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Directional complexity of the hypercubic billiard

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ABSTRACT

We consider a minimal rotation on the torus $\mathbb{T}^d$ of direction ω. A natural cellular decomposition of the torus is associated to this map. We consider an infinite orbit for this map. We compute the complexity of the associated word. Under some hypothesis on the direction, we obtain an exact formula which shows that the order of magnitude is n^d. This result is related to the billiard map inside a hypercube of $\mathbb{R}^{d+1}$.

1. Introduction

Sturmian words are infinite words over a two-letter alphabet that have exactly $n+1$ factors of length n for each integer n. The number of factors, of a given length, of an infinite word is called the complexity function. These words have been introduced by Morse–Hedlund, [14]. Consider a rotation of angle α on the torus $\mathbb{T}^1$. Consider a two-letter alphabet corresponding to the intervals $(0; 1-\alpha)$ and $(1-\alpha; 1)$. Then, the orbit of any point under the rotation is coded by an infinite word. This word is a sturmian word if and only if α is an irrational number. If α is rational, then the rotation is periodic and the word is periodic.

In this paper we consider rotation on the torus $\mathbb{T}^d$ of direction ω, see Section 4.4. We want to compute the complexity function of this map related to the natural partition. In the case of $d = 2$ the function has been computed by Arnoux, Mauduit, Shiokawa and Tamura [1] (dimension 3). Unfortunately this result was false, and we need some additional hypothesis on the direction, see [3,4] for a classification of complexity along the direction. The computation has been done by Baryshnikov [2] in any dimension under some hypothesis on the direction. The complexity function is a polynomial in n of degree d. Here we present a new proof of this result under some more general hypothesis on the direction.

This problem is related to the complexity of a billiard trajectory inside a cube of $\mathbb{R}^{d+1}$. The definition of the billiard map inside a polyhedron P is the following: A billiard ball, i.e. a point mass, moves inside a polyhedron P with unit speed along a straight line until it reaches the boundary ∂P, then instantaneously changes direction according to the mirror law, and continues along the new line. Label the sides of P by symbols from a finite alphabet $\mathcal{A}$ whose cardinality equals the number of faces of P. The orbit of a point corresponds to a word in the alphabet $\mathcal{A}$.

In the case of the cube where we code the parallel faces by the same letters, the infinite words obtained for an initial point of direction ω are equal to the infinite words obtained by a rotation on the torus $\mathbb{T}^d$ of direction ω, see Lemma 2.

In the context of the polygonal billiard some results are known on the complexity function. For the square (coded with two letters) we obtain Sturmian words and complexity $n+1$, see the famous paper of Morse and Hedlund [14]. It has been generalized to any rational polygon by Hubert [11]. He proves that the complexity is always linear in n. For an irrational

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polygon the only general result is that the billiard in a polygon has zero entropy, see Katok [13] or [10], and thus the complexity grows subexponentially. For any convex polyhedron the same fact is true, see [5].

Our result is the following, the definitions are given in the upcoming sections.

Theorem 1. *Consider the unit cube of $\mathbb{R}^{d+1}$, and code it by an alphabet with $d+1$ letters. Let ω be a B direction, and consider a billiard word in the direction ω. Denote the complexity of this word by $p(n, d, \omega)$.*

For $n, d \in \mathbb{N}$, the map $\omega \mapsto p(n, d, \omega)$ is constant on the set of B directions. Moreover if we denote it by $p(n, d)$ we have

$$p(n+2, d) - 2p(n+1, d) + p(n, d) = d(d-1)p(n, d-2) \quad \forall n, d \in \mathbb{N}.$$

Corollary 1. *For a B direction, we have*

$$p(n, d, \omega) = \sum_{i=0}^{\min(n,d)} \frac{n!d!}{(n-i)!(d-i)!\,i!} \quad \forall\, n, d \in \mathbb{N}.$$

Convention: We assume that $p(n, 0) = p(0, d) = 1$ for all integers n, d.

2. Overview of the proof

In Section 3 we define the different notions of a direction and give the precise statement of the theorem. In Section 4 we recall different facts about word combinatorics, billiard maps and the relationship between the billiard map and rotations on the torus. In Section 5 we prove two lemmas used at the end of the proof. In Section 6 the proof of Theorem 1 begins. The computation of the complexity function can be reduced to the computation of the number of bispecial words, see Lemma 1. In Proposition 1 we relate the number of the bispecial words to the number of words associated to generalized diagonals of the direction ω. In Proposition 2 we show that a diagonal is given by two subspaces of dimension $d-2$. Different properties of diagonals are studied in Section 8. In Proposition 4 we prove that the number of words associated to certain diagonals can be computed using a projection. Moreover, this number is proportional to the complexity function corresponding to a fixed direction in a hypercube of dimension $d-2$. Proposition 3 allows us to compute the number of diagonals. To finish the proof of Proposition 4, we prove the following fact in Corollary 4: The projection of a diagonal onto an appropriate subspace does not change the number of words associated to this diagonal. The end of the proof of Theorem 1 consists of a series of inductions on d and n. To start the induction we recall in Section 10 the result obtained in the case $d = 2$.

3. Definitions

In this section we give some definitions useful for the statement of the different theorems.

Definition 1. We define several notions of independence for a vector in $\mathbb{R}^{d+1}$. Let d be an element of $\mathbb{N}$.

- The real numbers $(a_i)_{1\leq i\leq d+1}$ are independent over $\mathbb{Q}$ if and only if
$$\sum_{i=1}^{d+1} r_i a_i = 0,\, r_i \in \mathbb{Q} \Longrightarrow r_i = 0 \quad \forall i \in [1, d+1].$$
- A vector $\omega = (\omega_i)_{1\leq i\leq d+1} \in \mathbb{R}^{d+1}$ is called an irrational direction if and only if:
  The real numbers $(\omega_i)_{1\leq i\leq d+1}$ are independent over $\mathbb{Q}$.
- A vector $\omega = (\omega_i)_{1\leq i\leq d+1} \in \mathbb{R}^{d+1}$ is called a totally irrational direction if and only if:
  The real numbers $(\omega_i)_{1\leq i\leq d+1}$ are independent over $\mathbb{Q}$, and
  the real numbers $(\omega_i^{-1})_{1\leq i\leq d+1}$ are independent over $\mathbb{Q}$.
- A vector $\omega = (\omega_i)_{1\leq i\leq d+1} \in \mathbb{R}^{d+1}$ is called a B direction if and only if:
  The real numbers $(\omega_i)_{1\leq i\leq d+1}$ are independent over $\mathbb{Q}$, and
  for each subset $I \subset \{1 \ldots d+1\}$ of cardinality three, the real numbers $(\omega_i^{-1})_{i\in I}$ are independent over $\mathbb{Q}$.

