On the computability of Walsh functions

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Abstract

The Haar and the Walsh functions are proved to be computable with respect to the Fine-metric $d_F$ which is induced from the infinite product $\Omega = \{0, 1\}\{\frac{1}{2}, \ldots\}$ with the weighted product metric $d_C$ of the discrete metric on $\{0, 1\}$. Although they are discontinuous functions on $[0, 1]$ with respect to the Euclidean metric, they are continuous functions on $(\Omega, d_C)$ and on $([0, 1], d_F)$.

On $(\Omega, d_C)$, computable real-valued cylinder functions, which include the Walsh functions, become computable and every computable function can be approximated effectively by a computable sequence of cylinder functions. The metric space $([0, 1], d_F)$ is separable but not complete nor effectively complete. We say that a function on $[0, 1]$ is uniformly Fine-computable if it is sequentially computable and effectively uniformly continuous with respect to the metric $d_F$. It is proved that a uniformly Fine-computable function is essentially a computable function on $\Omega$.

It is also proved that Walsh–Fourier coefficients of a uniformly Fine-computable function $f$ form a computable sequence of reals and there exists a subsequence of the Walsh–Fourier series which Fine-converges effectively uniformly to $f$. © 2002 Elsevier Science B.V. All rights reserved.

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0. Introduction

The definition of computable function proposed by Grzegorczyk and Lacombe is formulated in [7] as follows:

(i) (Sequential computability) If $\{x_n\}$ is a computable sequence of reals then $\{f(x_n)\}$ is also a computable sequence of reals.

(ii) $f$ is effectively uniformly continuous.

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A fundamental property of continuous functions is that they are determined by their values at a dense subset of the domain of definition. For a definition of computable functions, it is natural to require that they are determined effectively by their values at computable reals as well as that they map every computable sequence to a computable sequence of reals. The set of computable reals includes all rationals. Hence, it is a dense subset but it is still a countable set. Let \( A \) be the set of all computable reals in \([0,1]\), and suppose that a function \( f \) is defined on \( A \). To obtain a continuous extension of \( f \), it is necessary and sufficient that \( f \) is uniformly continuous on \( A \). Therefore, it is necessary that computable functions are effectively uniformly continuous with respect to some topology, under which the set of all computable reals becomes a dense subset.

According to the definition of Pour-El and Richards, a discontinuous function cannot be a computable function. Although the Haar function restricted to \([0,1]\), which is defined by

\[
h(x) = \begin{cases} 
1 & \text{if } x < \frac{1}{2}, \\
-1 & \text{if } \frac{1}{2} \leq x < 1, \\
1 & \text{if } x = 1
\end{cases}
\]

is a very simple function both in definition and in calculation, it is discontinuous and cannot be a computable function.

Pour-El and Richards also introduced the concept of a computability structure on a Banach space based on the effective convergence [7]. As an example of this formulation, they defined the intrinsic \( L^p \)-computability (Section 3 in Chapter 2). It is stated in the same section that “For \( L^p \)-functions, pointwise evaluation is not well-defined, since an \( L^p \)-function is determined only almost everywhere.” \( L^p \)-theory is a very powerful tool to deal with quantities which are represented as bounded linear functionals. In general, a bounded linear functional \( T(f) \) on \( L^p([0,1]) \) is represented as \( T(f) = \int_0^1 fg \, dx \) for some function \( g \) in \( L^q([0,1]) \), where \( 1/p + 1/q = 1 \).

However, there are many cases where pointwise evaluation is necessary. For example, to draw a graph of a function, pointwise evaluation is essentially needed. In the theory of Markov processes, pointwise evaluation of a sample path is essentially necessary. A sample path of a Markov process is right continuous and has a left limit at each time.

Besides pointwise evaluation, it is important and necessary in some cases to approximate a computable function effectively by a sequence of simple computable functions both in analysis and in drawing graphs.

On the other hand, Fine introduced a new metric on \([0,1]\) [3]. Let \( \psi(x) \) be the binary expansion of real \( x \) with the convention that we take one with finitely many 1’s for a binary rational \( x \) and \( d_F(x,y) = d_C(\psi(x),\psi(y)) \) for \( x,y \in [0,1] \). It is well known that a function is Fine-continuous if and only if \( f \) is continuous at a binary irrational point and right continuous at a binary rational. If we adopt an expansion with finitely many 0’s for a binary rational, then right continuity is replaced by left continuity.

Mori, Tsujii and Yasugi had proposed a metric space with a computability structure based on effective convergence [6]. Yasugi et al. [9] treated the Gaussian function and made a computability structure of functions to include the Gaussian function.
The computability theory of metric spaces has also been investigated by Weihrauch in the general formulation of Type 2 Effectivity [12]. He proposes many definitions of computability. Recently, Kamo and Brattka have proved independently that the computability of [6] is equivalent to the strong one of Weihrauch.

In this article, we employ the Fine-metric which excludes the convergence to any binary rational from the left-hand side so that the Haar and Walsh functions on the unit interval become uniformly continuous with respect to this metric. We also prove that they are computable with respect to the Fine-metric in the sense of [6].

The basic definitions of computable reals and a metric space with computability structure are summarized in Section 1.

In Section 2, we treat the space \( \Omega = \{0, 1\}^N \) with the Cantor topology, where \( N = \{1, 2, \ldots\} \). In this space, the Walsh function of the first degree

\[
w_1(\sigma) = (-1)^{\sigma_1}, \quad \sigma = (\sigma_1, \sigma_2, \ldots)
\]

is a continuous function of \( \sigma \). The Cantor topology is equivalent to the topology induced by the metric

\[
d_C(\sigma, \tau) = \sum_{k=1}^{\infty} \frac{|\sigma_k - \tau_k|}{2^k}
\]

for \( \sigma, \tau \in \Omega \).

If we define

\[
\mathcal{S}_C = \{\{\sigma_n\} \mid \sigma_n = (z(n, 1), z(n, 2), \ldots) \text{ for some recursive function } z\},
\]

then \( (\Omega, d_C, \mathcal{S}_C) \) becomes an effectively compact metric space with a computability structure.

A function on \( \Omega \) is called a cylinder function if there exists an integer \( n \) such that \( f(\sigma) \) depends only on \((\sigma_1, \ldots, \sigma_n)\) for every \( \sigma \in \Omega \). Every cylinder function which takes only computable values is computable, and every computable function on \( \Omega \) can be approximated effectively by a computable sequence of cylinder functions (Theorem 1).

