.4, this implies that $\xi = \eta$, $s = r$ and $C = 1$. □

Now, let β be a measure on $Q = D \times (0,T)$ defined by

$$d\beta(x,t) = a(x)b(t)m(x)\,d\nu(x)\,dt.$$

Here $a(x)$ is a nonnegative continuous function on D as described in the beginning of Section 4, and $b(t)$ is a nonnegative continuous function on $\mathbf{R}$ such that $b(t) > 0$ on $(T/2,T)$ and $b(t) = 0$ on $\mathbf{R} \setminus (T/2,T)$. Let $K_\beta(\cdot;\Xi)$, $\partial_M^\beta Q$, $\partial_m^\beta Q$, and $Q^{\beta *}$ be the β-Martin kernel, β-Martin boundary, minimal β-Martin boundary, and β-Martin compactification for $(Q, \partial_t + L)$ with $Q = D \times (0,T)$, respectively. The following proposition is an analogue of Proposition 4.5, and is shown in the same way.

Proposition 5.3.

(i) *The β-Martin boundary $\partial_M^\beta Q$ of Q for $\partial_t + L$ is equal to the disjoint union of $D \times \{0\}$, $\partial_M D \times [0,T)$ and the one point set $\{\varpi\}$:*

$$\partial_M^\beta Q = D \times \{0\} \cup \partial_M D \times [0,T) \cup \{\varpi\}. \tag{5.5}$$

In particular, $\partial_M^\beta Q$ does not depend on β.

(ii) *The β-Martin compactification $Q^{\beta *}$ of Q for $\partial_t + L$ is homeomorphic to the disjoint union of the topological product $D^* \times [0,T)$ and the one point set $\{\varpi\}$, where a fundamental neighborhood system of ϖ is given by the family $\{\varpi\} \cup D^* \times (T-\varepsilon,T)$, $0 < \varepsilon < T/2$. In particular, $Q^{\beta *}$ does not depend on β.*

(iii) *The β-Martin kernel K_β is given as follows*: *for $(x,t) \in Q$,*

$$K_\beta(x,t;y,0) = \frac{p(x,t;y,0)}{\beta(p(\cdot;y,0))}, \quad (y,0) \in D \times \{0\}, \tag{5.6}$$

$$K_\beta(x,t;\xi,s) = \frac{q(x,t;\xi,s)}{\beta(q(\cdot;\xi,s))}, \quad (\xi,s) \in \partial_M D \times [0,T), \tag{5.7}$$

and $K_\beta(x,t;\varpi) = 0$.

Lemma 5.4. *Let $(\xi,s) \in (\partial_M D \setminus \partial_m D) \times [0,T)$. Then $q(\cdot;\xi,s)$ is not minimal.*

Proof. Suppose that $q(\cdot;\xi,s)$ is minimal. Then we obtain from (5.3) that

$$q(x,\xi,t-s) = Cq(x,\eta,t-s), \quad x \in D,\ 0 < t < T,$$

for some $\eta \in \partial_m D$ and $C > 0$. By Lemma 5.2, this is a contradiction. □

Lemma 5.5. *Let $(\xi,s) \in \partial_m D \times [0,T)$. Then $q(\cdot;\xi,s)$ is minimal.*

Proof. Let u be a nonnegative solution of $(\partial_t + L)u = 0$ in Q such that $u(\cdot) \leqslant q(\cdot;\xi,s)$ in Q. For $r \in (s,T)$, let u_r be the minimal extension of u from $t = r$. By Proposition 5.1,

$$u_r(x,t) \leqslant q(x,t;\xi,s), \quad x \in D,\ t > 0.$$

By Lemma 4.10, there exists a constant C_r such that $u_r(x,t) = C_r q(x,t;\xi,s)$ for $t > 0$. But $u_r(x,t) = u(x,t)$ for $0 < t < r$. Thus C_r is independent of r; and so $u(\cdot) = Cq(\cdot;\xi,s)$ in Q for some constant C. □