Remark 1. We have the implications:

ω is a totally irrational direction $\Longrightarrow \omega$ is a B direction $\Longrightarrow \omega$ is an irrational direction.

Now we recall the theorem of Baryshnikov [2].

Theorem 2 (*Baryshnikov*). *Consider an unit cube of $\mathbb{R}^{d+1}$, we code it by an alphabet with $d+1$ letters. Let ω be a totally irrational direction, consider a billiard word in the direction ω, denote the complexity of this word by $p(n, d, \omega)$. Then we have*

$$p(n, d, \omega) = \sum_{i=0}^{\min(n,d)} \frac{n!d!}{(n-i)!(d-i)!i!} \quad \forall\, n, d \in \mathbb{N}.$$

4. Background

4.1. Combinatorics

For this section a general reference is [9].

Definition 2. Let $\mathcal{A}$ be a finite set called the alphabet. By a language L over $\mathcal{A}$ we always mean a factorial extendable language: a language is a collection of sets $(L_n)_{n\geq 0}$, where the only element of L_0 is the empty word. Each L_n consists of words of the form $a_1a_2\dots a_n$ with $a_i \in \mathcal{A}$, such that for each $v \in L_n$ there exist $a, b \in \mathcal{A}$ with $av, vb \in L_{n+1}$. For all $v \in L_{n+1}$, if $v = au = u'b$ with $a, b \in \mathcal{A}$, then $u, u' \in L_n$.

The complexity function $p : \mathbb{N} \to \mathbb{N}$ is defined by $p(n) = \text{card}(L_n)$.

First of all we recall a result of Cassaigne concerning combinatorics of words [7].

Definition 3. An infinite word v over the alphabet $\mathcal{A}$ is a sequence $(v_n)_{n\in\mathbb{N}}$ such that $v_n \in \mathcal{A}$ for every integer n. A subword w of v of length n is a finite word such that there exist $n_0 \in \mathbb{N}$ and $w = v_{n_0}v_{n_0+1}\dots v_{n_0+n-1}$. The set of subwords of length n is denoted by $\mathcal{L}_n$. If v is an infinite word defined over a finite alphabet, then the union $L = \bigcup \mathcal{L}_n$ forms a language. The complexity of u is by definition the complexity of L.

Definition 4. Let $\mathcal{L}(n)$ be an extendable, factorial language. For any $n \geq 1$ let $s(n) := p(n + 1) - p(n)$. For $v \in \mathcal{L}(n)$ let

$$m_l(v) = \text{card}\{u \in \mathcal{A}, uv \in \mathcal{L}(n + 1)\},$$
$$m_r(v) = \text{card}\{w \in \mathcal{A}, vw \in \mathcal{L}(n + 1)\},$$
$$m_b(v) = \text{card}\{u \in \mathcal{A}, w \in \mathcal{A}, uvw \in, \mathcal{L}(n + 2)\}.$$

A word is call right special if $m_r(v) \geq 2$, left special if $m_l(v) \geq 2$ and bispecial if it is right and left special. Let $\mathcal{BL}(n)$ be the set of the bispecial words.

Cassaigne [7] has shown:

Lemma 1. *For any language $\mathcal{L}$ the complexity function satisfies for all integer n:*

$$s(n + 1) - s(n) = \sum_{v\in\mathcal{BL}(n)} i(v),$$

where $i(v) = m_b(v) - m_r(v) - m_l(v) + 1$.

For the proof of the lemma we refer to [7] or [8].

4.2. Billiard maps

We recall some facts from billiard theory. Additional details can be found in [16] or [15].

Let C be an unit cube of $\mathbb{R}^{d+1}$. A billiard ball, i.e. a point mass, moves inside C with unit speed along a straight line until it reaches the boundary ∂C, then instantaneously changes direction according to the mirror law, and continues along the new line. More precisely, the billiard map T is defined on a subset X of $\partial C \times \mathbb{RP}^d$ by the following method (where $\mathbb{RP}^d$ is the projective space of dimension $d \geq 1$):

First we define the set $X' \subset \partial C \times \mathbb{RP}^d$. A point (m, ω) belongs to X' if and only if one of the following two conditions holds:

(1) The line $m + \mathbb{R}[\omega]$ intersects a face of C of dimension less than $d - 1$, where $[\omega]$ is a vector of $\mathbb{R}^{d+1}$ which represents ω.
(2) A segment of the line $m + \mathbb{R}[\omega]$ is included inside the face of C which contains m.

We define X as the set

$$X = (\partial C \times \mathbb{RP}^d) \setminus X'.$$

Now we define the map T: Consider $(m, \omega) \in X$, then we have $T(m, \omega) = (m', \omega')$ if and only if the segment mm' is collinear to $[\omega]$, and if $[\omega'] = s[\omega]$, where s is the linear reflection over the face which contains m'.

$$T : X \to \partial C \times \mathbb{RP}^d$$
$$T : (m, \omega) \mapsto (m', \omega').$$

Remark 2. In the following we identify $\mathbb{RP}^d$ with the unit vectors of $\mathbb{R}^{d+1}$ (*i.e* we identify ω and $[\omega]$).

4.3. Notations for the billiard map

Label the faces of C by $d + 1$ symbols from a finite alphabet $\mathcal{A}$ such that the two opposite faces of the cube are coded by the same symbols. To the orbit of a point in a direction ω we associate the word in the alphabet $\mathcal{A}$ which is given by the sequence of faces of the billiard trajectory.

The set of points (m, ω) is such that for all integers n, $T^n(m, \omega) \in X$ is denoted by X_∞. The infinite word associated to a point (m, ω) in X_∞ is denoted by $v_{m,\omega}$.

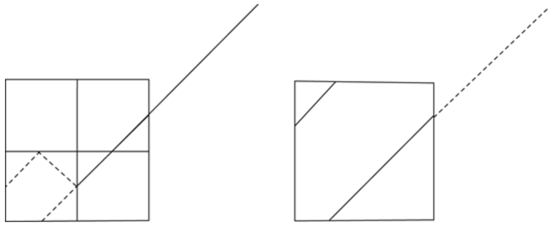

Fig. 1. Unfolding.