In Section 3, we first define the Fine-metric and discuss the Fine-continuity. Subsequently, Fine-computable sequences of reals are defined starting with computable sequences of binary rationals using effective Fine-convergence.

In Section 4, we consider the computability of the functions as well as their Fine-continuity. With respect to the metric \( d_F \), the Haar function \( h(x) \) is continuous and there exists a Fine-continuous function which diverges. A function on \( ([0, 1], d_F) \) is said to be uniformly Fine-computable if it satisfies (i) sequential computability and (ii) effective uniform Fine-continuity. It is proved that a function \( f \) is uniformly Fine-computable if and only if there exists a computable function \( g \) on \( (\Omega, d') \) such that \( f(x) = g(\psi(x)) \) (Theorem 2). Finally, we prove that Walsh–Fourier coefficients \( \{c_n\} \) of a uniformly Fine-computable function form a computable sequence of reals and \( S_{2^n}f \), where \( S_nf \) are the partial sums of the Walsh–Fourier series, Fine-converges effectively uniformly to \( f \) (Proposition 4.5).
1. Preliminaries

In this section we summarize definitions in [6, 7, 10] which we will need. We assume separability for a metric space in this article. The set of all real numbers is denoted by $\mathbb{R}$.

Definition 1.1 (Computable sequences of binary rationals and reals in $[0,1]$). (i) A sequence of binary rationals is said to be computable if there exist recursive functions $\alpha(n)$ and $\beta(n)$ such that

$$r_n = \frac{\beta(n)}{2^{\alpha(n)}}.$$

(ii) A double sequence $\{x_{n,m}\}$ of reals is said to converge effectively to a sequence of reals $\{x_n\}$ if there exists a recursive function $\beta$ such that, for all $k$ and $m \geq \alpha(n,k)$,

$$|x_{n,m} - x_n| \leq 1/2^k.$$

(iii) A sequence of reals, say $\{x_n\}$, is said to be computable, if there exists a computable double sequence of binary rationals which converges effectively to $\{x_n\}$.

A real number $x$ is called computable if $\{x,x,\ldots\}$ is a computable sequence.

Definition 1.2 (Effective convergence in metric spaces). Let $\langle X, d \rangle$ be a metric space, $\{x_n\}$ be a sequence from $X$ and $\{x_{n,m}\}$ be a double sequence from $X$.

(i) $\{x_{n,m}\}$ is said to converge effectively to $\{x_n\}$ if there exists a recursive function $\beta$ such that, for all $k$ and $m \geq \beta(n,k)$, $d(x_{n,m},x_n) \leq 1/2^k$.

(ii) $\{x_n\}$ is said to be effectively Cauchy if there exists a recursive function $\alpha$ such that, for all $p$ and $m, n \geq \alpha(p)$, $d(x_m,x_n) \leq 1/2^p$.

Definition 1.3 (Computability structure). $\mathcal{S}$ will be called a computability structure on $\langle X, d \rangle$ if it satisfies the three axioms below. A sequence in $\mathcal{S}$ is called a computable sequence (relative to $\mathcal{S}$).

Axiom M1 (Metrics). If $\{x_n\}, \{y_m\} \in \mathcal{S}$, then $\{d(x_n,y_m)\}_{n,m}$ forms a computable double sequence of reals.

Axiom M2 (Reenumerations). If $\{x_n\} \in \mathcal{S}$, then $\{x_{\alpha(n)}\} \in \mathcal{S}$ for any recursive function $\alpha$.

Axiom M3 (Limits). If $\{x_{n,m}\} \in \mathcal{S}$, $\{x_n\} \subset X$ and $\{x_{n,m}\}$ converges effectively to $\{x_n\}$, then $\{x_n\} \in \mathcal{S}$.

Definition 1.4 (Effective separability). $\langle X, d, \mathcal{S} \rangle$ is said to be effectively separable (with respect to $\mathcal{S}$) if there exists a sequence $\{e_n\}$ in $\mathcal{S}$ which is dense in $X$.

We define $B_d(a,r) = \{x | d(a,x) < r\}$.

Definition 1.5 (Effective compactness). $\langle X, d, \mathcal{S}, \{e_n\} \rangle$ is said to be effectively totally
bounded if there exists a recursive function \( z \) such that 
\[ X = \bigcup_{n=1}^{n(p)} B_d(e_n, 1/2^p) \]
for all \( p \).

We say that \( \langle X, d, \mathcal{S}, \{e_n\} \rangle \) is effectively compact if it is effectively totally bounded and complete.

We define computable functions, computable sequences of functions and effective uniform convergence of functions. We use the term “function” as a mapping from some metric space to \( R \) with the ordinary metric. Therefore, the convergence of the sequence \( \{f(x_n)\} \) is the ordinary convergence as a sequence of reals.

**Definition 1.6** (*Uniformly computable functions*). A function \( f \) is said to be uniformly computable if it satisfies the following conditions.

(i) \( f \) maps \( \mathcal{S} \) into the set of computable sequences of reals.

(ii) \( f \) is effectively uniformly continuous on \( X \), that is, there exists a recursive function \( z(n,k) \) such that, for all \( n \) and for all \( x, y \in X \),
\[
d(x, y) = \frac{1}{2^{n(k)}} \text{ implies } |f(x) - f(y)| \leq 1/2^k.
\]

**Definition 1.7** (*Computable sequence of functions*). A sequence \( \{f_n\} \) of functions from \( X \) to \( R \) is said to be computable if

(i) (Sequential computability) a double sequence \( \{f_n(x_m)\} \) is computable for any \( \{x_m\} \in \mathcal{S} \) and

(ii) (Effective uniform continuity) there exists a recursive function \( z(n,k) \) such that, for all \( n, k \) and all \( x, y \in X \),
\[
d(x, y) \leq \frac{1}{2^{n(k)}} \text{ implies } |f_n(x) - f_n(y)| \leq \frac{1}{2^k}.
\]

**Definition 1.8** (*Effective uniform convergence of functions*). A computable sequence of functions \( \{f_n\} \) is said to converge effectively uniformly to a function \( f \) if there exists a recursive function \( z(n,k) \) such that, for all \( n \) and \( k \),
\[
n \geq z(k) \text{ implies } |f_n(x) - f(x)| \leq \frac{1}{2^k}.
\]

2. Dyadic group

In this section, we first summarize the definition of the dyadic group (see [2–5, 8, 11]).