Completion of the proof of Theorem 1.3 in the case $0 < T < \infty$. Put

$$\mathcal{L}_m^\beta Q = \partial_m^\beta Q \setminus \big(D \times \{0\} \cup \{\varpi\}\big).$$

By Proposition 5.3, Lemmas 5.4 and 5.5, we get

$$\mathcal{L}_m^\beta Q = \partial_m D \times [0,T).$$

Thus, Theorem 2.1 of [34] which is an analogue of Theorem 4.1 completes the proof. □

6. Integral representations; the case $I = (-\infty, 0)$

In this section we prove Theorem 1.4. We begin with the following proposition, which can be shown in the same way as in the proof of Theorem 1 of [9] (see also [39]).

Proposition 6.1. *Assume* [SIU]. *Then*

$$\lim_{t\to\infty} \frac{e^{\lambda_0 t} p(x,y,t)}{\phi_0(x)\phi_0(y)} = 1 \quad \textit{uniformly in } (x,y) \in K \times D \tag{6.1}$$

for any compact subset K of D.

In the rest of this section we assume [SSP]. We may and shall assume that $a = 0 < \lambda_0$. By Theorem 1.1, we have the following corollary of Proposition 6.1.

Corollary 6.2. *Assume* [SSP]. *Then, for any compact subset K of D and $N > 1$,*

$$\lim_{s\to-\infty} \frac{p(x,y,t-s)}{e^{\lambda_0 s}\phi_0(y)} = e^{-\lambda_0 t}\phi_0(x) \quad \text{uniformly in } (x,y,t) \in K \times D \times (-N,0).$$

Lemma 6.3. *The solution $e^{-\lambda_0 t}\phi_0(x)$ is minimal.*

Proof. Suppose that $e^{-\lambda_0 t}\phi_0(x)$ is not minimal. Then, in view of Corollary 6.2, the same argument as in the proof of Theorem 1.3 shows that for any nonnegative solution u of the equation

$$(\partial_t + L)u = 0 \quad \text{in } Q = D \times (-\infty, 0)$$

there exists a unique Borel measure λ on $\partial_M D \times (-\infty, 0)$ supported by the set $\partial_m D \times (-\infty, 0)$ such that

$$u(x,t) = \int\limits_{\partial_M D\times(-\infty,t)} q(x,\xi,t-s)\,d\lambda(\xi,s), \quad (x,t) \in Q.$$

Thus

$$e^{-\lambda_0 t}\phi_0(x) = \int\limits_{\partial_M D\times(-\infty,t)} q(x,\xi,t-s)\,d\lambda(\xi,s), \quad (x,t) \in Q, \tag{6.2}$$

for such a measure λ. Now, fix x. It follows from Theorems 1.1 and 1.2 that for any $\delta > 0$ there exists a positive constant C_δ such that

$$C_\delta{}^{-1} \leqslant \frac{q(x,\xi,\tau)}{e^{-\lambda_0\tau}\phi_0(x)} \leqslant C_\delta, \quad \tau \geqslant \delta,\ \xi \in \partial_M D. \tag{6.3}$$

By (4.4),

$$q(x,\xi,\tau) \leqslant Ce^{-\alpha/\tau}\phi_0(x), \quad \xi \in \partial_M D,\ 0 < \tau < 1, \tag{6.4}$$

for some positive constants α and C. By (6.2) and (6.3),

$$e^{\lambda_0}\phi_0(x) \geqslant \int\limits_{\partial_M D\times(-\infty,-2)} C_1^{-1} e^{-\lambda_0(-1-s)}\,d\lambda(\xi,s).$$

Thus

$$\int\limits_{\partial_M D\times(-\infty,-2)} e^{\lambda_0 s}\,d\lambda(\xi,s) \leqslant C_1\phi_0(x). \tag{6.5}$$

For $t < -2$ and $0 < \delta < 1$, we have

$$\phi_0(x) = \int\limits_{\partial_M D \times \{(-\infty, t-\delta] \cup (t-\delta, t)\}} e^{\lambda_0(t-s)} q(x, \xi, t-s) e^{\lambda_0 s}\, d\lambda(\xi, s). \tag{6.6}$$

