JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **86**, 542–561 (1982)

On Some New Inequalities in *n* Independent Variables

E. THANDAPANI AND RAVI P. AGARWAL

MATSCIENCE, The Institute of Mathematical Sciences, Madras 600020, India

Submitted by Peter D. Lax

1. INTRODUCTION

During the past few years there have been a number of linear and nonlinear generalizations of the well-known integral ineqality due to Gronwall–Bellman and Reid, which play an important role in studying the qualitative as well as the quantative properties of solutions of differential and integral equations. The two independent variable generalization of this inequality was given by Wendroff [2, p. 154], which is useful in the theory of partial differential and integral equations. For various motivations this inequality has been further generalised in many directions, e.g., see [3–6, 10, 11, 13–21].

In this paper we establish a number of new integral inequalities in n independent variables which are further generalizations of some known results. In Section 2 we extend the result of Young [21] to discuss the case when an inequality has repeated integrals. A unified result is also presented which covers several results of Pachpatte [15, 16]. Some Wendroff type inequalities are also obtained and known results are deduced or compared. In Section 3, some nonlinear inequalities are given which are more general than those obtained in [4, 12]. Some applications are given in Section 4. Throughout the paper we shall use the following notations. Let Ω be an open bounded set in R^n and let a point $(x_1^i,\ldots,x_n^i)$ in Ω be denoted by x^i. Let y and x $(y<x)$ be any two points in Ω and denote by D the parallelepiped defined by $y<s<x$, that is, $y_j<s_j<x_j$, $1\leqslant j\leqslant n$. The $\int_y^x \cdot\, ds$ indicates the n-fold integral $\int_{y_1}^{x_1}\cdots\int_{y_n}^{x_n}\cdot\, ds_1\cdots ds_n$, and $u_x(x)$ denotes $\partial^n u(x)/(\partial x_1\cdots\partial x_n)$.

In what follows we shall assume that the functions which appear in the inequalities are real-valued, nonnegative, continuous and defined in Ω.

2. LINEAR INEQUALITIES

LEMMA 2.1. *Let $p(s)$ be a continuous function in Ω. Then the characteristic initial value problem*

0022-247X/82/040542-20$02.00/0

$$(-1)^n v_s(s,x) - p(s)\,v(s,x) = 0 \qquad \text{in } \Omega, \tag{2.1}$$

$$v(s,x) = 1 \qquad \text{on } s_i = x_i,\ 1 \leqslant i \leqslant n \tag{2.2}$$

has a unique solution $v(s,x)$ near to x and satisfies $\prod_{i=1}^{n}(x_i - s_i) \geqslant 0$. This solution is continuous and if $p(s)$ is nonnegative, so is $v(s,x)$.

Proof. The function $v(s,x)$ is the Riemann function relative to point x. Problem (2.1), (2.2) is equivalent to the integral equation

$$v(s,x) = 1 + \int_s^x p(t)\,v(t,x)\,dt. \tag{2.3}$$

The existence, uniqueness and nonnegative property of $v(s,x)$ follows by successive approximation arguments as given in [10, 11, 18, 19] for $n = 2$ and systems. An explicit representation of $v(s,x)$ is given in [21]. Since $v(s,x)$ is continuous and $v = 1$ on $s_i = x_i$, $1 \leqslant i \leqslant n$ there is a domain D^+ containing x on which $v \geqslant 0$ even if $p(s)$ is not nonnegative.

LEMMA 2.2. *Suppose $p(x)$ and $q(x)$ are continuous functions in Ω. Let $v(s,x)$ be the solution of* (2.1), (2.2) *and let D^+ be a connected subdomain of Ω containing x such that $v \geqslant 0$ for all $s \in D^+$. If $D \subset D^+$ and*

$$u_x(x) - p(x)\,u(x) \leqslant q(x), \tag{2.4}$$

where u vanishes together with all its mixed derivatives up to order $n-1$ on $x_i = y_i$, $1 \leqslant i \leqslant n$. Then

$$u(x) \leqslant \int_y^x q(t)\,v(t,x)\,dt. \tag{2.5}$$

Proof. The proof of Lemma 2.2 follows from Young's theorem [21].

THEOREM 2.3. *Let $V(s,x)$ be the solution of characteristic initial value problem*

$$(-1)^n V_s(s,x) - \sum_{r=1}^{m} E_s^r(s,b)\,V(s,x) = 0 \qquad \text{in } \Omega, \tag{2.6}$$

$$V(s,x) = 1 \qquad \text{on } s_i = x_i,\ 1 \leqslant i \leqslant n \tag{2.7}$$

and let D^+ be a connected subdomain of Ω containing x such that $V \geqslant 0$ for all $s \in D^+$. If $D \subset D^+$ and

$$u(x) \leqslant a(x) + b(x) \sum_{r=1}^{m} E^r(x,u), \tag{2.8}$$

where

$$E^r(x, u) = \int_y^x f_{r1}(x^1) \int_y^{x^1} f_{r2}(x^2) \cdots \int_y^{x^{r-1}} f_{rr}(x^r)\, u(x^r)\, dx^r \cdots dx^1 \quad (2.9)$$

then

$$u(x) \leqslant a(x) + b(x) \int_y^x \sum_{r=1}^m E_s^r(s, a)\, V(s, x)\, ds. \quad (2.10)$$

Proof. Define a function $\phi(x)$ such that

$$\phi(x) = \sum_{r=1}^m E^r(x, u);$$

then we have

$$\phi_x(x) = \sum_{r=1}^m E_x^r(x, u)$$

and hence from (2.8)

$$\begin{aligned} \phi_x(x) &\leqslant \sum_{r=1}^m E_x^r(x, a + b\phi) \\ &= \sum_{r=1}^m E_x^r(x, a) + \sum_{r=1}^m E_x^r(x, b\phi). \end{aligned} \quad (2.11)$$

Using the nondecreasing nature of $\phi(x)$ in (2.11), we find

$$\phi_x(x) - \sum_{r=1}^m E_x^r(x, b)\, \phi(x) \leqslant \sum_{r=1}^m E_x^r(x, a),$$

where ϕ vanishes together with all its mixed derivatives up to order $n-1$ on $x_i = y_i$, $1 \leqslant i \leqslant n$.