Let \( \Omega \) be the infinite product space \( \{0, 1\}^N \), where \( N = \{1, 2, 3, \ldots\} \). For an element \( \sigma = (\sigma_\ell) \in \Omega \), we call the \( \ell \)-coordinate \( \sigma_\ell \) the \( \ell \)-bit of \( \sigma \). The distance \( d_C(\sigma, \tau) \) between \( \sigma = (\sigma_\ell) \) and \( \tau = (\tau_\ell) \) in \( \Omega \) is defined by
\[
d_C(\sigma, \tau) = \sum_{\ell=1}^{\infty} \frac{|\sigma_\ell - \tau_\ell|}{2^\ell}.
\]
The “addition” $\sigma \oplus \tau$ is defined to be an element of $\Omega$ whose $\ell$-bit is $|\sigma_\ell - \tau_\ell|$. $(\Omega, \oplus)$ becomes an abelian group and is called the dyadic group. The dyadic group was introduced by Fine [3, 4] to investigate the Walsh functions and the Walsh–Fourier series. He proved that the Walsh functions are characters of this group. Especially, they are continuous functions on this group.

We define the computability structure $\mathcal{S}_C$ on $(\Omega, d_C)$.

**Definition 2.1 (Computability structure $\mathcal{S}_C$).**

$$\mathcal{S}_C = \{ \{ \sigma_n \} \mid \sigma_n = \{ \sigma_{n,\ell} \}, \sigma_{n,\ell} = \pi(n, \ell)$$

for some recursive function $\pi: \mathbb{N} \times \mathbb{N} \rightarrow \{0, 1\}$.

$(\Omega, d_C, \mathcal{S}_C)$ becomes an effectively compact metric space with a computability structure in the sense of [6]. Let $\Omega^0$ be the set of elements of $\Omega$ which have finitely many 1’s. There is an enumeration of $\Omega^0$, say $\{ e_n \}$, which belongs to $\mathcal{S}_C$. This $\{ e_n \}$ is an effective separating set for $(\Omega, d_C, \mathcal{S}_C)$, that is, $\{ e_n \}$ is a dense subset.

From the definition of the metric $d_C$, the following lemma is obtained easily.

**Lemma 2.1.** If $\sigma$ and $\tau$ satisfy $d_C(\sigma, \tau) < 1/2^k$, then for $\ell \leq k \quad \sigma_\ell = \tau_\ell$. On the other hand, if $\sigma_\ell = \tau_\ell$ for $\ell \leq k$, then $d_C(\sigma, \tau) \leq 1/2^k$.

**Definition 2.2 (Cylinder functions).** A function $f$ on $\Omega$ is called a cylinder function if there exists an integer $n$ such that $\sigma_\ell = \tau_\ell$ for $\ell \leq n$ implies $f(\sigma) = f(\tau)$.

If $f$ is a cylinder function then $f$ takes at most $2^n$ values, and we obtain the next proposition.

**Proposition 2.1 (Computability of cylinder functions).** Every cylinder function which takes only computable values is a computable function.

**Proof.** Let $f$ be a cylinder function with $n$ which satisfies the requirement of Definition 2.2. If $d_C(\sigma, \tau) \leq 1/2^{n+1}$, then $\sigma_\ell = \tau_\ell$ for $\ell \leq n$ from Lemma 2.1. This implies $f(\sigma) = f(\tau)$, and $f$ is effectively uniformly continuous.

Let $\{ \sigma_m \}$ be a computable sequence, $f(\sigma_m)$ is determined by the first $n$ bits of $\sigma_m$, therefore $f(\sigma_m)$ is a computable sequence of reals. □

For each pair of integers $(n, j)$, $n > 0$ and $0 \leq j < 2^n$, let

$$\Gamma_{n,j} = \{ \sigma \mid \sum_{\ell=1}^{n} 2^{-\ell} \sigma_\ell = j \}.$$

Then, $f$ is a computable cylinder function if and only if there exist a positive integer $n$ and a finite sequence of computable reals $c_1, \ldots, c_{2^n-1}$ such that $f(\sigma) = c_j$ if $\sigma \in \Gamma_{n,j}$.
**Definition 2.3** (Computable sequence of cylinder functions). A sequence of cylinder functions \( \{f_n\} \) is said to be computable if there exist a recursive function \( b_{VT} \) and a computable double sequence of reals \( \{c_{n,j}\} \) such that

\[
f_n(\sigma) = c_{n,j} \quad \text{if} \quad \sigma \in \Gamma_{\alpha(n),j} \quad \text{and} \quad 0 \leq j < 2^{\alpha(n)}.
\]

**Theorem 1** (Necessary and sufficient condition for computable functions). A function \( f \) is computable if and only if there exists a computable sequence of cylinder functions \( \{f_n\} \) which converges effectively uniformly to \( f \).

**Proof.** The if part is an immediate consequence of the above definitions. For the only if part, we show the construction of \( \{f_n\} \). By the computability of \( f \) there exists a recursive function \( b_{VT} \) such that

\[
|f(\sigma) - f(\tau)| \leq \frac{1}{2^k} \quad \text{if} \quad d_C(\sigma, \tau) \leq \frac{1}{2^{\alpha(k)}}.
\]

For \( 0 \leq j < 2^{\alpha(n)+1} \), we take \((b_{ESC1}, \ldots, b_{ESCn})\) which satisfies \( \sum_{\ell=1}^{\alpha(n)+1} 2^{\alpha(n)+1-\ell} \ell = j \), and \( c_{n,j} \) is defined to be \( f((b_{ESC1}, \ldots, b_{ESCn+1}, 0, 0, \ldots)) \).

**Example 2.1.** The Walsh functions \( \{w_n(\sigma)\} \) on \( \Omega \) are defined by the next equation.

\[
w_n(\sigma) = (-1)^{\sum_{i=0}^{k} \sigma_{i+1} n_i}
\]

where \( n = n_0 + 2n_1 + \cdots + 2^kn_k \) \( (n_k \neq 0) \) is the binary representation of \( n \). From the definition, \( w_n(\sigma) \) is determined by the first \( k+1 \) bits of \( \sigma \). For example, \( w_0(\sigma) \equiv 1 \), \( w_1(\sigma) = (-1)^{\sigma_1} \), \( w_2(\sigma) = (-1)^{\sigma_2} \), \( w_3(\sigma) = (-1)^{\sigma_1+\sigma_2} = w_1(\sigma)w_2(\sigma) \). It is an easy consequence of the definition and Eq. (1) that the Walsh functions form a computable sequence of cylinder functions.

