Phi-Accretive Operators and Ekeland's Theorem

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Let $X$ and $Y$ be Banach spaces, $\varphi: X \to Y^*$ and $P: X \to Y$; $P$ is said to be strongly \(\varphi\)-accretive if $\langle Px - Py, \varphi(x - y) \rangle \geq c \|x - y\|^2$ for some $c > 0$ and each $x, y \in X$. These mappings constitute a generalization simultaneously of monotone mappings (when $Y = X^*$) and accretive mappings (when $Y = X$). By applying a theorem of I. Ekeland, it is shown that a localized class of these mappings must be surjective under appropriate geometric assumptions on $Y^*$ and continuity assumptions on $P$. The results generalize two theorems of F. E. Browder and the proofs further refine the methodology for dealing with such mappings.

Let $X$ and $Y$ be Banach spaces with $Y^*$ the dual of $Y$, and let $\varphi$ be a mapping of $X$ into $Y^*$ such that,

1. $\varphi(X)$ is dense in $Y^*$,
2. for each $x \in X$ and each $\xi > 0$, $\|\varphi(x)\| \leq \|x\|$ and $\varphi(\xi x) = \xi \varphi(x)$.

A mapping $P$ from $X$ to $Y$ is said to be strongly $\varphi$-accretive ([1 or 8]) if there is a constant $c > 0$ such that, for all $x, u \in X$

$$\langle Px - Pu, \varphi(x - u) \rangle \geq c \|x - u\|^2.$$

The $\varphi$-accretive mappings were introduced in an effort to unify the theories for monotone mappings (when $Y = X^*$) and for accretive mappings (when $Y = X$). While the theorems obtained for the monotone and the accretive operators are very similar in character, the methods employed are fundamentally different and the goal in the study of $\varphi$-accretive operators is to develop a new methodology which is applicable to both the monotone and the accretive operators. Fundamental progress in this direction has been made by Browder ([1-4], e.g.); his techniques employed a nonconvex Bishop–Phelps lemma [3, Lemma 2] and rely upon some fairly deep observations on the geometry of arbitrary Banach spaces. More recently, Kirk in [7] has succeeded in clarifying some of Browder’s methods (as well as obtaining somewhat more general results) by applying a generalization of the Bishop–Phelps lemma due to Ekeland [6]. It is our purpose in this note to continue
this latter trend, and, in particular, to give elementary proofs of two theorems of Browder [4, Theorem 4(I,II)]; in addition, our methods enable us to extend these results to the class of locally \( \phi \)-accretive mappings introduced in [7] as well as to relax some of the geometric assumptions involved.

**Definition 1 [7].** Suppose \( \phi: X \rightarrow Y^* \) satisfies assumptions (1) and (2). The mapping \( P: X \rightarrow Y \) is said to be locally strongly \( \phi \)-accretive if, for each \( y \in Y \) and \( r > 0 \), there is a \( c > 0 \) such that the following condition holds:

\[
(3) \quad \text{If } \| P_x - y \| \leq r, \text{ then, for all } u \in X \text{ sufficiently near to } x, \quad \langle P_u - P_x, \phi(u - x) \rangle \geq c \| x - u \|^2.
\]

Before stating our results, we need to recall some further definitions. For a Banach space \( Y \) we denote by \( J \) the duality mapping from \( Y \) to \( 2^{Y^*} \) given by

\[
J(y) = \{ f \in Y^*: \| f \|^2 = \| y \|^2 = \langle y, f \rangle \},
\]

where \( \langle \cdot, \cdot \rangle \) denotes the generalized duality pairing. If \( Y^* \) is strictly convex, then \( J \) is single valued, while if \( Y^* \) is uniformly convex, then \( J \) is uniformly continuous on bounded sets; moreover, each of these properties of \( J \) characterizes the convexity on the norm of \( Y^* \) (see [9, 10]).