Now an application of Lemma 2.2, provides

$$\phi(x) \leqslant \int_y^x \sum_{r=1}^m E_s^r(s, a)\, V(s, x)\, ds. \quad (2.12)$$

Result (2.10) now follows from (2.12) and $u(x) \leqslant a(x) + b(x)\, \phi(x)$.

Some particular cases of Theorem 2.3, $n = 2$ and m up to 3 have been considered recently by Pachpatte [15, Theorems 1–4; 16, Theorems 1–2], but his results cannot be compared with our result. In the next theorem we shall consider a particular case of (2.8); the obtained result unifies all his six theorems for the general n.

We shall denote $\sum_{r=1}^{r_1} b(x) f_r(x) \bigcup_{i=1}^{r_2} g_i(x)$ as the sum of all functions except when $b(x) f_k(x) = g_l(x)$ for some $1 \leqslant k \leqslant r_1$, $1 \leqslant l \leqslant r_2$; then $g_l(x)$ is taken to be zero, also $\bigcup_{i=1}^{0} g_i(x) = 0$.

THEOREM 2.4. *Let $V_i(s, x)$, $1 \leqslant i \leqslant m$, be the solutions of characteristic initial value problems*

$$(-1)^n V_{1s}(s, x) - \left(\sum_{r=1}^{m} b(s) f_r(s) \bigcup_{i=1}^{m-1} g_i(s) \right) V_1(s, x) = 0 \qquad \text{in } \Omega,$$

$$(-1)^n V_{js}(s, x) - \left(\sum_{r=1}^{m-j+1} b(s) f_r(s) \bigcup_{i=1}^{m-j} g_i(s) - g_{m-j+1}(s) \right)$$
$$\times V_j(s, x) = 0 \qquad \text{in } \Omega,\ 2 \leqslant j \leqslant m,$$
$$V_j(s, x) = 1 \qquad \text{on } s_i = x_i,\ 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m$$

and let D^+ be a connected subdomain of Ω containing x such that $V_j \geqslant 0$, $1 \leqslant j \leqslant m$ for all $s \in D^+$. If $D \subset D^+$ and (2.8) is satisfied where $f_{ii}(x) = f_i(x)$, $1 \leqslant i \leqslant m$; $f_{i+1,i}(x) = f_{i+2,i}(x) = \cdots = f_{m,i}(x) = g_i(x)$, $1 \leqslant i \leqslant m-1$; then

$$u(x) \leqslant a(x) + b(x) P_j(x), \qquad 1 \leqslant j \leqslant m, \tag{2.13}$$

where

$$P_1(x) = \int_y^x a(x^1) \sum_{r=1}^{m} f_r(x^1) V_1(x^1, x)\, dx^1,$$

$$P_j(x) = \int_y^x \left[a(x^1) \sum_{r=1}^{m-j+1} f_r(x^1) + g_{m-j+1}(x^1) P_{j-1}(x^1) \right]$$
$$\times V_j(x^1, x)\, dx^1, \qquad 2 \leqslant j \leqslant m.$$

Proof. Inequality (2.8) with functions $f_{ij}(x)$ is equivalent to the system

$$u_1(x) \leqslant a(x) + b(x) \int_y^x [f_1(s) u_1(s) + g_1(s) u_2(s)]\, ds, \tag{2.14}$$

$$u_{j-1}(x) = \int_y^x [f_{j-1}(s) u_1(s) + g_{j-1}(s) u_j(s)]\, ds, \qquad 3 \leqslant j \leqslant m, \quad (2.15)_j$$

$$u_m(x) = \int_y^x f_m(s) u_1(s)\, ds. \tag{2.16}$$

Define

$$\phi_1(x) = \int_y^x [f_1(s) u_1(s) + g_1(s) u_2(s)]\, ds,$$

$$\phi_{j-1}(x)=\int_y^x [f_{j-1}(s)\,u_1(s)+g_{j-1}(s)\,u_j(s)]\,ds, \qquad 3\leqslant j\leqslant m,$$

$$\phi_m(x)=\int_y^x f_m(s)\,u_1(s)\,ds.$$

Then, from (2.14), $(2.15)_j$, (2.16) it follows that

$$\phi_{1x}(x)\leqslant f_1(x)[a(x)+b(x)\,\phi_1(x)]+g_1(x)\,\phi_2(x) \tag{2.17}$$

$$\phi_{j-1x}(x)\leqslant f_{j-1}(x)[a(x)+b(x)\,\phi_1(x)]+g_{j-1}(x)\,\phi_j(x), \qquad 3\leqslant j\leqslant m, \quad (2.18)_j$$

$$\phi_{mx}(x)\leqslant f_m(x)[a(x)+b(x)\,\phi_1(x)]. \tag{2.19}$$

We add (2.17), $(2.18)_j$, $3\leqslant j\leqslant m$, (2.19) to obtain

$$\left(\sum_{r=1}^m \phi_r(x)\right)_x \leqslant a(x)\sum_{r=1}^m f_r(x)+b(x)\sum_{r=1}^m f_r(x)\,\phi_1(x) + \sum_{r=1}^m g_r(x)\,\phi_{r+1}(x)$$

and hence

$$\left(\sum_{r=1}^m \phi_r(x)\right)_x - \left(\sum_{r=1}^m b(x) f_r(x) \bigcup_{i=1}^{m-1} g_i(x)\right)\left(\sum_{r=1}^m \phi_r(x)\right) \leqslant a(x)\sum_{r=1}^m f_r(x). \tag{2.20}$$