**Example 2.2.**

\[
g_n(\sigma) = \begin{cases} 
\sum_{\ell=1}^{\infty} \frac{\sigma_{\ell}}{2^{\ell}} - \left(1 - \frac{1}{2^{k-1}}\right) & \text{if} \quad \exists k < n \text{ s.t. } \sigma_k = 0 \\
& \quad \text{and } (\sigma_1, \ldots, \sigma_n) = (1, \ldots, 1, 0, *, \ldots, *), \\
0 & \quad \text{if } (\sigma_1, \ldots, \sigma_n) = (1, \ldots, 1) 
\end{cases}
\]

is a computable sequence of cylinder functions and converges effectively uniformly to

\[
g(\sigma) = \begin{cases} 
\sum_{\ell=1}^{\infty} \frac{\sigma_{\ell}}{2^{\ell}} - \left(1 - \frac{1}{2^{k-1}}\right) & \text{if} \quad \exists k \text{ s.t. } \sigma = (1, \ldots, 1, 0, *, *, \ldots), \\
0 & \quad \text{if } \sigma = (1, 1, \ldots).
\end{cases}
\]

From Theorem 1, \( g \) is a computable function on \( (\Omega, d) \).
3. Fine-metric and Fine-computability structure on the unit interval

We define the mapping \( \varphi \) from \( bLF \) to \([0, 1]\) by

\[
\varphi(\sigma) = \sum_{\ell=1}^{\infty} \sigma_{\ell} 2^{-\ell}.
\]

It is obvious that \( \varphi \) satisfies

\[
|\varphi(\sigma) - \varphi(\tau)| \leq d_C(\sigma, \tau) = \varphi(\sigma \oplus \tau),
\]

so \( \varphi \) is a continuous function on \((bLF, d)\).

Let \( bLF_0 \) be the set of all elements in \( bLF \) with finitely many 1's, \( bLF_1 \) be the set of all elements in \( bLF \) with finitely many 0's and \( Q_2 \) be the set of all binary rationals in \([0, 1]\).

Then the restriction of \( \varphi \) to \( bLF_0 \) \((bLF_1)\) is a one-to-one correspondence between \( bLF_0 \) \((bLF_1)\) and \( Q_2 \setminus \{1\} \) \((Q_2 \setminus \{0\})\) and the same holds for \( Q_2 \setminus (bLF_0 \cup bLF_1) \) and \([0, 1] \setminus Q_2\).

We define the mapping \( \psi : [0, 1] \to \Omega \) as follows:

\[
\psi(x) = \begin{cases} 
\sigma & \text{if } x \in Q_2, \sigma \in \Omega^0 \text{ and } \varphi(\sigma) = x, \\
\sigma & \text{if } x \notin Q_2 \text{ and } \varphi(\sigma) = x.
\end{cases}
\]

\( \psi(x) \) satisfies \( \varphi(\psi(x)) = x \) for all \( x \in [0, 1] \).

Now, we define the metric \( d_F \) on \([0, 1]\) as the induced metric on \([0, 1]\) by \( \psi \).

**Definition 3.1** (Fine-metric). For \( x, y \in [0, 1] \) the Fine-metric \( d_F(x, y) \) is defined by

\[
d_F(x, y) = \begin{cases} 
d_C(\psi(x), \psi(y)) & \text{if } x, y < 1, \\
2 & \text{if } x < 1 = y \text{ or } y < 1 = x, \\
0 & \text{if } x = y = 1.
\end{cases}
\]

The right-end point 1 is an isolated point under metric \( d_F \). \( d_F \) does not preserve the ordinary relations between the usual metric and order relation.

For a sequence of reals \( \{x_n\} \) and a real \( x \), we say that \( x_n \) Fine-converges to \( x \) or effectively Fine-converges to \( x \) if \( x_n \) converges to \( x \) or \( x_n \) converges effectively to \( x \), respectively, under the metric \( d_F \). In the same way, we use Fine–Cauchy or effective Fine–Cauchy.

**Definition 3.2** (Conjugate of \( \sigma \in \Omega^0 \)). For an element \( \sigma \in \Omega^0 \) we define its conjugate \( \sigma^* \) to be the element \( \tau \in \Omega^1 \) such that \( \varphi(\tau) = \varphi(\sigma) \).

**Example 3.1.** Let \( x = \frac{1}{2} \), then \( \psi(x) = (1, 0, 0, \ldots) \), \( \psi(x)^* = (0, 1, 1, \ldots) \) and \( d(\psi(x), \psi(x)^*) = 1 \).

From the definition of the metric \( d_F \) and inequality (3), it follows that

\[
d_F(x, y) = d_C(\psi(x), \psi(y)) \geq |x - y|.
\]
The following lemma is an immediate consequence of this inequality (4).

**Lemma 3.1.** If a sequence of real numbers \( \{x_n\} \) Fine-converges to a real number \( x \), then \( \{x_n\} \) converges to \( x \).

**Example 3.2.** Let \( x = \frac{1}{2} \) and \( x_n = \frac{1}{2} - 1/2^n \), then \( x_n \) converges effectively to \( x \). If \( x_n \) Fine-converges to \( y \), then \( x_n \) converges to \( y \) by Lemma 3.1, and \( y = x \). From the definition of \( \psi \), conjugate and the metric \( d_F \),

\[
\psi(x) = (1, 0, 0, \ldots), \quad \psi(x_n) = (0, 1, \ldots, 1, 0, 0, \ldots) \text{ and } d_F(x_n, x) = \sum_{k=1}^{n} \frac{1}{2^k} \to 1.
\]

Therefore, \( x_n \) does not Fine-converge to \( x \).

On the other hand, \( \{x_n\} \) forms an effective Fine–Cauchy sequence, so the metric space \( ([0,1], d_F) \) is not complete. In this case, \( \psi(x_n) \) converges to \( \psi(x)^* \) in \( \Omega \). \( (\Omega, d_C) \) can be regarded as a completion of the metric space \( ([0,1], d_F) \).

**Example 3.3.** Let \( x = \frac{1}{2} \) and \( x_n = \frac{1}{2} + (-1)^n/2^n \), then \( x_n \) converges effectively to \( x \) and forms an effective Cauchy sequence. On the other hand, \( \psi(x_{2k}) \) converges to \( \psi(x) \) and \( \psi(x_{2k-1}) \) converges to \( \psi(x)^* \), so \( \psi(x_n) \) does not Fine-converge, and \( \{x_n\} \) is not a Fine–Cauchy sequence.