**Theorem 1.** Let \( X \) and \( Y \) be Banach spaces and \( P \) a locally Lipschitzian and locally strongly \( \phi \)-accretive mapping of \( X \) into \( Y \). If \( Y^* \) is strictly convex and \( J \) is continuous and \( P(X) \) is closed in \( Y \), then \( P(X) = Y \).

Under somewhat stronger assumptions on \( \phi \) and \( J \), the continuity assumptions on \( P \) can be relaxed.

**Theorem 2.** Let \( X \) and \( Y \) be Banach spaces and \( P \) a locally strongly \( \phi \)-accretive mapping of \( X \) into \( Y \). Suppose further that \( Y^* \) is uniformly convex, \( J \) is Lipschitzian on bounded subsets of \( Y \), \( \phi(X) = Y^* \) and \( P(X) \) is closed in \( Y \). If \( P \) satisfies the following lip\(-\frac{1}{2}\) condition

\[
(4) \quad \| x - u \|^{-1/2} \| P_x - P_u \| \rightarrow 0 \text{ as } x \rightarrow u, \text{ then } P(X) = Y.
\]

We note that if \( P \) is (globally) strongly \( \phi \)-accretive, then it follows routinely from the definitions that \( P(X) \) is closed in \( Y \) since \( c \| x - u \| \leq \| P_x - P_u \| \) for all \( x, u \in X \). This need not be the case for mappings of the localized class, hence the assumption that \( P(X) \) is closed. Besides extending Browder's results to the locally \( \phi \)-accretive mappings, each of the above theorems somewhat weakens the geometric assumptions made on the space \( Y \). In particular, in Browder's version of our Theorem 1 (Theorem 4(I)) it is assumed that \( Y^* \) is uniformly convex, and hence, that \( J \) is uniformly continuous on bounded sets. In Theorem 4(II) Browder assumes that both \( Y \) and \( Y^* \) are uniformly convex in addition to our assumptions in Theorem 2.
We note also that Kirk has obtained a version of Theorem 2 [7, Theorem 4] for locally $\varphi$-accretive mappings which need not have closed range; however, he retains the original assumption that $Y$ is uniformly convex.

While we are able to obtain somewhat sharper results than in [4 or 7], we feel our major contribution is in further refinement of the methodology for obtaining mapping theorems for $\varphi$-accretive operators. Our approach, which is fairly elementary and direct, uses the following formulation of Ekeland's theorem due to Caristi [5]:

**Theorem C.** Let $(D, \rho)$ be a complete metric space, $g$ an arbitrary mapping of $D$ into itself and $\psi$ a lower-semicontinuous mapping of $D$ into the nonnegative reals. Suppose for each $x \in D$

$$\rho(x, g(x)) \leq \psi(x) - \psi(g(x)).$$

(5)

Then $g$ has a fixed point in $D$.

We will also use the fact that if $g(y) = \frac{1}{2} \|y\|^2$, then $J$ is the subgradient of $g$, i.e.,

$$\|x\|^2 \leq \|y\|^2 - 2\langle y - x, J(x) \rangle$$

(6)

for all $x, y \in Y$. Finally, for $E = X$ or $E = Y$, we denote by $B(x; r)$ the set

$$B(x; r) = \{w \in E : \|w - x\| \leq r\}.$$ proof of Theorem 1. We will show $P(X)$ is open in $Y$, from which $P(X) = Y$. Fix $x_0 \in X$ and select $\varepsilon_1 > 0$ so small that $P$ is Lipschitzian with constant $M$ on $B(x_0, 2\varepsilon_1)$. Choose $c > 0$ and $\varepsilon_2 > 0$ so that (3) holds on $B(Px_0; 2M\varepsilon_1)$ whenever $\|u - x_0\| \leq 2\varepsilon_1$; set $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$ and set $r = \min\{\varepsilon c/2, M\varepsilon\}$. It suffices to show that $P$ maps onto $B(Px_0; r)$; thus, we fix $y \in B(Px_0; r)$ and show $y \in P(X)$.