Using Lemma 2.2, we find

$$\sum_{r=1}^m \phi_r(x)\leqslant P_1(x) \tag{2.21}$$

and hence

$$\phi_m(x)\leqslant P_1(x)-\sum_{r=1}^{m-1}\phi_r(x). \tag{2.22}$$

Adding (2.17), $(2.18)_j$, $3\leqslant j\leqslant m$, and making use of (2.22), we obtain

$$\left(\sum_{r=1}^{m-1} \phi_r(x)\right)_x \leqslant a(x)\sum_{r=1}^{m-1} f_r(x)+b(x)\sum_{r=1}^{m-1} f_r(x)\,\phi_1(x) + \sum_{r=1}^{m-2} g_2(x)\,\phi_{r+1}(x)+g_{m-1}(x)\left[P_1(x)-\sum_{r=1}^{m-1}\phi_r(x)\right]:$$

thus it follows that

$$\left(\sum_{r=1}^{m-1} \phi_r(x)\right)_x - \left(\sum_{r=1}^{m-1} b(x) f_r(x) \bigcup_{i=1}^{m-2} g_i(x) - g_{m-1}(x)\right)\left(\sum_{r=1}^{m-1} \phi_r(x)\right)$$
$$\leqslant a(x) \sum_{r=1}^{m-1} f_r(x) + g_{m-1}(x) P_1(x).$$

Using again Lemma 2.2, we get

$$\sum_{r=1}^{m-1} \phi_r(x) \leqslant P_2(x) \tag{2.23}$$

or

$$\phi_{m-1}(x) \leqslant P_2(x) - \sum_{r=1}^{m-2} \phi_r(x). \tag{2.24}$$

We add (2.17), $(2.18)_j$, $3 \leqslant j \leqslant m-1$, and use (2.24) to find

$$\left(\sum_{r=1}^{m-2} \phi_r(x)\right)_x - \left(\sum_{r=1}^{m-2} b(x) f_r(x) \bigcup_{i=1}^{m-3} g_i(x) - g_{m-2}(x)\right)\left(\sum_{r=1}^{m-2} \phi_r(x)\right)$$
$$\leqslant a(x) \sum_{r=1}^{m-2} f_r(x) + g_{m-2}(x) P_2(x)$$

and hence from Lemma 2.2, we obtain

$$\sum_{r=1}^{m-2} \phi_r(x) \leqslant P_3(x). \tag{2.25}$$

Continuing in this way, we find

$$\sum_{r=1}^{m-j+1} \phi_r(x) \leqslant P_j(x), \qquad 4 \leqslant j \leqslant m. \qquad (2.26)_j$$

Since $u(x) = u_1(x) \leqslant a(x) + b(x)\,\phi_1(x)$ and $\phi_1(x) \leqslant \sum_{r=1}^{m-j+1} \phi_r(x)$, $1 \leqslant j \leqslant m$, result (2.13) follows from (2.21), (2.23), (2.25), $(2.26)_j$.

For the particular case $m = 2$, $b = 1$, $f_{11} = f_{21} = f_1, f_{22} = f_2$ in (2.8), estimate (2.13) takes the form

$$u(x) \leqslant a(x) + \int_y^x f_1(x^1)$$
$$\times \left[a(x^1) + \int_y^x a(x^2)(f_1(x^2) + f_2(x^2))\, V_1(x^2, x^1)\, dx^2\right] dx^1, \tag{2.27}$$

where $V_1(s, x)$ is the solution of characteristic initial value problem

$$(-1)^n V_{1s}(s, x) - (f_1(s) + f_2(s))\, V_1(s, x) = 0 \qquad \text{in } \Omega, \tag{2.28}$$

$$V_1(s, x) = 1 \qquad \text{on } s_i = x_i,\ 1 \leqslant i \leqslant n. \tag{2.29}$$

In the next result we shall show that estimate (2.27) can be improved uniformly. The improved version of Theorem 1 in [15] is the following (here we have taken $\sigma = 0$ since it does not play any role, the term $\int_y^x b(s)\,\sigma(s)\,ds$ can always be merged in $a(x)$).

THEOREM 2.5. *Let $V_1(s,x)$ be the solution of (2.28), (2.29) and let D^+ be a connected subdomain of Ω containing x such that $V_1 \geqslant 0$ for all $s \in D^+$. If $D \subset D^+$ and (2.8) is satisfied where $m = 2$, $b = 1$, $f_{11} = f_{21} = f_1$, $f_{22} = f_2$, then*

$$u(x) \leqslant a(x) + \int_y^x f_1(x^1) \tag{2.30}$$

$$\times \left[a(x^1) + \int_y^{x^1} \{a(x^2)(f_1(x^2) + f_2(x^2)) - c(x^2)\}\, V_1(x^2, x_1)\, dx^2\right] dx^1,$$

where

$$c(x) = f_2(x) \int_y^x a(x^1) f_2(x^1)\, dx^1.$$

Proof. Define

$$\phi_1(x) = \int_y^x f_1(x^1)\, u(x^1)\, dx^1 + \int_y^x f_1(x^1) \int_y^{x^1} f_2(x^2)\, u(x^2)\, dx^2\, dx^1; \tag{2.31}$$

then, from (2.8), it follows that

$$\phi_{1x}(x) \leqslant f_1(x) \left[a(x) + \phi_1(x) + \int_y^x f_2(x^1)[a(x^1) + \phi_1(x^1)]\, dx^1\right]. \tag{2.32}$$

