Note

Polynomials and packings: a new proof of de Bruijn's theorem

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Abstract

In 1969 de Bruijn published a proof of the following fact: An \( a \times ab \times abc \) brick can be used to pack an \( A \times B \times C \) box if, and only if, the integers \( A, B, C \) are in some order a multiple of \( a \), a multiple of \( ab \), and a multiple of \( abc \). We give a quick proof of this result based on the following elementary lemma. The polynomial \( (x^a - 1)(x^{ab} - 1)(x^{abc} - 1) \) divides \( (x^A - 1)(x^B - 1)(x^C - 1) \) if, and only if, the integers \( A, B, C \) are in some order a multiple of \( a \), a multiple of \( ab \), and a multiple of \( abc \).

The following two results, in which \( a, b, c, A, B, C \) denote positive integers.

**Theorem 1** (de Bruijn [1]). An \( A \times B \times C \) box can be filled with \( a \times ab \times abc \) bricks if, and only if, the numbers \( A, B, C \) are (in some order) a multiple of \( a \), a multiple of \( ab \), and a multiple of \( abc \).

**Lemma 2.** The polynomial \( (x^a - 1)(x^b - 1)(x^c - 1) \) is divisible by \( (x^A - 1)(x^B - 1)(x^C - 1) \) if, and only if, the numbers \( A, B, C \) are (in some order) a multiple of \( a \), a multiple of \( ab \), and a multiple of \( abc \).

In this note we give a short proof of de Bruijn's theorem using Lemma 2. We also include a brief discussion of the history of the idea of using polynomials in packing problems.

**Proof of Lemma 2.** If \( a \) divides \( A \), \( ab \) divides \( B \), and \( abc \) divides \( C \), then clearly \( (x^a - 1)(x^{ab} - 1)(x^{abc} - 1) \) divides \( (x^A - 1)(x^B - 1)(x^C - 1) \). Assume now that

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(x^a - 1)(x^b - 1)(x^{abc} - 1) divides F(x) = (x^A - 1)(x^B - 1)(x^C - 1). We must show that abc divides at least one of A, B, C, that ab divides at least two of A, B, C, and that a divides all of A, B, C.

Let \( \rho_a \) denote a primitive \( a \)th root of unity. Then \( \rho_a \) is a zero of \( F(x) \) of multiplicity three. Since the polynomials \( (x^a - 1), (x^b - 1), (x^{abc} - 1) \) each have distinct roots, it follows that \( \rho_a \) is a root of all three, and hence that \( a \) divides each of \( A, B, C \). Similar arguments using the primitive roots \( \rho_{ab} \) and \( \rho_{abc} \) of unity show that \( ab \) (respectively \( abc \)) must divide at least two (one) of \( A, B, C \).

Proof of Theorem 1. If \( a \) divides \( A \), \( ab \) divides \( B \), and \( abc \) divides \( C \), then clearly the box can be filled with bricks. Assume now that the \( A \times B \times C \) box can be filled with \( a \times ab \times abc \) bricks. Consider the box as consisting of \( ABC \) unit cubes, which we will call 'cells'. Give each cell three coordinates \( (i, j, k) \) with \( 0 \leq i < A \), \( 0 \leq j < B \), \( 0 \leq k < C \). Let \( x \) be an indeterminate, and to each cell \( (i, j, k) \) assign the 'weight' \( x^i + j + k \). If a brick is laid somewhere inside the box, we define the weight of the brick to be the sum of the weights of the cells which it occupies.

Let \( s_d(x) \) denote the polynomial \( 1 + x + x^2 + \cdots + x^{n-1} \). The sum of the weights of all the cells in the box is \( s_A(x)s_B(x)s_C(x) \). The weight of a brick situated in the box is a multiple of \( s_a(x)s_{ab}(x)s_{abc}(x) \). Hence \( s_a(x)s_{ab}(x)s_{abc}(x) \) must divide \( s_A(x)s_B(x)s_C(x) \). Multiplying both polynomials by \( (x - 1)^3 \) gives us the equivalent condition that \( (x^a - 1)(x^{ab} - 1)(x^{abc} - 1) \) divides \( (x^A - 1)(x^B - 1)(x^C - 1) \). We conclude from Lemma 2 that the numbers \( A, B, C \) are (in some order) a multiple of \( a \), a multiple of \( ab \), and a multiple of \( abc \).

The polynomial \( s_A(x)s_B(x)s_C(x) \) and its near relative appear in Katona and Szász [2] and de Bruijn [1] respectively. However the authors use these polynomials in the case of a \( 1 \times 1 \times c \) brick, and evaluate them at \( x = \rho_c \). The conclusion obtained is that \( c \) must divide one of \( A, B, C \).

The practice of assigning a polynomial to each cell of a box appears in [3] (where the phrase 'the associated polynomial' is used) and in [4] (from which we have taken the term 'weight'). In both of these papers, a different variable is used for each dimension — that is, the weight of the cell \( (i, j, k) \) is \( x^iy^jz^k \). In this situation, the weight of a brick laid in the box is a multiple of one of the six polynomials obtained by permuting \( x, y, z \) in the expression \( s_a(x)s_{ab}(y)s_{abc}(z) \). This precludes the argument of Lemma 2 involving the multiplicity of roots.

This result has an obvious generalization to \( k \) dimensions, and this generalization can be proved in exactly the same way. Consider a \( k \)-dimensional brick with dimensions \( a_1 \times a_2 \times \cdots \times a_k \) written in nondecreasing order, and a \( k \)-dimensional box with dimensions \( A_1 \times A_2 \times \cdots \times A_k \), again written in nondecreasing order. de Bruijn calls the brick 'harmonic' if \( a_1 | a_{i+1} \) for \( i = 1, \ldots, k - 1 \), and he calls the box a 'multiple' of the brick if \( a_i | A_i \) for \( i = 1, \ldots, k \). He proved that a harmonic brick can pack only those boxes which are multiples of itself (as we have done here), and he went on to show that a nonharmonic brick can pack a box which is not a multiple of itself.
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References