Definition 5. Consider the billiard map T inside the cube, and a point $(m, \omega) \in X_\infty$. We define the complexity $p(n, m, \omega)$ by the complexity of the infinite word $v_{m,\omega}$ (see Definition 3). We call it the *directional complexity*.

4.4. Unfolding: Definition and example

The *unfolding* is a very useful tool in the study of billiard behavior. Consider a billiard trajectory in a polyhedron. To draw the orbit, we must reflect the line each time it hits a face of the polyhedron. The unfolding consists of reflecting the polyhedron through the face while continuing along the same line.

Although we deal with the cube of $d + 1$ dimensions, the figures are made for the square.

Example 1. The billiard orbit of (m, ω) appears as sequence of intersections of the line $m + \mathbb{R}\omega$ with the lattice $\mathbb{Z}^{d+1}$, see Fig. 1. In the left-hand picture we represent a billiard orbit inside the square by a dotted curve. It is unfolded into the line which intersects $\mathbb{Z}^2$.

In the right-hand picture we exemplify how the study of the billiard orbit can be performed on the big square, where we identify the opposite sides. Thus we obtain a torus, and the map is a translation on this torus:

For $\omega \in \mathbb{R}^{d+1}$, a rotation T_ω of direction ω on the torus is a map defined as follows.

$$\mathbb{R}^{d+1}/\mathbb{Z}^{d+1} \to \mathbb{R}^{d+1}/\mathbb{Z}^{d+1}$$
$$T_\omega : (x_i)_{i\leq d+1} \mapsto (x_i + \omega_i)_{i\leq d+1}.$$

Fig. 1 explains the following result:

Lemma 2. *Let $\omega \in \mathbb{RP}^d$, and consider the billiard map T in the cube of $\mathbb{R}^{d+1}$. Then this is equivalent to studying the orbit $(T^n(m, \omega))_n$ or the orbit $(T^n_\omega(m))_n$.*

4.5. Minimality

Definition 6. A direction $\omega \in \mathbb{RP}^d$ is called minimal if, for all points m, the projection of the sequence $(T^n(m, \omega))_{n\in\mathbb{N}}$ in ∂C is dense in ∂C.

The following lemma deals with minimality of billiard words. This minimality depends on algebraic properties of the translation direction.

Lemma 3. *Let $\omega = (\omega_i)_{1\leq i\leq d+1}$ be an unit vector of $\mathbb{R}^{d+1}$. Consider the billiard map in the cube of $\mathbb{R}^{d+1}$. Then the direction ω is minimal if and only if ω is an irrational direction.*

The proof of this lemma is based on Kronecker's lemma, see [12].

4.6. Remarks

The following lemma is very useful in what follows.

Lemma 4. *Consider an orthogonal projection on a face of the cube. The orthogonal projection of a billiard trajectory is a billiard trajectory inside the face which itself is a cube of lower dimension.*

Definition 7. Let $v = v_0 \dots v_n$ be a billiard word. We define the cell of v as the subset of $\{(m, \omega) \in \partial P \times \mathbb{RP}^d\}$ given by the conditions that

$$\forall i, 0 \leq i \leq |v| - 1, \pi_1(T^i(m, \omega)) \in v_i.$$

In this formula π_1 represents the projection into the first variable. v_i represents the letter and the face of the cube coded by this letter.

Throughout the proof we consider billiard trajectories inside the cube as lines on $\mathbb{R}^{d+1}$ with the above introduced unfolding.

5. Combinatorial lemmas

In this part we prove different results which will be used in the end of the proof.

Lemma 5. *For all $n > d$ we have:*

$$\sum_{i=0}^{d} \frac{n!d!(d+1-i)}{(n-i)!(d-i)!i!} = \sum_{i=0}^{d} \frac{(n+1)!d!}{(n+1-i)!(d-i)!i!}.$$

Proof. Consider the vector space given by all polynomials of degree less than or equal to d. This space has the following two bases

$$(e_i)_{-1\le i\le d-1} = (1, X, X(X-1), \dots, X(X-1)\dots(X-d+1)),$$
$$(e'_i)_{-1\le i\le d-1} = (1, X+1, (X+1)X, \dots, (X+1)X\dots(X-d+2)),$$

where $e_i = X(X-1)\dots(X-i)$, $e'_i = (X+1)X\dots(X+1-i)$ if $i \ge 0$ and $e_{-1} = e'_{-1} = 1$ by convention.

For $i \ge 0$, since we have $e'_j = (X+1)X\dots(X-(j-1)) = (X-j+j+1)X\dots(X-(j-1))$, we deduce:

$$e'_j = e_j + (j+1)e_{j-1} \quad \forall j \ge 0.$$

Now consider a polynomial P of degree d. It can be expressed as

$$P = \sum_{i=-1}^{d-1} b_i e'_i.$$

$$P = b_{-1} + \sum_{i=0}^{d-1} b_i e'_i.$$

The preceding formula gives

$$P = b_{-1} + \sum_{i=0}^{d-1} b_i e_i + \sum_{i=0}^{d-1} b_i (i+1) e_{i-1},$$

$$P = b_{-1} + \sum_{i=0}^{d-1} b_i e_i + \sum_{i=-1}^{d-2} b_{i+1}(i+2) e_i,$$

$$P = b_{-1} + b_0 + b_{d-1}e_{d-1} + \sum_{i=0}^{d-2} [b_i + (i+2)b_{i+1}] e_i.$$

$$P = b_{d-1}e_{d-1} + \sum_{i=-1}^{d-2} [b_i + (i+2)b_{i+1}] e_i. \tag{1}$$

Thus we have an expression of P inside the basis $(e_i)_i$.