Let \( r \in \mathbb{Q}_2 \), \( \psi(r) = (\sigma_1, \ldots, \sigma_k, 0, 0, \ldots) \) and \( d_F(r, x) < 1/2^k \). Then the first \( k \) bits of \( \psi(x) \) coincide with that of \( \psi(r) \) by Lemma 2.1, so \( \psi(x) = (\sigma_1, \ldots, \sigma_k, *, *, \ldots) \) and \( r \preceq x \). This means that Fine-convergence excludes the convergence to a binary rational from the left-hand side.

For the relation between Fine-convergence and ordinary convergence with respect to the Euclidean metric, the following lemma is essentially well known [2, 8].

**Lemma 3.2.** Let \( \{x_n\} \) be a sequence of real numbers which converges to \( x \).

(i) If \( x \) is not a binary rational, then \( x_n \) Fine-converges to \( x \).

(ii) If \( x \) is a binary rational and \( x_n \geq x \) for all \( n \), then \( x_n \) Fine-converges to \( x \).

(iii) If \( x \) is a binary rational and \( x_n < x \) for all \( n \), then \( \psi(x_n) \) converges to \( \psi(x)^* \) in \( (\Omega, d_C) \).

We define a computability structure in the metric space \( ([0,1], d_F) \) in the same way as the ordinary computability structures, starting from computable double sequences of binary rationals using effective Fine-convergence instead of using effective convergence.

**Definition 3.3** (Computable double sequences of binary rationals). A double sequence of binary rationals \( \{r_{n,m}\} \) is said to be computable if there exist recursive functions
\(z(n,m)\) and \(\beta(n,m)\) which satisfy
\[r_{n,m} = \frac{\beta(n,m)}{2^z(n,m)}, \quad 0 \leq \beta(n,m) \leq 2^z(n,m).\]

**Definition 3.4 (Fine-computable sequences).** A sequence of reals \(\{x_n\}\) is said to be Fine-computable if there exists a computable double sequence of binary rationals \(\{r_{n,m}\}\) which Fine-converges effectively to \(\{x_n\}\), that is, there exists a recursive function \(\gamma(n,k)\) such that
\[d_F(r_{n,m}, x_n) \leq \frac{1}{2^k} \text{ for } m \geq \gamma(n,k).\]

**Proposition 3.1.** If a sequence \(\{x_n\}\) of reals does not contain 1, then \(\{x_n\}\) is Fine-computable if and only if \(\{\psi(x_n)\}\) is computable in \((\Omega, d_C)\).

A real number \(x < 1\) is Fine-computable if and only if its binary representation, with the convention that we take one with finitely many 1’s for a binary rational, is determined recursively.

Let \(\mathcal{S}_F\) be the set of all Fine-computable sequences, then \(\mathcal{S}_F\) is a computability structure, that is, \(\mathcal{S}_F\) satisfies 3 axioms in Definition 1.3 with respect to the metric \(d_F\). Every Fine-computable sequence is a computable sequence, since the Fine-convergence implies the ordinary convergence. As a special case, every Fine-computable number is a computable number.

**Proposition 3.2.** If \(x\) is computable, then \(x\) is also Fine-computable.

**Proof.** Let \(\{r_k\}\) be a computable sequence of binary rationals which converges effectively to \(x\). By Definition 3.3, there exist recursive functions \(z(k)\) and \(\beta(k)\) such that \(r_k = \beta(k)/2^z(k)\) converges effectively to \(x\). We can assume without loss of generality that \(r_k > x\) for all \(k\).

If we denote \(\psi(x) = \sigma = (\sigma_\ell)\) and \(\psi(r_k) = \sigma_k = (\sigma_{k,\ell})\), then \(\sigma_{k,\ell} = 0\) for \(\forall \ell > z(k)\).

Let \(m(k) = \max\{\ell \mid \sigma_{k,\ell} = 1\}\), then \(m(k)\) is a recursive function, since \(m(k)\) is determined by \((\sigma_{k,1}, \ldots, \sigma_{k,z(k)})\).

To prove the Fine-computability of \(x\), it is sufficient to construct a sequence \(\{\tau_n, \ell\} \subset \Omega^0\) such that \(\tau_{n,\ell} = 0\) for \(\ell > n + 3\) and \(d_C(\tau_n, \sigma) < 1/2^n\).

From effective convergence of \(\{r_k\}\) to \(x\), there exists a recursive function \(\gamma(n)\) such that
\[0 < r_k - x < \frac{1}{2^n} \text{ for } k > \gamma(n).\]

Therefore, if we define \(k = \gamma(n + 3) + 1\), then \(0 < r_k - x < 1/2^{n+3}\).

**Case (i) (n + 3 \geq m(k)).** There exists \(i\) such that \(r_k = i/2^{n+3}\) and \((i-1)/2^{n+3} < x < i/2^{n+3}\) holds. This implies that the first \(n + 3\) bits of \(\sigma\) and those of \(\sigma^*_k\) coincide, and accordingly \(d_C(\sigma, \sigma^*_k) < 1/2^{n+2}\).
If we take 
\[ \tau_n = (\sigma_{k,1}, \ldots, \sigma_{k,m(k)-1}, 0, 1, \ldots, 1, 0, 0, \ldots), \]
then 
\[ d(\sigma_k^*, \tau_n) < \frac{1}{2^{n+2}} \]
and 
\[ d(\sigma, \tau_n) < \frac{1}{2^{n+1}}. \]

Case (ii) \((n + 3 < m(k))\). From
\[ 0 < r_k - x = \sum_{\ell=1}^{m(k)} \frac{\sigma_{k,\ell}}{2^\ell} - \sum_{\ell=1}^{\infty} \frac{\sigma_\ell}{2^\ell} = \sum_{\ell=1}^{m(k)} \frac{\sigma_{k,\ell} - \sigma_\ell}{2^\ell} - \sum_{\ell=m(k)+1}^{\infty} \frac{\sigma_\ell}{2^\ell} < \frac{1}{2^{n+3}}, \]
it holds that
\[ 0 < \sum_{\ell=1}^{m(k)} \frac{\sigma_{k,\ell} - \sigma_\ell}{2^\ell} < \frac{2}{2^{n+3}} = \frac{1}{2^{n+2}}. \]  
(5)

From the left inequality, there exists \(j(k) \leq m(k)\) such that \(\sigma_{k,j(k)} = 1\), \(\sigma_{j(k)} = 0\) and \(\sigma_{k,i} = \sigma_i\) for \(i < j(k)\).