Let

$$\phi_2(x) = \phi_1(x) + \int_y^x f_2(x^1)[a(x^1) + \phi_1(x^1)]\, dx^1; \tag{2.33}$$

then, it follows that

$$\phi_{2x}(x) = \phi_{1x}(x) + f_2(x)[a(x) + \phi_1(x)],$$

which is, from (2.32) and (2.33),

$$\phi_{2x}(x) \leqslant f_1(x)[a(x) + \phi_2(x)] + f_2(x) \left[a(x) + \phi_2(x) - \int_y^x a(x^1) f_2(x^1)\, dx^1 \right].$$

Using Lemma 2.2, we obtain

$$\phi_2(x) \leqslant \int_y^x \{a(x^1)(f_1(x^1) + f_2(x^1)) - c(x^1)\} V_1(x^1, x)\, dx^1.$$

Substituting this in (2.32), we find

$$\phi_1(x) \leqslant \int_y^x f_1(x^1) \left[a(x^1) + \int_y^{x^1} \{a(x^2)(f_1(x^2) + f_2(x^2)) - c(x^2)\} \times V_1(x^2, x^1)\, dx^2 \right] dx^1$$

and now result (2.30) follows from $u(x) \leqslant a(x) + \phi_1(x)$.

In our next result we shall obtain a Wendroff type estimate for (2.8).

THEOREM 2.6. *Let inequality* (2.8) *be satisfied in* Ω, *where* (i) $a(x)$ *is positive and nondecreasing and* (ii) $b(x) \geqslant 1$. *Then*

$$u(x) \leqslant a(x)\, b(x) \exp\left(\sum_{r=1}^{m} E^r(x, b) \right). \tag{2.34}$$

Proof. Inequality (2.8) can be written as

$$\phi_1(x) \leqslant 1 + \sum_{r=1}^{m} E^r(x, b\phi_1), \tag{2.35}$$

where

$$\phi_1(x) = \frac{u(x)}{a(x)\, b(x)}.$$

Let $\phi_2(x)$ be the right member of (2.35), then

$$\phi_{2x}(x) \leqslant \sum_{r=1}^{m} E^r_x(x, b\phi_1) \leqslant \sum_{r=1}^{m} E^r_x(x, b\phi_2) \tag{2.36}$$

and $\phi_2(x_1,..., x_{i-1}, y_i, x_{i+1},..., x_n) = 1$; all the partial derivatives up to order $n-1$ vanish when $x_i = y_i$, for any i, $1 \leqslant i \leqslant n$.

Since $\phi_2(x)$ is nondecreasing, it follows from (2.36) that

$$\phi_{2x}(x) \leqslant \sum_{r=1}^{m} E_x^r(x, b)\, \phi_2(x)$$

or

$$\frac{\phi_{2x}(x)}{\phi_2(x)} \leqslant \sum_{r=1}^{m} E_x^r(x, b) + \frac{(\phi_{2x_n}(x))(\phi_{2x_1 \cdots x_{n-1}}(x))}{\phi_2^2(x)}$$

and hence

$$\left(\frac{\phi_{2x_1 \cdots x_{n-1}}(x)}{\phi_2(x)}\right)_{x_n} \leqslant \sum_{r=1}^{m} E_x^r(x, b).$$

Keeping $x_1, \ldots, x_{n-1}$ fixed in the above inequality and setting $x_n = s_n$ and integrating with respect to s_n from y_n to x_n, we obtain

$$\begin{aligned}\left(\frac{\phi_{2x_1 \cdots x_{n-1}}(x)}{\phi_2(x)}\right) &\leqslant \int_{y_n}^{x_n} \sum_{r=1}^{m} E^r_{x_1 \cdots x_{n-1} s_n}(x_1, \ldots, x_{n-1}, s_n, b)\, ds_n \\ &= \sum_{r=1}^{m} E^r_{x_1 \cdots x_{n-1}}(x, b).\end{aligned}$$

Repeating the above argument for x_{n-1}, x_{n-2}, to x_2, we obtain

$$\frac{\phi_{2x_1}(x)}{\phi_2(x)} \leqslant \sum_{r=1}^{m} E^r_{x_1}(x, b).$$

Integrating the above inequality with respect to x_1 and using $\phi_2(y_1, x_2, \ldots, x_n) = 1$, we find

$$\phi_2(x) \leqslant \exp\left(\sum_{r=1}^{m} E^r(x, b)\right).$$

Result (2.34) now follows from $\phi_1(x) \leqslant \phi_2(x)$ and the definition of $\phi_1(x)$.

Estimate (2.34) for $n = 2$, $m = 1$ is sharper than that given in [2, p. 154] and the same as that obtained by Kasture and Deo [13, Theorem 9]. Some results are given in [5] for $n = 2$, $b = 1$, m up to 2 with different assumptions on $a(x)$. In our next result we do not require any condition on $a(x)$ and $b(x)$ as in Theorem 2.6, also estimate (2.34) can be reobtained.

THEOREM 2.7. *Let inequality* (2.8) *be satisfied in* Ω. *Then*

$$u(x) \leqslant a(x) + b(x) \int_y^x \sum_{r=1}^{m} E_s^r(s, a) \exp\left(\int_s^x \sum_{r=1}^{m} E_t^r(t, b)\, dt\right) ds. \quad (2.37)$$

Proof. Define

$$w(s, x) = \exp\left(\int_s^x \sum_{r=1}^m E_t^r(t, b)\, dt\right);$$

then, it follows that

$$\begin{aligned} &(-1)^n w_s(s, x) - \sum_{r=1}^m E_s^r(s, b)\, w(s, x) \geqslant 0, \\ &w(s, x) = 1 \qquad \text{on} \quad s_i = x_i,\ 1 \leqslant i \leqslant n. \end{aligned} \tag{2.38}$$

Consequently, $w(s, x)$ satisfies a differential inequality (2.38) of which $V(s, x)$ is the exact solution (Theorem 2.3). It follows from [20, pp. 126, 130] that $w(s, x) \geqslant V(s, x)$, and now (2.37) follows from (2.10).