We first observe that if $x \in B(x_0; 2\varepsilon)$ and $Px \in B(y; r)$, then

$$\|x - x_0\| \leq (1/c) \|Px - Px_0\| \leq (1/c)(\|P_x - y\| + \|y - Px_0\|) \leq 2r/c \leq \varepsilon,$$

i.e.,

$$\|x - x_0\| \leq \varepsilon \quad \text{if} \quad x \in B(x_0; 2\varepsilon) \quad \text{and} \quad Px \in B(y; r).$$

(7)

Now set $d = \text{dist}(y, P(X))$ and suppose that $d > 0$. Set

$$D = \{x \in B(x_0; \varepsilon) : \|y - Px\| \leq r\}$$
and fix $x \in D$. Choose $h \in X$ such that $\|h\| \geq 1$ and

$$\|\phi(h) - y - Px\|^{-1} J(y - Px) \leq c/2M; \tag{8}$$

set $x_t = x + th$. We will show that, for $t > 0$ sufficiently small, $x_t \in D$.

First observe that, for $t$ sufficiently small, (3) implies

$$\langle Px_t - Px, \phi(x_t - x) \rangle \geq c \|x_t - x\|^2,$$

which in turn implies

$$\langle Px_t - Px, \phi(h) \rangle \geq ct\|h\|^2.$$

Now if $t$ is so small that $\|x_t - x\| \leq \varepsilon$ (so $x_t \in B(x_0; 2\varepsilon, \varepsilon)$) it follows from the above, the fact that $\|h\| \geq 1$ and $\|x_t - x\| = \varepsilon\|x\|$ that

$$\langle Px_t - Px, \phi(h) \rangle \geq (c/M) \|Px_t - Px\|.$$

Thus, applying (8),

$$\langle Px_t - Px, J(y - Px) \rangle = \langle Px_t - Px, \|y - Px\| \phi(h) - \|y - Px\| \phi(h) + J(y - Px) \rangle$$

$$\geq (c/M) \|y - Px\| \|Px_t - Px\|$$

$$- (c/2M) \|y - Px\| \|Px_t - Px\|$$

i.e.,

$$\langle Px_t - Px, J(y - Px) \rangle \geq (c/2M) \|y - Px\| \|Px_t - Px\|. \tag{9}$$

We now apply (6) and (9) to estimate $\|y - Px_t\|$.

$$\|y - Px_t\|^2 \leq \|y - Px\|^2 - 2\langle Px_t - Px, J(y - Px) \rangle$$

$$\leq \|y - Px\|^2 - 2\langle Px_t - Px, J(y - Px) - J(y - Px) + J(y - Px) \rangle$$

$$\leq \|y - Px\|^2 - (cd/M) \|Px_t - Px\|$$

$$+ 2 \|Px_t - Px\| \|J(y - Px) - J(y - Px)\|.$$  

Now continuity of $J$ implies $J(y - Px_t) \to J(y - Px)$ as $t \to 0$, and thus, we may choose $t > 0$ so small that

$$\|J(y - Px) - J(y - Px_t)\| \leq cd/4M;$$

this yields

$$\|y - Px_t\|^2 \leq \|y - Px\|^2 - (cd/M) \|Px_t - Px\| + (cd/2M) \|Px_t - Px\| \tag{10}$$

$$= \|y - Px\|^2 - (cd/2M) \|Px_t - Px\|.$$
From this \( \| y - Px \| < \| y - P_x \| \leq r \). Now \( x_i \in B(x_0; 2\epsilon) \) by assumption, so (7) implies \( x_i \in D \). We have also chosen \( t \) so small that (3) holds, and, hence, \( c \| x_t - x \| \leq \| Px_t - P_x \| \). Setting \( g(x) = x_t \), we see that \( g: D \to D \), and moreover, (10) implies (5) holds:

\[
\| g(x) - x \| \leq (2M/c^2d) [\| y - P_x \|^2 - \| y - Pg(x) \|^2],
\]

and so Theorem C, with \( \psi(x) = (2M/c^2d) \| y - P_x \|^2 \), implies \( g \) has a fixed point. Since \( \| x_t - x \| = t \| h \| > 0 \), this is in contradiction to our assumption that \( d > 0 \), and so the theorem is proved.