Case (iia) \((j(k) \geq n + 3)\). Take \(\tau_n = (\sigma_1, \ldots, \sigma_{n+3}, 0, 0, \ldots)\), then 
\[ d(\tau_n, \sigma) < 1/2^{n+2}. \]

Case (iib) \((j(k) < n + 3)\). \(\sigma\) and \(\sigma_k^*\) are as follows.
\[
\begin{align*}
\sigma_k &= (\sigma_1 \cdots \sigma_{j-1} 1 \ast \ast \cdots), \\
\sigma &= (\sigma_1 \cdots \sigma_{j-1} 0 \ast \ast \cdots).
\end{align*}
\]

From the right inequality of (5), it holds that \(\sigma_{k,\ell} = 0\) for \(j + 1 \leq \ell \leq n + 1\) and \(\sigma_\ell = 1\) for \(j + 1 \leq \ell \leq n + 1\). Therefore, if we take
\[
\tau_n = (\sigma_1, \ldots, \sigma_{j-1}, 0, 1, \ldots, 1, 0, \ldots),
\]
then 
\[ d(\sigma, \tau_n) < 1/2^n \] holds.

The space \(\Omega = \{0,1\}^{\{1,2,\ldots\}}\) is essentially the same as \(\{0,1\}^\omega\). If we denote by \(\rho_F\) the inverse of \(\psi\), then \(\rho_F\) maps \(\Omega \setminus \Omega^1\) to \([0,1)\). Therefore, \(\rho_F\) is a representation of \([0,1)\) and Fine-computability is identical to the \(\rho_F\) computability in the sense of Weihrauch [13].

As to the relation between Fine-computability and \(\rho_2\) computability, where \(\rho_2\) is the binary representation, Brattka has proved the following proposition.

**Proposition 3.3** (Brattka [1]). The representations \(\rho_F\) and \(\rho_2\) are related as follows:

(i) \(\rho_F\) is reducible to \(\rho_2\), but \(\rho_2\) is not reducible to \(\rho_F\),
(ii) a real number \(x \in [0,1]\) is \(\rho_F\)-computable, if and only if it is \(\rho_2\)-computable, if and only if it is computable,
(iii) each \(\rho_F\)-computable sequence is \(\rho_2\)-computable, but there exists a \(\rho_2\)-computable sequence which is not \(\rho_F\)-computable.
4. Fine-continuous functions, the class $\mathcal{D}$ and uniformly Fine-computable functions

A function $f$ is called Fine-continuous if it is a continuous function with respect to the metric $d_F$. For the Fine-continuity, the following proposition is well known [2, 8].

**Proposition 4.1** (Necessary and sufficient condition for Fine-continuity). A function $f$ on $[0, 1]$ is Fine-continuous if and only if it satisfies the following two conditions.

(i) $f$ is continuous at $x \not\in Q_2$.
(ii) $f$ is right continuous at $x \in Q_2$.

Let $g$ be a continuous function on $(\Omega, d_C)$. Then any function $f$ on $[0, 1]$, which satisfies $f(x) = g(\psi(x))$ for $x \in [0, 1)$, is a Fine-continuous function. We define a subclass $\mathcal{D}$ of the set of all Fine-continuous functions.

**Definition 4.1** (Class $\mathcal{D}$).

$$\mathcal{D} = \{ f \mid \exists g: \text{continuous on } (\Omega, d) \text{ such that } f(x) = g(\psi(x)) \text{ for } x \in [0, 1) \}. $$

For the characterization of the class $\mathcal{D}$, the following proposition is well known [2, 8].

**Proposition 4.2** (Characterization of $\mathcal{D}$). Let $f$ be a Fine-continuous function on $[0, 1]$. Then $f$ belongs to $\mathcal{D}$ if and only if it has a left limit at every $x \in Q_2$.

We call a uniformly continuous function with respect to the metric $d_F$ a uniformly Fine-continuous function.

**Proposition 4.3** (Another characterization of $\mathcal{D}$). A function $f$ on $[0, 1]$ belongs to $\mathcal{D}$ if and only if $f$ is a uniformly Fine-continuous function.

**Proof.** The proof of the only if part is obvious since $(\Omega, d_C)$ is compact. The proof of the if part consists of the construction of a continuous function $g$ on $\Omega$, which satisfies $f(x) = g(\psi(x))$ for $x \in [0, 1)$. Let $f$ be a uniformly Fine-continuous function on $[0, 1]$. For $\sigma \in \Omega^1$ we define $g(\sigma)$ to be $f(\psi(\sigma))$. By the uniform Fine-continuity of $f$, $g$ is uniformly continuous on $\Omega \setminus \Omega^1$. Since $\Omega \setminus \Omega^1$ is a dense subset of $\Omega$, $g$ can be extended to a continuous function on $\Omega$. $\square$

**Remark 4.1.** For $\sigma^* \in \Omega^1$, $g(\sigma^*)$ is obtained as the limit of $\{g(\sigma_n)\}$ for an arbitrary sequence $\{\sigma_n\}$ which converges to $\sigma^*$. For example, if $\sigma^* = (\sigma_1, \ldots, \sigma_k, 0, 1, 1, \ldots)$, then we can take $\sigma_n$ to be $(\sigma_1, \ldots, \sigma_k, 0, 1, 1, \ldots, 0, 0, \ldots)$.

We have defined the computability structure $\mathcal{S}_F$ on the metric space $([0, 1], d_F)$. As stated in Section 3, this space is separable but not complete. In this section, we define uniformly Fine-computable functions.
Definition 4.2 (Uniformly Fine-computable functions). A function $f$ on $[0, 1]$ is said to be uniformly Fine-computable if it satisfies the following conditions:

(i) (Sequential computability) if $\{x_n\}$ is a Fine-computable sequence then $\{f(x_n)\}$ is a computable sequence of reals and

(ii) (Effective uniform Fine-continuity) there exists a recursive function $z$ such that

$$|f(x) - f(y)| \leq \frac{1}{2^n} \text{ if } d_F(x, y) \leq \frac{1}{2^{n+1}}.$$

Remark 4.2. Since $1$ is an isolated point with respect to the Fine-metric, the above definition requires only that $f(1)$ is computable for the point $1$.

Theorem 2 (Necessary and sufficient condition for uniformly Fine-computable function). A function $f$ on $[0, 1]$ is a uniformly Fine-computable function if and only if there exists a computable function $g$ on $(\Omega, d_F)$ such that $f(x) = g(\psi(x))$ for all $x \in [0, 1]$ and $f(1)$ is a computable real.