In case the conditions on $a(x)$ (can be nonnegative) and $b(x)$ of Theorem 2.6 are satisfied then, from (2.37), we get

$$u(x) \leqslant a(x)\, b(x) \left[1 + \int_y^x \sum_{r=1}^m E_s^r(s, b) \exp\left(\int_s^x \sum_{r=1}^m E_t^r(t, b)\, dt\right) ds\right]. \tag{2.39}$$

We use (2.38) in (2.39), to obtain

$$u(x) \leqslant a(x)\, b(x) \left[1 + (-1)^n \int_y^x w_s(s, x)\, ds\right]. \tag{2.40}$$

Now, using the fact that the partial derivatives of $w(s, x)$ up to order $n-1$ vanishes on $s_i = x_i$, $1 \leqslant i \leqslant n$, it follows from (2.40) that

$$u(x) \leqslant a(x)\, b(x) \left[1 + (-1)^{2n-1} \int_{y_1}^{x_1} w_{s_1}(s_1, y_2, \dots, y_n, x)\, ds_1\right]$$

and hence

$$u(x) \leqslant a(x)\, b(x)[1 + (-1)^{2n-1}(w(x_1, y_2, \dots, y_n, x) - w(y, x))]$$

or

$$u(x) \leqslant a(x)\, b(x)\, w(y, x),$$

which is the same as (2.34).

Thus to obtain (2.34) in Theorem 2.7, we require $a(x)$ to be nonnegative and nondecreasing.

3. Nonlinear Inequalities

Our first result for the nonlinear case is connected with the inequality

$$u(x) \leqslant a(x)\left[c + \sum_{r=1}^{m} H^r(x, u)\right], \tag{3.1}$$

where

$$H(x, u) = \int_y^x f_{r1}(x^1)\, u^{\alpha_{r1}}(x^1) \cdots \int_y^{x^{r-1}} f_{rr}(x^r)\, u^{\alpha_{rr}}(x^r)\, dx^r \cdots dx^1$$

and α_{ri}, $1 \leqslant i \leqslant r$, $1 \leqslant r \leqslant m$ are nonnegative real numbers and the constant $c > 0$.

In the following result we shall denote $\alpha_r = \sum_{i=1}^{r} \alpha_{ri}$ and $\alpha = \max_{1 \leqslant r \leqslant m} \alpha_r$.

THEOREM 3.1. *Let inequality* (3.1) *be satisfied in* Ω. *Then*

$$u(x) \leqslant ca(x) \exp\left(\int_y^x Q(s)\, ds\right), \qquad \text{if } \alpha = 1, \tag{3.2}$$

$$u(x) \leqslant a(x)\left[c^{1-\alpha} + (1-\alpha)\int_y^x Q(x)\, ds\right]^{1/1-\alpha}, \qquad \text{if } \alpha \neq 1, \tag{3.3}$$

where

$$Q(x) = \sum_{r=1}^{m} H_x^r(x, a)\, c^{\alpha_r - \alpha}$$

and when $\alpha > 1$, *we assume*, $c^{1-\alpha} + (1-\alpha)\int_y^x Q(s)\, ds > 0$.

Proof. Inequality (3.1) can be written as

$$u(x) \leqslant a(x)\, \phi(x), \tag{3.4}$$

where

$$\phi(x) = c + \sum_{r=1}^{m} H^r(x, u).$$

Thus, on using the nondecreasing nature of $\phi(x)$ and (3.4), we find

$$\phi_x(x) \leqslant \sum_{r=1}^{m} H_x^r(x, a)[\phi(x)]^{\alpha_r}.$$

Since $\phi(x) \geqslant c$, we get

$$\phi_x(x) \leqslant \sum_{r=1}^{m} H_x^r(x, a)\, c^{\alpha_r - \alpha} \phi^\alpha(x)$$
$$= Q(x)\, \phi^\alpha(x).$$

Now following the proof of Theorem 2.6, it is easy to show that

$$\frac{\phi_{x_1}(x)}{\phi^\alpha(x)} \leqslant \int_{y_2}^{x_2} \cdots \int_{y_n}^{x_n} Q(x_1, s_2,..., s_n)\, ds_2 \cdots ds_n. \tag{3.5}$$

Since $\phi(y_1, x_2,..., x_n) = c$, the result follows on integrating (3.5).

For $n = m = 1$, $a(x) = 1$, $\alpha_{11} = 2$ Theorem 3.1 reduces to first result in this direction by Freedman [9]; also for m up to 2 see [14].

For the next result we shall need the following class of functions:

DEFINITION. A function W: $[0, \infty) \to (0, \infty)$ is said to belong to the class S if

(i) $W(u)$ is positive, nondecreasing, continuous and $W_{x_k}(u(x_1,..., x_n)) \geqslant 0$ for all $2 \leqslant k \leqslant n$ and $u \geqslant 0$,

(ii) $(1/v)\, W(u) \leqslant W(u/v)$ for all $u \geqslant 0$, $v \geqslant 1$.

This class has been modified here as given and used for $n = 1$ in [7, 8] to avoid the triviality $W(u) = uW(1)$; also see [1].

THEOREM 3.2. *Let the inequality*

$$u(x) \leqslant a(x) + \sum_{r=1}^{m} E^r(x, u) + \sum_{i=1}^{l} g_i(x) \int_y^x h_i(s)\, W_i(u(s))\, ds \tag{3.6}$$

be satisfied, where

(i) $a(x) \geqslant 1$ *and nondecreasing,*

(ii) $g_i(x) \geqslant 1$, $1 \leqslant i \leqslant l$,

(iii) $W_i \in S$, $1 \leqslant i \leqslant l$.