The sum $A = \sum_{i=0}^{d} \frac{(n+1)!d!}{(n+1-i)!(d-i)!i!}$ is a polynomial on n of degree d. Moreover we have, for all $i \ge 1$:

$$\frac{(n+1)!}{(n+1-i)!} = e'_{i-1}(n).$$

We denote $a_i = \frac{d!}{(d-i)!i!}$. We will obtain the expression of A inside the basis (e_i):

$$A = 1 + \sum_{i=1}^{d} a_i e'_{i-1}(n).$$

$$A = 1 + \sum_{i=0}^{d-1} a_{i+1} e'_i(n).$$

$$A = \sum_{i=-1}^{d-1} \frac{d!}{(d-i-1)!(i+1)!} e'_i.$$

Then, by Eq. (1), we deduce the following formula:

$$A = b_{d-1}e_{d-1}(n) + \sum_{i=-1}^{d-2} \left[\frac{d!}{(i+1)!(d-i-1)!} + \frac{d!(i+2)}{(i+2)!(d-i-2)!}\right] e_i(n),$$

$$A = b_{d-1}e_{d-1}(n) + d! \sum_{i=-1}^{d-2} \left[\frac{i+2+(i+2)(d-i-1)}{(i+2)!(d-i-1)!}\right] e_i(n),$$

$$A = a_d e_{d-1}(n) + d! \sum_{i=-1}^{d-2} \frac{(i+2)(d-i)}{(i+2)!(d-i-1)!} e_i(n),$$
$$A = e_{d-1}(n) + d! \sum_{i=-1}^{d-2} \frac{(d-i)}{(i+1)!(d-i-1)!} e_i(n),$$
$$A = d! \sum_{i=-1}^{d-1} \frac{(d-i)}{(i+1)!(d-i-1)!} e_i(n),$$
$$A = d! \sum_{i=0}^{d} \frac{(d-i+1)}{(i)!(d-i)!} e_{i-1}(n),$$
$$A = \sum_{i=0}^{d} \frac{(d+1-i)d!n!}{(d-i)!i!(n-i)!}. \quad \square$$

Lemma 6. *Consider a sequence $(p(n,d))_{n,d\in\mathbb{N}}$, where n, d are two integers such that $p(0,d) = 1$, $p(1,d) = d+1$ for all d, and $p(n,0) = 1$ for all n. Define $s(n,d) = p(n+1,d) - p(n,d)$ for all $d > 0$. Assume that for all integers $n, d \geq 2$ we have*

$$s(n+1,d) - s(n,d) = d(d-1)p(n,d-2).$$

Then we have:

$$s(n,d) = dp(n,d-1) \quad \forall n \geq 0, \forall d \geq 2.$$

Proof. We give a proof by induction on n.

The equality is true for $n = 0$.

We have

$$dp(n,d-1) = d + d \sum_{i\leq n-1} s(i,d-1).$$

By induction we deduce

$$dp(n,d-1) = d + d(d-1) \sum_{i\leq n-1} p(i,d-2).$$

Then we apply the hypothesis and obtain

$$dp(n,d-1) = d + \sum_{i\leq n-1} [s(i+1,d) - s(i,d)].$$
$$dp(n,d-1) = s(n,d).$$

The induction process is finished, and the lemma is proved. $\square$

6. First part

Remark that the hypothesis on the direction implies that a billiard word in direction ω is not in one of the $d+1$ coordinates' hyperplane.

6.1. Notations

We want to relate the bispecial words to the generalized diagonals.

Definition 8. A diagonal is the set of all trajectories from a face of dimension $d-1$ to a face of the same dimension.

Definition 9. We say that a diagonal, between A and B, is of combinatorial length n if the orbit segment passes through n cubes. We denote it by $d(A,B) = n$. We denote the diagonals of direction ω and combinatorial length n by $\mathrm{Diag}(n,d,\omega)$ (see Fig. 2).

$$\mathrm{Diag}(n,d,\omega) = \{(A,B) \text{ faces of dimension } d-1, \exists p,q \in A,B\ \vec{pq}//\omega\ \ d(A,B) = n\}.$$

In the following we only consider diagonals of combinatorial length n whose initial segment is in the cube $[0,1]^{d+1}$. Moreover, all the diagonals will be in the direction ω.

We denote the fact that an orbit in the diagonal γ has code v by $v \in \gamma$.

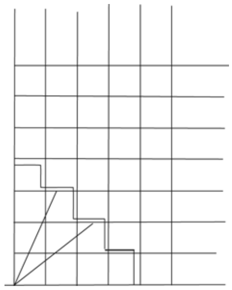

Fig. 2. Unfolding of billiard trajectories and diagonals.

This section is devoted to the proof of the following result:

Proposition 1. *Let d be an integer greater than* 1*.*

$$\sum_{v\in\mathcal{BL}(n,d,\omega)} i(v) = \sum_{\gamma\in\mathrm{Diag}(n,d,\omega)} \sum_{v\in\gamma} 1. \tag{2}$$

6.2. Lemmas

For the proof we need the following lemmas.

Lemma 7. *We consider a word v in $\mathcal{L}(n,d)$ with $n \geq 2$, consider the unfolding of the billiard trajectories which are coded by v and start inside the cube $[0;1]^{d+1}$. Then for all i, $0 \leq i \leq n-1$, there exists only one face corresponding to the letter v_i.*

Proof. First we consider the intersection of the cell of v with $\mathbb{RP}^d$. This set is a proper subset of an octant since $n \geq 2$. Now we make the proof by contradiction. We consider the first time j where two different faces appear. There exist two lines starting from a face (corresponding to v_{j-1}) which pass through these two different faces. These faces are different but are coded by the same letter, thus they are in two different hypercubes. Thus the two directions are in different octants, a contradiction. □

Lemma 8. *Let v be a bispecial word in $\mathcal{BL}(n,d,\omega)$, then there exists only one diagonal, of direction ω, associated to this word.*

Proof. We consider a bispecial word v. We consider the faces which prolong v into a word of length $n+1$. We claim that these faces always intersect. By Lemma 7 these faces are in the same hypercube. They correspond to different letters of the coding, thus these faces intersect (by definition of the coding). The claim is proved.

Thus those faces have a non-empty intersection. Thus there exists a trajectory which has v as coding and starts on the face of dimension $d-1$. Consider the same intersection with the prefix, we have built a diagonal associated to this word. By construction it is unique. □

Now we can prove

Lemma 9. *Consider a word v element of $\mathcal{BL}(n,d,\omega)$. We claim that*

$$i(v) = 1.$$

Proof. We know that there is only one diagonal associated to this word.

Let γ be a diagonal, and v the associated word. Since the faces A, B are of dimension $d-1$ they are at the intersection of two faces of dimension d. Since we have a B direction, it cannot pass through the boundary of A or B.