In view of (6.4) and (6.5), we choose δ so small that the integral on $\partial_M D \times (t-\delta, t)$ of the right-hand side of (6.6) is smaller than $\phi_0(x)/3$. Then, in view of (6.3) and (6.5), we choose $t < -2$ with $|t|$ being so large that the integral on $\partial_M D \times (-\infty, t-\delta]$ of the right-hand side of (6.6) is smaller than $\phi_0(x)/3$. This is a contradiction. □

Completion of the proof of Theorem 1.4. By virtue of Corollary 6.2 and Lemma 6.3, the same argument as in the proof of Theorem 1.3 shows Theorem 1.4. □

7. Examples

In this section we give two examples in order to illustrate a scope of Theorem 1.3. Throughout this section L_0 is a uniformly elliptic operator on $\mathbf{R}^n$ of the form

$$L_0 u = -\sum_{i,j=1}^{n} \partial_i \big(a_{ij}(x)\, \partial_j u\big),$$

where $a(x) = [a_{ij}(x)]_{i,j=1}^n$ is a symmetric matrix-valued measurable function on $\mathbf{R}^n$ satisfying, for some $\Lambda > 0$,

$$\Lambda^{-1}|\xi|^2 \leqslant \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leqslant \Lambda|\xi|^2, \quad x, \xi \in \mathbf{R}^n.$$

7.1. Let $V(x)$ be a measurable function in $L^\infty_{\mathrm{loc}}(\mathbf{R}^n)$, and $L = L_0 + V(x)$ on $D = \mathbf{R}^n$.

Theorem 7.1. *Suppose that there exist a positive constant $c < 1$ and a positive continuous increasing function ρ on $[0, \infty)$ such that*

$$c\big[\rho\big(|x|\big)\big]^2 \leqslant V(x) \leqslant \big[\rho\big(|x|\big)\big]^2, \quad x \in \mathbf{R}^n, \tag{7.1}$$

$$c\rho\left(r + \frac{c}{\rho(r)}\right) \leqslant \rho(r), \quad r \geqslant 0. \tag{7.2}$$

Assume that

$$\int_1^\infty \frac{dr}{\rho(r)} < \infty. \tag{7.3}$$

Then 1 is a small perturbation of L on $\mathbf{R}^n$. Thus Theorem 1.3 holds true.

Remark. Compare this theorem with a non-uniqueness theorem of [26].

Proof. We first note that (7.2) yields

$$c\rho(r) \leqslant c\rho\left(r - \frac{c}{\rho(r)} + \frac{c}{\rho(r - \frac{c}{\rho(r)})}\right) \leqslant \rho\left(r - \frac{c}{\rho(r)}\right), \quad r \geqslant \frac{c}{\rho(0)},$$

since ρ is increasing. We show the theorem by using the same approach as in the proof of Theorem 5.1 of [31]. Put $b = c^{-2}$ and

$$\ell = \inf\{j \in \mathbf{Z};\ \rho(0) < b^j\}.$$

For $k \geqslant \ell$, put $r_k = \sup\{r \geqslant 0;\ \rho(r) \leqslant b^k\}$. By the continuity of ρ and (7.3), $\rho(r_k) = b^k$ and $\lim_{k\to\infty} r_k = \infty$. By (7.2),

$$\rho\left(r_k + cb^{-k}\right) \leqslant c^{-1}\rho(r_k) = b^{1/2}b^k < b^{k+1} = \rho(r_{k+1}).$$