Then

$$u(x) \leqslant a(x)\, \psi(x)\, e(x) \prod_{i=1}^{l} F_i(x),$$

where

$$\psi(x) = \exp\left(\sum_{r=1}^{m} E^r(x, e)\right),$$

$$e(x) = \prod_{i=1}^{l} g_i(x),$$

$$F_k(x) = G_k^{-1}\left[G_k(1) + \int_y^x h_k(s)\,\psi(s)\,e(s) \prod_{j=1}^{k-1} F_j(s)\,ds\right],$$

$$F_0(x) = 1,\ 1 \leqslant k \leqslant l,$$

$$G_k(\theta) = \int_{\theta_0}^{\theta} \frac{ds}{W_k(s)}, \qquad 0 < \theta_0 \leqslant \theta$$

as long as

$$G_k(1) + \int_y^x h_k(s)\,\psi(s)\,e(s) \prod_{j=1}^{k-1} F_j(s)\,ds \in \mathrm{Dom}(G_k^{-1}), \qquad 1 \leqslant k \leqslant l.$$

Proof. From inequality (3.6), we have

$$\frac{u(x)}{e(x)} \leqslant a^*(x) + \sum_{r=1}^{m} E^r\left(x, \frac{eu}{e}\right),$$

where

$$a^*(x) = a(x) + \sum_{i=1}^{l} \int_y^x h_i(s)\,W_i(u(s))\,ds.$$

Since $a^*(x)$ is nondecreasing, from Theorem 2.6 it follows that

$$\frac{u(x)}{e(x)} \leqslant a^*(x)\,\psi(x)$$

and hence on using the definition of class S

$$y(x) \leqslant 1 + \sum_{i=1}^{l} \int_y^x h_i(s)\,e(s)\,\psi(s)\,W_i(y(s))\,ds,$$

where

$$y(x) = \frac{u(x)}{a(x)\,\psi(x)\,e(x)}.$$

Thus it is sufficient to show that $y(x) \leqslant \prod_{i=1}^{l} F_i(x)$; this we shall prove by finite induction. For $l = 1$, we have

$$y(x) \leqslant 1 + \int_y^x h_1(s)\, e(s)\, \psi(s)\, W_1(y(s))\, ds.$$

Let $\phi_1(x)$ be the right member of the above inequality; then on using nondecreasing nature of W_1, we find

$$\phi_{1x}(x) \leqslant h_1(x)\, e(x)\, \psi(x)\, W_1(\phi_1(x))$$

or

$$\left(\frac{\phi_{1x_1 \cdots x_{n-1}}(x)}{W_1(\phi_1(x))}\right)_{x_n} \leqslant h_1(x)\, e(x)\, \psi(x)$$

and hence as in Theorem 2.6

$$\frac{\phi_{1x_1 \cdots x_{n-1}}(x)}{W_1(\phi_1(x))} \leqslant \int_{y_n}^{x_n} h_1(x_1, ..., x_{n-1}, s_n)\, e(x_1, ..., x_{n-1}, s_n) \times \psi(x_1, ..., x_{n-1}, s_n)\, ds_n.$$

Repeating the procedure, we obtain

$$\frac{\phi_{1x_1}(x)}{W_1(\phi_1(x))} \leqslant \int_{y_2}^{x_2} \cdots \int_{y_n}^{x_n} h_1(x_1, s_2, ..., s_n)\, e(x_1, s_2, ..., s_n) \times \psi(x_1, s_2, ..., s_n)\, ds_2 \cdots ds_n. \tag{3.7}$$

From the definition of G_1, we have

$$\begin{aligned} G_1(\phi_1(x)) - G_1(\phi_1(y_1, x_2, ..., x_n)) &= \int_{\phi_1(y_1, x_2, ..., x_n)}^{\phi_1(x)} \frac{ds}{W_1(s)} \\ &= \int_{y_1}^{x_1} \frac{\phi_{1s_1}(s_1, x_2, ..., x_n)}{W_1(\phi_1(s_1, x_2, ..., x_n))}\, ds_1. \end{aligned} \tag{3.8}$$

We use (3.7) in (3.8) to obtain

$$\phi_1(x) \leqslant G_1^{-1}\left[G_1(1) + \int_y^x h_1(s)\, e(s)\, \psi(s)\, ds\right] = F_1(x).$$

Now assuming that the result is true for some k such that $1 \leqslant k \leqslant l-1$, then for $k+1$, we are given

$$y(x) \leqslant \left[1 + \int_y^x h_{k+1}(s)\, e(s)\, \psi(s)\, W_{k+1}(y(s))\, ds\right] + \sum_{i=1}^{k} \int_y^x h_i(s)\, e(s)\, \psi(s)\, W_i(y(s))\, ds.$$

Since the part inside the bracket is nondecreasing, we find

$$y(x) \leqslant \left[1 + \int_y^x h_{k+1}(s)\, e(s)\, \psi(s)\, W_{k+1}(y(s))\, ds\right] \prod_{i=1}^{k} F_i(x)$$

or

$$\frac{y(x)}{\prod_{i=1}^{k} F_i(x)} \leqslant 1 + \int_y^x h_{k+1}(s)\, e(s)\, \psi(s) \times \prod_{i=1}^{k} F_i(s)\, W_{k+1}\left(y(s) \Big/ \prod_{i=1}^{k} F_i(s)\right) ds$$

and from this $y(x) \leqslant \prod_{i=1}^{k+1} F_i(x)$ follows on using the same arguments as for the case $l = 1$. This completes the proof.