Thus we have $m_r(v) = m_l(v) = 2$. Clearly the diagonal is in the interior of the cell, thus a small perturbation of the diagonal still exists in the interior of the cell. Thus all the possibilities exist and $m_b = 4$, and thus $i(v) = 1$. □

The phase space is the set of points of the boundary of the cube with a direction. Thus it is of dimension $2d$. In the phase space a word corresponds to a cell which is a manifold of dimension $2d$.

A set of points in the cell such that their orbits intersect a face of dimension $d-1$ is called discontinuity of v. Let us remark that a discontinuity is of dimension at most $2d-1$, and that a diagonal is in the intersection of two discontinuities.

6.3. Proof of Proposition 1

We consider the map

$$f : \mathcal{BL}(n,d,\omega) \to \mathrm{Diag}(n,d,\omega).$$
$$f : v \mapsto \gamma.$$

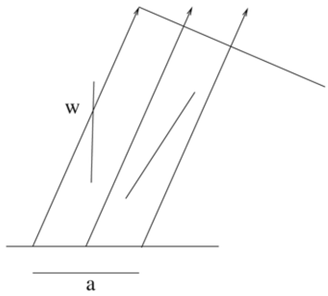

Fig. 3. Diagonal and words.

Lemma 8 implies that f is well defined and onto, thus

$$\mathrm{card}(\mathcal{BL}(n,d,\omega)) = \sum_{\gamma\in \mathrm{Diag}(n,d,\omega)} \mathrm{card}(f^{-1}(\gamma)).$$

By definition we have $\mathrm{card}(f^{-1}(\gamma)) = \sum_{v\in\gamma} 1$, we deduce

$$\sum_{v\in\mathcal{BL}(n,d,\omega)} i(v) = \sum_{\gamma\in\mathrm{Diag}(n,d,\omega)} \sum_{v\in\gamma} i(v).$$

Then Lemma 9 finishes the proof.

7. Diagonals

Consider a diagonal between two faces A, B of dimension $d-1$, the aim of this section is to prove:

Proposition 2. *For each diagonal γ of direction ω between two faces A, B of dimension $d-1$, there exist two subspaces a, b of dimension $d-2$ such that:*

For all points in a there exists a point in b which belongs to the orbit of the initial point in the billiard flow of direction ω, moreover we have:

$$\sum_{\gamma}\sum_{v\in\gamma(A,B)} i(v) = \sum_{a,b}\sum_{v\in\gamma} 1.$$

See Fig. 3.

The diagonal γ is the collection of trajectories in the direction ω which passes through two faces A, B of dimension $d-1$. If we consider any face A, B with the good distance, it is possible that an associated diagonal does not exist: Indeed the direction is fixed and each orbit can pass through a third edge, see the example in the cube of $\mathbb{R}^3$ [3]. The case where the direction is not fixed is treated in [6].

Lemma 10. *Let A, B be two faces of dimension $d-1$ and ω a direction. We consider*

$$\gamma_{A,B} = \{m \in A, m+\mathbb{R}\omega \cap B \neq \emptyset\}.$$

Then $\gamma_{A,B}$ has one of the following equations

(1) *There exist $i, j \in [1\ldots d+1]$ such that $n\omega_i = p\omega_j$, with $n, p \in \mathbb{N}$.*
(2) *There exist $i, j \in [1\ldots d+1]$ such that $m_i + \frac{n\omega_i}{\omega_j} = p$ with $n, p \in \mathbb{N}$.*
(3) *There exist $i, j \in [1\ldots d+1]$ such that $\omega_j m_i - \omega_i m_j = n\omega_i - p\omega_j$ with $n, p \in \mathbb{N}$.*

Proof. First we can assume that the point $m \in A$ has coordinates of the following form

$${}^t\begin{pmatrix} m_1, & \ldots & m_{d-1}, & 0, & 0\end{pmatrix}.$$

Then each point of B has two coordinates equal to integers n, p. Thus its coordinates are of the form:

$${}^t\begin{pmatrix} b_1, & \ldots & n, & \ldots & p, & \ldots & b_{d-1}\end{pmatrix}.$$

If the line $m+\mathbb{R}\omega$ intersects B it means that there exists λ such that $m+\lambda\omega \in B$. Then there are three choices, depending on the positions of n, p in the coordinates.

- If n, p are at positions $d, d+1$ we obtain a system of the form

$$\begin{cases}\lambda\omega_d = n\\ \lambda\omega_{d+1} = p.\end{cases}$$

This gives Eq. (1).

- If n is at a position i less than or equal to $d-1$, and p is at position d or $d+1$, we obtain

  $$\begin{cases}\lambda\omega_d = p\\ m_i + \lambda\omega_i = n.\end{cases}$$

  This gives the second equation.
- If n and p are at positions less than $d-1$, we obtain a system of two equations

  $$\begin{cases}m_i + \lambda\omega_i = n\\ m_j + \lambda\omega_j = p.\end{cases}$$

  We eliminate λ and we obtain the equation of case (3). □

Corollary 2. *Let $A, B, C_i, i = 1 \dots l$ be $l+2$ faces of dimension $d-1$ and ω a minimal direction. We have the equivalence*

$$\gamma_{A,B} = \bigcup_i \gamma_{A,C_i} \iff \omega \text{ is not a } B \text{ direction}.$$

Proof. We consider the three functions which appear in Lemma 10.

$$\begin{cases}f(m) = n\omega_i - p\omega_j,\\ g(m) = m_i + \dfrac{n\omega_i}{\omega_j} - p,\\ h(m) = \omega_j m_i - \omega_i m_j - (n\omega_i + p\omega_j).\end{cases}$$

The diagonals $\gamma_{A,B}$, γ_{A,C_i} have equations of the type f, g, h by the preceding Lemma (with different n, p, i, j). Now without loss of generality we treat the case $l = 1$. The sets $\gamma_{A,B}$ and $\gamma_{A,C}$ are equal if and only if two of the preceding functions are equal on a set of positive measures. If two functions are equal for all m, it implies that they are of the same form. For example f cannot be equal to g on a set of positive measures, thus we have different cases:

- If we have several times the map f, it implies the relation

  $$n\omega_i - p\omega_j = n'\omega_k - p'\omega_l.$$

  The coefficients of the direction are dependent over $\mathbb{Q}$: contradiction with the minimality of the direction.
- If we have several times the map h, we obtain

  $$n\omega_i + p\omega_j = n'\omega_i + p'\omega_j$$

  which is impossible for the same argument.
- Thus the two equations are of the second form. We obtain

  $$q\frac{\omega_i}{\omega_j} - q'\frac{\omega_i}{\omega_k} = p - p',$$

  $$\frac{q}{\omega_j} - \frac{q'}{\omega_k} = \frac{p-p'}{\omega_i}.$$

  Thus ω is not a B direction. The converse is easy by the same argument. □

7.1. Diagonal and words

Now we consider a B direction, and a diagonal in this direction. We know it exists by Corollary 2. We denote by A, B the faces related to the diagonal, and define the sets a, b by

Definition 10. With the same notations, we denote

$$a = \{m \in A, (m + \mathbb{R}\omega) \cap B \neq \emptyset.\}.$$

$$b = \{m \in B, (m + \mathbb{R}\omega) \cap A \neq \emptyset.\}.$$

Lemma 11. *Let γ be a diagonal corresponding to the faces A, B, then we have* $\dim a = \dim b = d - 2$.

Proof. We make the computation for b, (it is exactly the same method for a). It represents the points which form a diagonal. The cylinder $A + \mathbb{R}\omega$ is of dimension $d = 1 + d - 1$. We use the dimension formula

$$\dim E \cap F = \dim E + \dim F - \dim (E + F).$$

In this case we have $E = A + \mathbb{R}\omega$, $F = B$. We deduce that the intersection of this cylinder with B is of dimension

$$d + d - 1 - (d+1) = d - 2. \quad \square$$

7.2. Notations

This lemma shows that a diagonal is in bijection with the two subspaces a, b. Thus in the following we will denote a diagonal by (a, b), if necessary.

8. Calculus on diagonals

8.1. Length of a diagonal

Lemma 12. *Let A, B be two faces of dimension less than or equal to $d-1$. Assume A, B are at combinatorial length n, see Definition 9, in a direction ω. Assume that the elements of A are of the form*

$$^t(m_1, \quad \dots \quad m_{d-1}, \quad 0, \quad 0).$$

Then we have:

- *Either A, B are in a subspace of dimension $d-1$ then there exist $n_d, n_{d+1} \in \mathbb{N}$ such that each point $b_0 \in B$ has coordinates*
$$b_0 = {}^t(b_1 \quad \dots \quad b_{d-1}, \quad n_d, \quad n_{d+1}),$$
with $\gcd(n_{d+1}, n_d) = 1$ and $\sum_{i=1}^{d-1} \lfloor b_i \rfloor + n_{d+1} + n_d = n$;
- *Or there exist $i, j \in [1 \dots d+1]$ with $(i, j) \neq (d, d+1)$ such that each point $b_0 \in B$ has the following coordinates:*
$$b_0 = {}^t(b_1 \quad \dots \quad n_i, \quad \dots \quad n_j, \quad \dots \quad b_{d+1}),$$
with $n_i, n_j \in \mathbb{N}$ and $n_i + n_j + \sum_{k=1}^{d+1} \lfloor b_k \rfloor = n$.

Proof. • First of all we consider the faces of dimension d which are at combinatorial length n of A. We claim that the points $(b_i)_{i \leq d+1}$ of these faces verify $\sum_{i=1}^{d+1} \lfloor b_i \rfloor = n$.

The proof is made by induction on n. It is clear for $n = 1$, now consider a billiard trajectory of length n, it means that just before the last face we intersect another face of the same cube. This face is at combinatorial length $n-1$, and we can apply the induction process. Now consider a point of these faces, denote by $(c_i)_{i \leq d+1}$ its coordinates. We verify easily that $\sum_{i=1}^{d+1} \lfloor b_i \rfloor - \sum_{i=1}^{d+1} \lfloor c_i \rfloor = 1$ for all points b_0, c. This finishes the proof of the claim.

• Now there are also two cases if A, B are in the same hyperplane or not. If they are not in the same hyperplane the coordinates of points in B have the form given in the second point of the Lemma. Now assume A, B are contained in a hyperplane. The fixed coordinates of all points in A and B are at the same places. Then we project on the plane generated by these coordinates. On the plane of projection the images of A, B are two points with integer coordinates $(0; 0)$ and $(n_d; n_{d+1})$. The diagonal projects on a line passing through these two points. To ensure that $\gamma_{A,B}$ is not the union of γ_{A,C_i} we can verify that this line does not contain integer points. The condition $gcd(n_{d+1}, n_d) = 1$ is equivalent to this property. □

Corollary 3. *A diagonal in a B direction is of the second form.*

Proof. Consider a diagonal in a B direction. By the preceding lemma there are two cases for the coordinates of the faces of start and go. In the first case, we have a rational relation between ω_d and ω_{d+1} see Eq. (1) of Lemma 10. □

8.2. Number of diagonals

We consider the different diagonals of length n in direction ω.

Proposition 3. *Let ω be a B direction, then we obtain:*

$$\mathrm{card}(\mathrm{Diag}(n, d, \omega)) = d(d-1) \quad \forall n \in \mathbb{N}^*.$$

Proof. First consider the face A of dimension $d-1$. We can always assume that the points of A have the following coordinates

$$^t(a_1 \quad \dots \quad a_{d-1}, \quad 0, \quad 0).$$

We use Lemma 10, and we must compute the number of different faces B. Since the direction is a B direction, we cannot have case (1), see the preceding Corollary. Now case (2) corresponds to the choice of one integer inside the first $(d-1)$ coordinates, and one integer inside the last two coordinates. It gives $2(d-1)$ possibilities.

Case (3) corresponds to the case of two integers inside the first $d-1$ coordinates, it gives $(d-1)(d-2)$ possibilities.

The total number of faces B is finally the sum of these numbers:

$$(d-1)(d-2) + 2(d-1) = d(d-1). \quad \square$$

9. Projections

First we define the notion of coordinate spaces:

Definition 11. The space $\mathbb{R}^{d+1}$ has a basis (e_i) such that the edges of the cube are parallel to the vectors e_i. Then we say that a linear space H is a coordinate space if there exists $I \subset \{1 \dots d+1\}$ such that $(e_i)_{i \in I}$ is a basis of H.

We denote by π_H the orthogonal projection on H. We recall that a diagonal is given by two subspaces a, b.