Proof. The proof of the if part is obvious. To prove the only if part, let $f$ satisfy conditions (i) and (ii), then $f$ is a uniformly Fine-continuous function. It is sufficient to prove that the function $g$ defined in the proof of Proposition 4.3 is computable on $(\Omega, d)$.

Effective uniform Fine-continuity: If $\sigma, \tau \in \Omega \setminus \Omega^1$, then $d_F(\sigma, \tau) = d_F(\phi(\sigma), \phi(\tau))$.

From (ii), there exists a recursive function $z(n)$ such that

$$d(\sigma, \tau) \leq \frac{1}{2^{z(n)}} \implies |g(\sigma) - g(\tau)| \leq \frac{1}{2^n}.$$ 

We can assume that $z(n)$ is strictly increasing.

Let $\sigma^*, \tau^* \in \Omega^1$ and $\{\sigma_n\}, \{\sigma_m\}$ be the approximating sequence as in Remark 4.1.

Assume that $d_F(\sigma^*, \tau^*) \leq 1/2^{z(n+1)}$, then the first $z(n)$ bits of $\sigma^*$ coincide with those of $\tau^*$ by Lemma 2.1. If $m > z(n+1)$, the first $z(n)$ bits of $\sigma^*$ coincide with those of $\sigma_m$. The same holds for $\tau^*$ and $\tau_m$. These imply that

$$d_F(\sigma_m, \tau_m) \leq \frac{1}{2^{z(n)}} \text{ and } |g(\sigma_m) - g(\tau_m)| \leq \frac{1}{2^n}.$$ 

If we take $m$ to $\infty$, we obtain

$$|g(\sigma^*) - g(\tau^*)| \leq \frac{1}{2^n}.$$ 

Sequential computability: Let $\{\sigma_n\}$ be a computable sequence. Then, there exists a computable double sequence $\{\tau_{n,k}\}$ of elements in $\Omega^0$, which converges effectively to $\{\sigma_n\}$. $\phi(\{\tau_{n,k}\})$ is a computable sequence of binary rationals and $\{g(\tau_{n,k})\} = \{f(\phi(\tau_{n,k}))\}$ Fine-converges effectively to $\{g(\sigma_n)\}$. □
On \(\langle \Omega, d_c, \mathcal{S}_c \rangle\), we have defined cylinder functions as an elementary class of functions and proved that every computable function is an effective limit of a computable sequence of cylinder functions (Definition 2.3 and Theorem 1). On \(\langle [0, 1], d_F, \mathcal{S}_F \rangle\), we define binary step functions. Let \(A_{n,j} = [j/2^n, (j+1)/2^n)\) for \(0 \leq j < 2^n\) and we call such an interval a dyadic interval.

**Definition 4.3 (Binary step functions).** A function \(f\) on \([0, 1]\) is called a binary step function if there exist dyadic intervals \(I_1, \ldots, I_k\) which are mutually disjoint, their union is \([0, 1)\) and \(f\) is constant on each interval \(I_i\).

As in the case of cylinder functions on \(\Omega\), every binary step function which takes only computable real values is uniformly Fine-computable. It is obvious that a binary step function \(f\) on \([0, 1]\) is computable if and only if there exists a cylinder function \(g\) on \(\Omega\) such that \(f(x) = g(\psi(x))\) for \(x \in [0, 1)\) and \(f(1)\) is a computable real.

**Definition 4.4 (Fine-computable sequences of binary step functions).** A sequence of functions \(\{f_n\}\) on \([0, 1]\) is said to be a Fine-computable sequence of binary step functions if \(\{f_n(1)\}\) is a computable sequence of reals and there exists a recursive function \(a\) and a computable double sequence of reals \(\{s_{n,j}\}\) such that

\[
f_n(x) = s_{n,j} \quad \text{if} \quad x \in A_{a(n),j} \quad \text{and} \quad 0 \leq j < 2^{a(n)}
\]

\(\{f_n\}\) is a Fine-computable sequence of binary step functions if and only if there exists a computable sequence of cylinder functions \(g_n\) on \(\Omega\) such that \(f_n(x) = g_n(\psi(x))\) for \(x \in [0, 1)\) and \(\{f_n(1)\}\) is a computable sequence of reals.

The next theorem is a restatement of Theorem 1.

**Theorem 3 (Necessary and sufficient condition for uniformly Fine-computable function).** A function \(f\) on \([0, 1]\) is uniformly Fine-computable if and only if there exists a Fine-computable sequence of binary step functions which converges effectively uniformly to \(f\).

Pour-El and Richards introduced the concept of intrinsic \(L^p\)-computability.

**Definition 4.5 (Intrinsic \(L^p\)-computability).** A function \(f \in L^p[0, 1]\) is \(L^p\)-computable if there exists a sequence \(\{g_k\}\) of continuous functions which is computable (in the sense of Chapter 0 in [7]) and such that the \(L^p\)-norm \(\|g_k - f\|_p\) converges to zero effectively.

Let \(a, b\) be binary rationals and \(\chi\) be the indicator function of the interval \([a, b)\). If we define

\[
f_n(x) = \begin{cases} 
  nx - na + 1 & \text{if} \quad a - \frac{1}{n} < x < a, \\
  -nx + nb & \text{if} \quad b - \frac{1}{n} < x < b, \\
  \chi(x) & \text{otherwise},
\end{cases}
\]
then \( \{ f_n \} \) is a computable sequence of functions in the sense of [7] and \( \| f_n - \chi \|_p \) converges to zero effectively. Since a Fine-computable binary step function is a finite linear combination with computable coefficients of such \( \chi \)'s, we obtain the following proposition.

**Proposition 4.4.** A uniformly Fine-computable function is also an \( L^p \)-computable function.

**Example 4.1.** Let \( g \) be the function defined in Example 2.2. Then,

\[
 f(x) = g(\psi(x)) = \begin{cases} 
 x - (1 - \frac{1}{2^{n-1}}) & \text{if } 1 - \frac{1}{2^{n-1}} \leq x < 1 - \frac{1}{2^n}, \\
 0 & \text{if } x = 1
\end{cases}
\]

is uniformly Fine-computable.

**Example 4.2.** We have defined the Walsh functions \( w_n(x) \) on \( \Omega \) in Example 2.1. The Walsh functions on \([0, 1]\) are defined using \( w_n(\sigma) \), as follows:

\[
 W_n(x) = \begin{cases} 
 w_n(\psi(x)) & \text{if } x \in [0, 1), \\
 1 & \text{if } x = 1.
\end{cases}
\]

\( \{ W_n(x) \} \) forms a Fine-computable sequence of binary step functions.