THEOREM 3.3. *In addition to the hypothesis of Theorem* 3.2 *let* $g_i(x)$, $1 \leqslant i \leqslant l$ *be nondecreasing. Then*

$$u(x) \leqslant a(x)\, \psi_1(x) \prod_{i=1}^{l} F_i(x),$$

where

$$\psi_1(x) = \exp\left(\sum_{r=1}^{m} E^r(x, 1)\right),$$

$$F_k(x) = g_k(x)\, G_k^{-1}\left[G_k(1) + \int_y^x h_k(s)\, \psi_1(s)\, g_k(s) \prod_{i=1}^{k-1} F_i(s)\, ds\right], \quad 1 \leqslant k \leqslant l,\ F_0(x) = 1,$$

as long as

$$G_k(1) + \int_y^x h_k(s)\, \psi_1(s)\, g_k(s) \prod_{i=1}^{k-1} F_i(s)\, ds \in \mathrm{Dom}(G_k^{-1}), \qquad 1 \leqslant k \leqslant l.$$

THEOREM 3.4. *Let the inequality*

$$u(x) \leqslant a(x) + \sum_{r=1}^{m} E^r(x, u) + \sum_{i=1}^{l} E^i(x, W(u)) \tag{3.9}$$

be satisfied, where

(i) $a(x) \geqslant 1$ *and nondecreasing,*

(ii) $W \in S$.

Then

$$u(x) \leqslant a(x)\, \psi_1(x)\, G^{-1}\left[G(1) + \int_y^x \sum_{i=1}^{l} E_s^i(s, \psi_1)\, ds\right], \tag{3.10}$$

where $\psi_1(x)$ *is same as in Theorem* 3.3 *and the term inside the bracket of* $(3.10) \in \mathrm{Dom}(G^{-1})$.

The proofs of Theorems 3.3 and 3.4 are similar to the proof of Theorem 3.2.

THEOREM 3.5. *Let inequality* (3.9) *be satisfied, where*

(i) $a(x)$ *is positive and nondecreasing,*

(ii) W *is positive, continuous, nondecreasing, submultiplicative and* $W_{x_k}(u(x_1, \ldots, x_n)) \geqslant 0$ *for all* $2 \leqslant k \leqslant n$.

Then

$$u(x) \leqslant a(x)\, \psi_1(x)\, G^{-1}\left[G(1) + \int_y^x \sum_{r=1}^{l} E_s^r\left(s, \frac{W(a\psi_1)}{a}\right) ds\right], \tag{3.11}$$

where $\psi_1(x)$ *is the same as that in Theorem* 3.3 *and the term inside the bracket of* $(3.11) \in \mathrm{Dom}(G^{-1})$.

Proof. We apply Theorem 2.6 for inequality (3.9), to obtain

$$u(x) \leqslant \left[a(x) + \sum_{i=1}^{l} E^i(x, W(u))\right] \psi_1(x)$$

or

$$\frac{u(x)}{a(x)\, \psi_1(x)} \leqslant 1 + \sum_{i=1}^{l} E^i\left(x, W\left(\frac{u}{a\psi_1}\, a\psi_1\right)\Big/ a\right). \tag{3.12}$$

Let $\phi(x)$ be the right-hand side of (3.12), then

$$\phi_x(x) = \sum_{i=1}^{l} E_x^i\left(x, W\left(\frac{u}{a\psi_1} a\psi_1\right)\Big/a\right).$$

Now using the fact that W is nondecreasing and submultiplicative, we get

$$\frac{\phi_x(x)}{W(\phi(x))} \leqslant \sum_{i=1}^{l} E_x^i(x, W(a\psi_1)/a).$$

Using the same arguments as those in Theorem 3.2, we find

$$\phi(x) \leqslant G^{-1}\left[G(1) + \int_y^x \sum_{i=1}^{l} E_s^i(s, W(a\psi_1)/a)\, ds\right]$$

and from this the result follows.

Some particular cases $n = 2$, m up to 2 with different assumptions on $a(x)$, have been discussed recently in [4].

4. Some Applications

The results obtained in Sections 2 and 3 can be directly used to prove the uniqueness and continuous dependence for the solutions of hyperbolic differential systems and hyperbolic integrodifferential equations of a more general type then those given in [3–5, 10, 11, 13–16, 18–20], since the arguments are similar the details are not repeated here. To show the importance of our results we shall use our Theorem 2.7 to provide an upper bound on the solutions of the nonlinear hyperbolic integrodifferential equation

$$u_x(x) = f\left(x, u(x), \int_y^x k(x, s, u(s))\, ds\right) \tag{4.1}$$

together with the given suitable boundary conditions $u(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n)$, $1 \leqslant i \leqslant n$.

The functions f and k are continuous on their respective domains of definitions and

$$|f(x, u(x), v(x))| \leqslant f_{11}(x)\,|u(x)| + f_{12}(x)\,|v(x)|, \tag{4.2}$$

$$|k(x, s, u(s))| \leqslant f_{22}(s)\,|u(s)|, \tag{4.3}$$

where f_{11}, f_{12}, f_{22} are the same as those appearing in (2.8).

Any solution $u(x)$ of (4.1) satisfying the boundary conditions is also a solution of the Volterra integral equation

$$u(x) = a(x) + \int_y^x f\left(x^1, u(x^1), \int_y^{x^1} k(x^1, x^2, u(x^2))\, dx^2\right) dx^1, \tag{4.4}$$

where $a(x)$ takes care of the boundary conditions.