Thus $r_k + cb^{-k} < r_{k+1}$ for $k \geqslant \ell$. Define a positive continuously differentiable increasing function $\widetilde{\rho}$ on $[0, \infty)$ as follows. Put $\widetilde{\rho}(r) = b^\ell$ for $r \leqslant r_\ell$,

$$\widetilde{\rho}(r) = b^{k+1} \quad \text{for } r_k + cb^{-k} \leqslant r \leqslant r_{k+1} \quad (k \geqslant \ell);$$

and $\widetilde{\rho}(r) = \rho_k(r)$ for $r_k \leqslant r \leqslant r_k + cb^{-k}$ $(k \geqslant \ell)$ by choosing a continuously differentiable function ρ_k on $[r_k, r_k + cb^{-k}]$ such that

$$\rho_k(r_k) = b^k, \qquad {\rho_k}'(r_k) = 0, \qquad \rho_k\left(r_k + cb^{-k}\right) = b^{k+1}, \qquad {\rho_k}'\left(r_k + cb^{-k}\right) = 0,$$

and

$$0 \leqslant {\rho_k}'(r) \leqslant Bb^{2k}, \qquad r_k \leqslant r \leqslant r_k + cb^{-k},$$

for some constant $B > 0$ independent of k. Then we have

$$C^{-1} \leqslant \frac{\widetilde{\rho}(r)}{\rho(r)} \leqslant C, \qquad 0 \leqslant \widetilde{\rho}'(r) \leqslant C\rho(r)^2, \qquad r \geqslant 0, \tag{7.4}$$

for some positive constant C. Introduce a Riemannian metric $g = (g_{ij})_{i,j=1}^n$ by $g_{ij} = \widetilde{\rho}(|x|)^2\delta_{ij}$. Then $M = \mathbf{R}^n$ with this metric g becomes a complete Riemannian manifold. Furthermore, by (7.2) and (7.4), M has the bounded geometry property (1.1) of [4]. The associated gradient ∇ and divergence div are written as

$$\nabla = \widetilde{\rho}\left(|x|\right)^{-2}\nabla^0, \qquad \operatorname{div} = \widetilde{\rho}\left(|x|\right)^{-n} \circ \operatorname{div}^0 \circ \widetilde{\rho}\left(|x|\right)^n,$$

where ∇^0 and div^0 are the standard gradient and divergence on $\mathbf{R}^n$. Put

$$\mathcal{L} = \widetilde{\rho}\left(|x|\right)^{-2} L,$$

$$m(x) = \widetilde{\rho}\left(|x|\right)^{2-n}, \qquad A(x) = \left[a_{ij}(x)\right]_{i,j=1}^n, \qquad \gamma(x) = \widetilde{\rho}\left(|x|\right)^{-2} V(x).$$

Then

$$\mathcal{L}u = -\frac{1}{m}\operatorname{div}(mA\nabla u) + \gamma = -\operatorname{div}(A\nabla u) - \left\langle \frac{1}{m}A\nabla^0 m, \nabla u\right\rangle^0 + \gamma,$$

where $\langle\cdot,\cdot\rangle^0$ is the standard inner product on $\mathbf{R}^n$. Since the inner product $\langle\cdot,\cdot\rangle$ associated with the metric g is written as

$$\langle X, Y\rangle = \langle \widetilde{\rho}^2 X, Y\rangle^0,$$

we have

$$\mathcal{L}u = -\operatorname{div}(A\nabla u) - \left\langle \widetilde{\rho}^{-2}\frac{A\nabla^0 m}{m}, \nabla u\right\rangle + \gamma. \tag{7.5}$$

By (7.4),

$$\left|\nabla^0 m(x)\right| \leqslant C^3|n-2|\widetilde{\rho}\big(|x|\big)m(x).$$

From this we have

$$\left\langle \widetilde{\rho}^{-2}\frac{A\nabla^0 m}{m}, \widetilde{\rho}^{-2}\frac{A\nabla^0 m}{m}\right\rangle \leqslant \widetilde{\rho}^{-2}\Lambda^2\big(C^3|n-2|\widetilde{\rho}\big)^2 \leqslant \big\{\Lambda\big(C^3|n-2|\big)\big\}^2.$$

By (7.1) and (7.4),

$$c\,C^{-2} \leqslant \gamma(x) \leqslant C^2.$$

Thus the operator $\mathcal{L} - c\,C^{-2}/2$ has the Green function; and $\mathcal{L}$ belongs to the class $\mathcal{D}_M(\theta, \infty, \epsilon)$ introduced by Ancona [4], where

$$\theta = \max\big(\Lambda, \Lambda\big(C^3|n-2|\big), C^2\big), \qquad \epsilon = c\,C^{-2}/2.$$