Here we prove:

Proposition 4. *For all coordinate spaces H of dimension $d-1$ there exist $n_0 \in \mathbb{N}$, $\omega' \in \mathbb{RP}^{d-2}$ and a B direction, ω' such that*

$$\sum_{a \in H} \sum_{v \in (a,b)} 1 = (d-1)p(n-n_0, d-2, \omega').$$

Lemma 13. *Assume that H does not contain the vectors e_d, e_{d+1} of the basis related to the cube. Denote the coordinates in this base by $(X_i)_{i \le d+1}$. Then the image by π_H of the linear space intersection of $X_d = c, c \in \mathbb{R}$ and $\langle a, b\rangle$, the space generated by a and b, is of dimension $d-2$.*

Proof. By assumption we have that the points of a have the following coordinates

$$^t\left(a_1, \quad a_2, \quad \dots \quad a_{d-1}, \quad 0, \quad 0\right).$$

Then the space $\langle a, b\rangle$ generated by a and b contains points with coordinates of the form

$$^t\left(\lambda a_1 + \mu\omega_1, \quad \lambda a_2 + \mu\omega_2, \quad \dots \quad \lambda a_{d-1} + \mu\omega_{d-1}, \quad \mu\omega_d, \quad \mu\omega_{d+1}\right).$$

The intersection with the plane $X_d = c$ gives points of coordinates

$$^t\left(\lambda a_1 + c\omega_1/\omega_d, \quad \lambda a_2 + c\omega_2/\omega_d, \quad \dots, \quad \lambda a_{d-1} + c\omega_{d-1}/\omega_d, \quad c, \quad c\omega_{d+1}/\omega_d\right).$$

Thus the projection by π_H gives an hyperplane parallel to a, thus of dimension $d-2$. □

9.1. Words

During the proof of Proposition 4, if γ is a diagonal between a and b we must compute the number of words in the diagonal. The set a is partitioned into several sets a_i, and each set a_i corresponds to a different word of the diagonal. We must compute the number of sets of these partitions to prove our result. We will project these trajectories inside the space H, and compute the number of words inside this subspace.

Definition 12. Each diagonal is associated to faces A, B. We denote by $\mathcal{L}_{A,B}$ the sets of the billiard words of direction ω between these faces. In the space H the billiard map is coded with d letters in the natural way. We denote by $\mathcal{L}_{\pi_H(A),\pi_H(B)}$ the sets of the billiard words of direction $\pi(\omega)$ between these faces.

All the trajectories of one diagonal are in a space of dimension $d-1$. If we project on H two letters vanish and $d-1$ letters are retained.

In the space $\mathbb{R}^{d+1}$ the letter i appears in a word $v \in \gamma$, if and only if the plane $X_i = c, c \in \mathbb{R}$ intersects $\langle a, b\rangle$. The preceding lemma shows that the projection of this set does not vanish. But this projection does not coincide with a letter of the natural coding of billiard inside H for all letters since $\dim H = d-1$. Thus we will add two letters to the natural coding of H to keep an alphabet with $d+1$ letters, and we denote by $\overline{\mathcal{L}_{\pi(A),\pi(B)}}$, the sets of all projections of the diagonal words.

Remark 3. In Fig. 4 we assume that H is of dimension two, and we draw the two codings on H. One is with $2 = d-1$ letters and one with $4 = d+1$ letters.

Lemma 14. *The map π_H can be extended to billiard words.*

$$\pi_H : \mathcal{L}_{A,B} \to \overline{\mathcal{L}_{\pi_H(A),\pi_H(B)}}$$
$$\pi_H : v \mapsto \pi_H(v).$$

The proof will explain the definition of $\pi_H(v)$.

Proof. We consider a billiard trajectory between A and B. Assume it has v for coding. Then we consider the image of the line by π. It is a billiard trajectory inside the unit cube of $\mathbb{R}^{d-1}$, since the projection is an orthogonal projection by Lemma 4. We denote its coding by $\pi_H(v)$. □

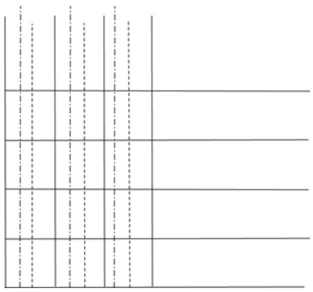

Fig. 4. Different codings in H.

Lemma 15. *Assume $d \geq 3$, let $v \in \mathcal{L}(m, d-2, \pi(\omega))$ be a billiard word between two faces A, B' of dimension $d-2$. Then for all integers $n \geq m+1$ there exists one and only one face B of dimension $d-2$ such that:*

$d(A, B) = n,$

$\gamma_{A,B}$ is an element of $\mathrm{Diag}(n, d, \omega)$,

$\pi(B) = B'.$

Proof. First remark that $\dim \pi(B) \geq d-2$. Indeed we have by definition $\dim \pi(B) \leq \dim B$. Now the global space is of dimension $d+1$, thus the orthogonal of $\langle A, B\rangle$ is of dimension 1 since $\dim \langle A, B\rangle = d$. It implies that the dimension of $\pi(B)$ is at least $d-1-1$.

By Lemma 12, we can always lift the face B' in a face B with $d(A, B) = n$. We just have to translate B' to the coordinate $x_d = n - m$. This face B is unique. The only point to prove is that the trajectories between A, B form a diagonal. We make a proof by contradiction. Then each trajectory between A, B intersects another face C_i. It implies that $\gamma_{A,B}$ is covered by some γ_{A,C_i}. This is a contradiction to Corollary 2. $\square$

Corollary 4. *The map π is a bijection.*

Proof. All the trajectories between A, B are inside the space $\langle a, b\rangle$. If there are several words in a diagonal $\gamma_{A,B}$, it means that the cylinder $\langle a, b\rangle$ is cut by different hyperplanes. The hyperplanes which cut $\langle a, b\rangle$ into a set of dimensions d are not interesting, since they correspond to letters which appear in each word of $\gamma_{A,B}$. Thus to count the number of words in $\gamma_{A,B}$ we must calculate the number of hyperplanes H which cut $E_{a,b}$ into a subspace H' of codimension 1 into $\langle a, b\rangle$. Since the projection π fulfills $\dim \mathrm{Im}\, \pi = d - 1 = \dim \langle a, b\rangle$, we deduce $\dim H' = \dim H - 1 = d - 1$, thus $\dim \pi(H') = d - 1$. Thus there is no erasure in the projection, and π is injective. Remark that some hyperplanes can project into spaces which do not have integer coordinates, see the preceding Lemma. Then we apply Lemma 15, and the map π is surjective. It suffices to consider the word v associated to the diagonal between A and B, with $A' = \pi(A)$ and m is the length of the word $\pi(v)$. $\square$