We use (4.2), (4.3) in (4.4), to obtain

$$|u(x)| \leqslant |a(x)| + \int_y^x \left[f_{11}(x^1)\, |u(x^1)| + f_{12}(x^1) \int_y^{x^1} f_{22}(x^2)\, |u(x^2)|\, dx^2 \right] dx^1. \tag{4.5}$$

From Theorem 2.7, we find

$$|u(x)| \leqslant |a(x)| + \int_y^x \left[f_{11}(x^1)\, |a(x^1)| + f_{12}(x^1) \int_y^{x^1} f_{22}(x^2)\, |a(x^2)|\, dx^2 \right] \times \exp\left(\int_{x^1}^x \left[f_{11}(x^2) + f_{12}(x^2) \int_y^{x^2} f_{22}(x^3)\, dx^3 \right] dx^2 \right) dx^1 \Bigg]. \tag{4.6}$$

If, $|a(x)| \leqslant M$, where $M > 0$ is a constant, then from (4.6) or (4.5) with Theorem 2.6, we get

$$|u(x)| \leqslant M \exp\left(\int_y^x \left[f_{11}(x^1) + f_{12}(x^1) \int_y^{x^1} f_{22}(x^2)\, dx^2 \right] dx^1 \right). \tag{4.7}$$

Further, if $f_{11} = f_{12}$ then from (4.7), we obtain

$$|u(x)| \leqslant M \exp\left(\int_y^x f_{11}(x^1) \left[1 + \int_y^{x^1} f_{22}(x^2)\, dx^2 \right] dx^1 \right). \tag{4.8}$$

Estimate (4.8) is not comparable with

$$|u(x)| \leqslant M \left[1 + \int_y^x f_{11}(x^1) \exp\left(\int_y^{x^1} [f_{11}(x^2) + f_{22}(x^2)]\, dx^2 \right) dx^1 \right] \tag{4.9}$$

as obtained in [5] for $n = 2$.

In order for $|u(x)|$ to remain bounded in (4.9) it is necessary to have

$$\int_y^x [f_{11}(x^1) + f_{22}(x^1)]\, dx^1 < \infty.$$

which is the same as

$$\int_y^x f_{11}(x^1)\, dx^1 < \infty, \int_y^x f_{22}(x^1)\, dx^1 < \infty. \tag{4.10}$$

In (4.8), we require

$$\int_y^x f_{11}(x^1) \left[1 + \int_y^{x^1} f_{22}(x^2)\, dx^2 \right] < \infty, \tag{4.11}$$

which is obviously satisfied if (4.10) holds, but in several cases (4.11) is more general than (4.10), for example, let $f_{22}(x) = \exp(\sum_{i=1}^n (x_i - y_i))$ and $f_{11}(x) = \exp(-2 \sum_{i=1}^n (x_i - y_i))$; for this (4.10) is not satisfied, whereas (4.11) holds. Thus the results obtained here will be applicable to more general situations.

Acknowledgments

The authors are thankful to the referee for his suggestions on the first draft of this paper.

References

1. P. R. Beesack, On Lakshmikantham's comparison method for Gronwall inequality, *Ann. Polon. Math.* **35** (1977), 187–222.
2. E. F. Beckenbach and R. Bellman, "Inequalities," Springer-Verlag, Berlin, 1961.
3. B. K. Bondge, "Some Contributions to Integral Inequalities," Ph. D. thesis, Marathwada University, India, 1979.
4. B. K. Bondge and B. G. Pachpatte, On nonlinear integral inequalities of the Wendroff type, *J. Math. Anal. Appl.* **70** (1979), 161–169.
5. B. K. Bondge, B. G. Pachpatte, and W. Walter, On generalized Wendroff type inequalities and their applications, *Nonlinear Anal. Theory Methods Appl.* **4** (1980), 491–495.
6. J. Chandra and P. W. Davis, Linear generalizations of Gronwall's inequality, *Proc. Amer. Math. Soc.* **60** (1976), 157–160.
7. S. G. Deo and U. D. Dhongade, Some generalizations of Bellman–Bihari integral inequalities, *J. Math. Anal. Appl.* **44** (1973), 218–226.
8. U. D. Dhongade and S. G. Deo, Pointwise estimates of solutions of some Volterra integral equations, *J. Math. Anal. Appl.* **45** (1974), 615–628.
9. H. Freedman, An explicity estimate on the norm of the inverse of a matrix, *SIAM Rev.* **11** (1967), 254–256.

10. S. GHOSHAL AND M. A. MASSOD, Gronwall's vector inequality and its applications to a class of non-self adjoint linear and nonlinear hyperbolic partial differential equations, *J. Indian Math. Soc.* **38** (1974), 383–394.
11. S. GHOSHAL, A. GHOSHAL, AND M. A. MASSOD, Generalized n-dimensional Gronwall's inequality and its applications to non-selfadjoint and non-linear hyperbolic equations, *Ann. Polon. Math.* **33** (1977), 223–233.
12. V. B. HEADLEY, A multidimensional non-linear Gronwall inequality, *J. Math. Anal. Appl.* **47** (1974), 250–255.
13. D. Y. KASTURE AND S. G. DEO, Inequalities of Gronwall type in two independent variables, *J. Math. Anal. Appl.* **58** (1977), 361–372.
14. B. G. PACHPATTE, A note on Gronwall–Bellman inequality, *J. Math. Anal. Appl.* **44** (1973), 758–762.
15. B. G. PACHPATTE, On some new integral and integrodifferential inequalities in two independent variables and their applications, *J. Differential Equations* **33** (1979), 249–272.
16. B. G. PACHPATTE, On some fundamental partial integral inequalities, *J. Math. Anal. Appl.* **73** (1980), 238–251.
17. R. P. SHASTRI AND D. Y. KASTURE, Wendroff type inequalities, *Proc. Amer. Math. Soc.* **72** (1978), 248–250.
18. D. R. SNOW, A two independent variable Gronwall type inequality, *in* "Inequalities" Vol. III, pp. 333–340, Academic Press, New York, 1972.
19. D. R. SNOW, Gronwall's inequality for systems of partial differential equations in two independent variables, *Proc. Amer. Math. Soc.* **33** (1972), 46–54.
20. W. WALTER, "Differential and Integral Inequalities", Springer-Verlag, Berlin, 1970.
21. E. C. YOUNG, Gronwall's ineequality in n independent variables, *Proc. Amer. Math. Soc.* **41** (1973), 241–244.