Put

$$\mathcal{L}_2 = \widetilde{\rho}\big(|x|\big)^{-2}(L+1) = \mathcal{L} + \widetilde{\rho}\big(|x|\big)^{-2}.$$

In order to apply the results of [4], we proceed to estimate $\widetilde{\rho}(|x|)^{-2}$. Let $d(x)$ be the Riemannian distance $\operatorname{dist}(0, x)$ from the origin 0 to x, and put

$$\psi(r) = \int_0^r \widetilde{\rho}(s)\,ds.$$

Then we see that $d(x) = \psi(|x|)$. Denote by ψ^{-1} the inverse function of ψ, and put

$$\Phi(s) = \big[\widetilde{\rho}\big(\psi^{-1}(s)\big)\big]^{-2}, \quad s \geqslant 0.$$

Then

$$0 < \widetilde{\rho}\big(|x|\big)^{-2} = \Phi\big(d(x)\big), \quad x \in M.$$

Furthermore,

$$\int_0^\infty \Phi(s)\,ds = \int_0^\infty \Phi\big(\psi(r)\big)\widetilde{\rho}(r)\,dr = \int_0^\infty \frac{dr}{\widetilde{\rho}(r)} \leqslant C \int_0^\infty \frac{dr}{\rho(r)}\,dr < \infty.$$

Hence, by virtue of Corollary 6.1, Theorems 1 and 2 of [4], $\widetilde{\rho}(|x|)^{-2}$ is a small perturbation of $\mathcal{L}$ on the manifold M. That is, for any $\varepsilon > 0$ there exists a compact subset K of $D = M$ such that

$$\int_{D\setminus K} H(x,z)\widetilde{\rho}\big(|z|\big)^{-2} H(z,y)\widetilde{\rho}\big(|z|\big)^{n}\,dz \leqslant \varepsilon H(x,y), \quad x, y \in D \setminus K,$$

where dz is the Lebesgue measure on $\mathbf{R}^n$, and $H(x,z)$ is the Green function of $\mathcal{L}$ on D with respect to the measure $\widetilde{\rho}(|z|)^n\,dz$. Denote by $G(x,z)$ the Green function of L on D with respect to the measure dz. Since $\mathcal{L} = \widetilde{\rho}(|x|)^{-2}L$, we have

$$H(x,z) = G(x,z)\widetilde{\rho}\big(|z|\big)^{2-n}$$

Thus

$$\int_{D\setminus K} G(x,z)\widetilde{\rho}\big(|z|\big)^{(2-n)-2} G(z,y)\widetilde{\rho}\big(|y|\big)^{2-n}\widetilde{\rho}\big(|z|\big)^{n}\,dz \leqslant \varepsilon G(x,y)\widetilde{\rho}\big(|y|\big)^{2-n}$$

for any $x, y \in D \setminus K$. Hence 1 is a small perturbation of L on $\mathbf{R}^n$. □

Remark. A sufficient condition for (7.2) is the following: ρ is a positive differentiable function on $[0,\infty)$ satisfying

$$0 \leqslant \rho'(r)\rho(r)^{-2} \leqslant C, \quad r \geqslant 0, \tag{7.6}$$

for some positive constant C. Indeed, from (7.6) we have

$$X(\delta) \equiv \rho\left(r + \frac{\delta}{\rho(r)}\right)\rho(r)^{-1} \leqslant \exp\big[C\delta X(\delta)\big], \quad r \geqslant 0,\ \delta > 0.$$

Put $\delta = (2Ce)^{-1}$, and let $\gamma \in (1, e)$ be the solution of the equation

$$\exp[X/2e] = X.$$

Then we get $1 \leqslant X(\delta) \leqslant \gamma$. Thus (7.2) holds with $c = \min(\delta, 1/\gamma)$.

Condition (7.3) is sharp, since Theorem 6.2 of [17] yields the following uniqueness theorem.