**Example 4.3.** Let \( f(x) \) be

\[
 f(x) = \begin{cases} 
 1 - \frac{1}{2^n} & \text{if } x < \frac{1}{2^n}, \\
 0 & \text{if } x \geq \frac{1}{2^n}.
\end{cases}
\]

Then, \( f \) is a Fine-continuous function but it diverges from the left-hand side at \( \frac{1}{2^n} \). Therefore, \( f \) does not belong to the class \( \mathcal{D} \) and it is not a uniformly Fine-computable function.

**Example 4.4.** Let \( f(x) \) be expressed as

\[
 f(x) = \begin{cases} 
 2^n x & \text{if } 1 - \frac{1}{2^{n-1}} \leq x < 1 - \frac{1}{2^n}, \\
 0 & \text{if } x = 1.
\end{cases}
\]

Then, \( f \) is also a Fine-continuous function. But it is not uniformly Fine-continuous and it does not belong to the class \( \mathcal{D} \).

The functions in Examples 4.3 and 4.4 are not uniformly Fine-computable functions as stated above, but they are still simple functions both in definition and in calculation. In order to obtain a definition of the computability including the functions which do not belong to the class \( \mathcal{D} \), it is necessary to replace the effective uniform continuity by a weaker effective continuity, since uniform continuity implies boundedness.
Definition 4.6 (Locally uniformly computable sequences of functions). Let \( \{e_i\} \) be an effectively separating set. A sequence of functions \( \{f_n\} \) is said to be locally uniformly Fine-computable if

(i) \( f_n \) is sequentially computable and

(ii) there exist a computable sequence of reals \( \{r_{n,i}\} \) and a recursive function \( z(n,i,k) \) such that

\[
\bigcup_{i=1}^{\infty} B_F(e_i, r_{n,i}) = X \quad \text{for each } n
\]

\[
|f_n(x) - f_n(y)| \leq \frac{1}{2^k} \quad \text{for } x, y \in B_F(e_i, r_{n,i}) \quad \text{and} \quad d_F(x, y) \leq \frac{1}{e^{2^k(n,i,k)}},
\]

where, \( B_F(e, r) = \{x \mid d_F(x, e) < r \} \).

This together with the computability structure and computable functions on the dyadic field, which was introduced by Fine [4], will be discussed in a forthcoming paper.

Recently, Brattka [1] has proved that \( f(x) \) in Example 4.3 or 4.4 is \( (\rho_F, \rho_E) \) computable in the sense of Weihrauch, where \( \rho_E \) is some admissible standard representation of the real numbers.

Finally, we treat the computability of the Walsh–Fourier coefficients and the effective uniform convergence of the partial sums for the uniformly Fine-computable functions. The Walsh–Fourier coefficients \( c_m \) are defined by

\[
c_m = \int_0^1 w_m(x) f(x) \, dx
\]

and the partial sum of the Walsh–Fourier series is defined by

\[
S_n f(x) = \sum_{m=0}^{n} c_m w_m(x).
\]

The following lemma is well known (cf. [3, 5]).

Lemma 4.1. If \( f \) is integrable and Fine-continuous then

\[
S_{2^n} f(x) - f(x) = 2^n \int_0^{2^{-n}} (f(x \oplus t) - f(x)) \, dt,
\]

holds, where \( x \oplus t = \varphi(\psi(x) \oplus \psi(t)) \) and \( dt \) is the Lebesgue measure on \([0,1]\).

Proposition 4.5 (Convergence of \( S_{2^n} f \) to \( f \) for uniformly Fine-computable functions). If \( f \) is a uniformly Fine-computable function then it holds that

(i) the Walsh–Fourier coefficients \( \{c_m\} \) form a computable sequence of reals and \( \{S_n f\} \) is a Fine-computable sequence of binary step functions and

(ii) \( \{S_{2^n} f\} \) Fish-converges effectively uniformly to \( f \).
Proof. Suppose that \( f \) is a uniformly Fine-computable function. Then there exist a recursive function \( \alpha(n) \) and computable double sequence \( \{ s_{n,j} \} \) such that \( \{ f_n \} \), which is defined by Eq. (5), converges effectively to \( f \). It is obvious that a binary step function is integrable. From the uniform convergence of \( \{ f_n \} \) to \( f \), \( f \) is also integrable.

To prove the computability of \( \{ c_m \} \), let

\[
\begin{align*}
c_{m,n} &= \int_0^1 w_m(x) f_n(x) \, dx.
\end{align*}
\]

If \( 2^k \leq m < 2^{k+1} \), then \( w_m(x) \) is constant on each interval \([i/2^k, (i+1)/2^k)\). \( f_n(x) \) is also constant on each interval \([j/2^k, (j+1)/2^n]\) if \( k \geq \alpha(n) \). Therefore,

\[
\begin{align*}
c_{m,n} &= \begin{cases} 
\frac{1}{2^n} \sum_{i=0}^{2^k-1} w_m \left( \frac{i}{2^k} \right) \left( \sum_{j=0}^{2^n-1} f_n \left( \frac{i}{2^k} + \frac{j}{2^n} \right) \right) & \text{if } k \leq \alpha(n), \\
\frac{1}{2^n} \sum_{j=0}^{2^n-1} f_n \left( \frac{j}{2^n} \right) \left( \sum_{i=0}^{2^k-1} w_m \left( \frac{j}{2^n} + \frac{i}{2^k} \right) \right) & \text{if } k > \alpha(n)
\end{cases}
\end{align*}
\]

and \( \{ c_{m,n} \} \) is a computable double sequence of reals. From the effective uniform convergence of \( \{ f_n \} \) to \( f \) and the inequality

\[
|c_{m,n} - c_m| = \left| \int_0^1 w_m(x) (f_n(x) - f(x)) \, dx \right| \leq \int_0^1 |f_n(x) - f(x)| \, dx,
\]

it follows that \( \{ c_{m,n} \} \) converges effectively to \( \{ c_m \} \), and \( \{ c_m \} \) is a computable sequence of reals.

The effective uniform Fine-convergence of \( \{ S_2^\alpha f \} \) to \( f \) follows from Eq. (6), the effective Fine-continuity of \( f \) and the inequality \( d_F(x \oplus t, x) \leq t \) if \( t > 0 \).

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