9.2. Proof of Proposition 4

By Lemma 4 the projection of the billiard trajectory inside H is a billiard trajectory. These trajectories start into $\pi(a)$ and finish into $\pi(b)$. By Lemma 12, their combinatorial lengths are equal to $n - m$. Moreover the B direction projects on a B direction $\pi(\omega)$, by definition. By Corollary 4 we have

$$\sum_{v\in\gamma_a} 1 = \sum_{v\in\pi(\gamma_a)} 1.$$

Now the space H is of dimension $d-1$, thus there are $d-1$ faces for the cube in this space. If we consider all the trajectories in direction $\pi(\omega)$ which start from these $d-1$ faces we have obtained all the billiard trajectories of length $n-m$ in this space. With preceding notations we can denote the complexity in the natural language by $p(n, d-2, \omega')$ and in the new coding by $\overline{p(n, d-2, \omega')}$. If we denote $n_0 = n - m$ we deduce

$$\sum_{\gamma} \sum_{v\in\gamma_a} 1 = \overline{p(n - n_0, d-2, \pi(\omega))}.$$

Now we claim $(d-1)p(n, d-2, \omega') = \overline{p(n, d-2, \omega')}$. When we pass from one coding to the other, it is similar to code the billiard inside the cube with the same letter for the parallel faces or not. Thus the two complexities are proportional, and the factor equals $d-1$. The proof finishes with this claim.

10. Dimension three

In this section we recall some facts about dimension three. This will be used in the next section, where we prove Theorem 1 by induction on d.

In [3] we prove the following result:

Theorem 3. *Assume ω is a B direction of $\mathbb{RP}^2$, then we have*

$$p(n, 2, \omega) = n^2 + n + 1.$$
$$s(n + 1, 2, \omega) - s(n, 2, \omega) = 2.$$

Proposition 3 has showed that there are two diagonals in this case, thus the second point of the theorem is proved here. In fact this was proved in [3] with another method for $d = 2$. It implies that for all $\gamma \in \mathrm{Diag}(n, 2, \omega)$ we have $\sum_{v\in\gamma} 1 = 1$. This finishes the computation of $p(n, 2, \omega)$. Remark that for $d = 2$, the set of B directions equals the set of totally irrational directions.

11. Proofs of the results

11.1. Proof of Theorem 1

By Lemma 1 we must compute $\sum_{\mathcal{BL}(n)} i(v)$. By Proposition 1, we have:

$$\sum_{\mathcal{BL}(n)} i(v) = \sum_{\gamma_{A,B}} \sum_{v\in\ \gamma_{A,B}} 1.$$

This can be written as

$$s(n + 1, d, \omega) - s(n, d, \omega) = \sum_{H} \sum_{\gamma\in H} \sum_{v\in\gamma} i(v).$$

By Proposition 4 we have for any (A, B) which forms a diagonal

$$\sum_{\gamma\in H} \sum_{v\in\gamma_{A,B}} 1 = (d - 1)p(n - n_0, d - 2, \omega').$$

Thus we have

$$\sum_{\mathcal{BL}(n)} i(v) = \sum_{H} (d - 1)p(n - n_0, d - 2, \omega').$$

In the following we denote the diagonal by the faces of start and end $(A; B)$.

• We make an induction on d. The hypothesis is:

The complexity map $p(n, d, \omega)$ is independent of ω for all n.

First the induction hypothesis is true for $d = 2$, see the preceding Section.

Now by the preceding Proposition we have

$$s(n + 1, d, \omega) - s(n, d, \omega) = \sum_{H} (d - 1)p(n - n_0, d - 2, \omega').$$

Then we use the induction hypothesis for $d - 2$, and choose a direction ω such that $n_0 = 0$, see Proposition 4. Remark that such a direction can depend on the integer n.

We deduce

$$s(n + 1, d, \omega) - s(n, d, \omega) = \sum_{H} (d - 1)p(n, d - 2).$$

Thus the induction process is finished, and the claim is proved.

Moreover Proposition 3 implies that

$$\mathrm{card}(A, B \in \mathrm{Diag}(n, \omega, d)) = d(d - 1).$$

Since $\dim H = d - 1$ we deduce that $d - 1$ projections of diagonals are in the same space H. Thus there are d different classes of diagonals, in each class every diagonal belongs to the same space H. We deduce

$$s(n + 1, d, \omega) - s(n, d, \omega) = d(d - 1)p(n, d - 2).$$

This finishes the proof of the theorem. Remark that in the case $d = 0$, there is only one letter and we have $p(n, 0) = 1$, thus the formula of Theorem 1 is true for $d = 2$.

11.2. Proof of Baryshnikov's formula

In this section we prove Corollary 1. First we can omit the direction in the notation, with the help of Theorem 1. We will prove the formula by induction on n for all d.

For $n = 0$ the formula is true.

It is clear that $n \mapsto p(n, d)$ is a polynomial function on n, thus we only compute its value for $n > d$, by analyticity it will be the same for $n \leq d$.

Then Lemma 6 gives:

$$p(n+1, d) = p(n, d) + dp(n, d-1).$$

Now the induction hypothesis gives

$$p(n+1, d) = \sum_{i=0}^{d} \frac{n!d!}{(n-i)!(d-i)!i!} + d\sum_{i=0}^{d-1} \frac{n!(d-1)!}{(n-i)!(d-1-i)!i!}.$$

$$p(n+1, d) = \sum_{i=0}^{d-1} \frac{n!d!}{(n-i)!(d-1-i)!i!}\left[1 + \frac{1}{d-i}\right] + \frac{n!}{(n-d)!}.$$

$$p(n+1, d) = \sum_{i=0}^{d-1} \frac{n!d!(d+1-i)}{(n-i)!(d-i)!i!} + \frac{n!}{(n-d)!}.$$

$$p(n+1, d) = \sum_{i=0}^{d} \frac{n!d!(d+1-i)}{(n-i)!(d-i)!i!}.$$

Now we use Lemma 5, and the induction process is finished.

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