Theorem 7.2. *Suppose that there exists a positive continuous increasing function ρ on $[0,\infty)$ such that*

$$\big|V(x)\big| \leqslant \rho\big(|x|\big)^2, \quad x \in \mathbf{R}^n. \tag{7.7}$$

Assume that

$$\int_1^\infty \frac{dr}{\rho(r)} = \infty. \tag{7.8}$$

Then [UP] *holds. Thus Fact* AT *holds true.*

7.2. Throughout this subsection we assume that D is a bounded domain of $\mathbf{R}^n$. Let L be an elliptic operator on D of the form

$$L = \frac{1}{w(x)} L_0,$$

where w is a positive measurable function on D such that $w, w^{-1} \in L^\infty_{\mathrm{loc}}(D)$.

Theorem 7.3. *Let D be a Lipschitz domain. Suppose that there exists a positive function ψ on $(0,\infty)$ such that $s^2\psi(s)$ is increasing and*

$$w(x) \leqslant \psi\big(\delta_D(x)\big), \quad x \in D, \tag{7.9}$$

where $\delta_D(x) = \mathrm{dist}(x, \partial D)$. Assume that

$$\int_0^1 s\psi(s)\, ds < \infty. \tag{7.10}$$

Then 1 *is a small perturbation of L on D. Thus Theorem* 1.3 *holds true.*

Remark. (i) The first assertion of this theorem is implicitly shown in [17] (see Theorem 7.11 and Remark 7.12(ii) there).

(ii) The Lipschitz regularity of the domain D is assumed only for the Hardy inequality to hold for any function in $C_0^\infty(D)$. Thus, for this theorem to hold, it suffices to assume (for example) that D is uniformly Δ-regular John domain or a simply connected domain of $\mathbf{R}^2$ (see [3,4]).

Proof of Theorem 7.3. For $x \in D$, put

$$D_x = \left\{ y \in D;\ |x-y| < \frac{\delta_D(x)}{2} \right\}.$$

Then

$$\frac{1}{2}\delta_D(x) \leqslant \delta_D(y) \leqslant \frac{3}{2}\delta_D(x), \quad y \in D_x.$$

Thus

$$\delta_D(x)^2\, w(y) \leqslant 4\delta_D(y)^2 \psi\big(\delta_D(y)\big) \leqslant 4\left(\frac{3}{2}\delta_D(x)\right)^2 \psi\left(\frac{3}{2}\delta_D(x)\right).$$

Put $\Psi(s) = 9s^2\psi((3/2)s)$. Then $\Psi(s)$ is increasing, and satisfies

$$\delta_D(x)^2\Big(\sup_{y\in D_x} w(y)\Big) \leqslant \Psi\big(\delta_D(x)\big), \qquad \int_0^1 \frac{\Psi(s)}{s}\,ds < \infty.$$

Hence, by virtue of Proposition 9.2, Theorem 9.1′ and Corollary 6.1 of [4], w is a small perturbation of L_0 on D. This implies that 1 is a small perturbation of L on D. □

Condition (7.10) is sharp, since Theorem 7.8 and Lemma 7.6 of [17] yield the following uniqueness theorem.

Theorem 7.4. *Suppose that there exists a positive continuous increasing function* ψ *on* $(0,\infty)$ *such that*

$$c\psi\big(\delta_D(x)\big) \leqslant w(x) \leqslant \psi\big(\delta_D(x)\big), \quad x \in D, \tag{7.11}$$

for some positive constant c, and

$$\nu \leqslant \frac{\psi(\eta s)}{\psi(s)} \leqslant \nu^{-1}, \quad s > 0,\ \frac{1}{2} \leqslant \eta \leqslant 2, \tag{7.12}$$

for some positive constant ν. *Assume*

$$\int_0^1 \Big[\psi(s)\Big(\inf_{s\leqslant r\leqslant 1} r^2\psi(r)\Big)\Big]^{1/2} ds = \infty. \tag{7.13}$$

Then [UP] *holds. Thus Fact* AT *holds true.*

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