The proof of Theorem 2 is, in general outline, similar to that of Theorem 1; we consequently omit some of the details.

**Proof of Theorem 2.** As in Theorem 1, we show \( P(X) \) is open in \( Y \). Fix \( x_0 \in X \) and \( r > 0 \); choose \( \epsilon > 0 \) so small and \( c > 0 \) so that (3) holds on \( B(Px_0; 2r) \) whenever \( \| u - x_0 \| \leq 2\epsilon \). If \( \| y - Px_0 \| \leq \hat{r} \) and \( \| x - x_0 \| \leq 2\epsilon \), then, as in Theorem 1, \( \| x - x_0 \| \leq \epsilon \) provided that \( \hat{r} = \min\{r, c\epsilon/2\} \). Fix \( y \in B(Px_0; \hat{r}) \); it suffices to show \( y \in P(X) \).

Set \( d = \text{dist}(y, P(X)) \) and suppose \( d > 0 \). Set

\[
D = \{ x \in B(x_0; \epsilon); \| y - P_x \| \leq \hat{r} \}
\]

and fix \( x \in D \). Choose \( h \in X \) so that \( \varphi(h) = J(y - P_x) \) and, for \( t > 0 \), set \( x_t = x + th \). We will show, for \( t \) sufficiently small, that \( x_t \in D \).

As in Theorem 1, it easily follows that

\[
\langle Px_t - P_x, J(y - P_x) \rangle \geq c \| x_t - x \| \| y - P_x \| \quad (11)
\]

provided that \( \| x_t - x \| \) is sufficiently small. Now let \( M \) be the Lipschitz constant of \( J \) on \( B(0; 2\hat{r}) \) and apply (6) and (11)

\[
\| y - P_x \|^2 \leq \| y - P_x \|^2 - 2 \langle Px_t - P_x, J(y - P_x) \rangle \\
= \| y - P_x \|^2 - 2 \langle Px_t - P_x, J(y - P_x) - J(y - P_x) + J(y - P_x) \rangle \\
\leq \| y - P_x \|^2 - 2c \| x_t - x \| \| y - P_x \| \\
+ 2 \| Px_t - P_x \| \| J(y - P_x) - J(y - P_x) \| \\
\leq \| y - P_x \|^2 - 2cd \| x_t - x \| + 2M \| Px_t - P_x \|^2
\]

provided that \( \| y - P_x \| \leq 2\hat{r} \). Now by condition (4),

\[
\| Px_t - P_x \|^2 \leq \epsilon(t) \| x_t - x \|,
\]

where \( \epsilon(t) \) is chosen so that

\[
\| x_t - x \| \leq \epsilon(t) \| x_t - x \|
\]

for all \( t > 0 \). The proof is complete.
where $\varepsilon(t) \to 0$ as $t \to 0$. In particular, by choosing $t$ sufficiently small, $\varepsilon(t) \leq cd/2M$, and thus,

$$\|y - Px_t\|^2 \leq \|y - Px\|^2 - cd \|x_t - x\|.$$ 

This implies that $x_t \in D$ and that

$$\|x_t - x\| \leq (1/cd)[\|y - Px\|^2 - \|y - Px_t\|^2]$$

and we again obtain a contradiction via Caristi's theorem.

We conclude by remarking that the above theorems, in assuming that $P(X)$ is closed, belong to the general class of "normal solvability" results developed by Browder and numerous other authors. While Browder obtains his Theorem 4(I,II) by means of some of these more general normal solvability results, our approach is somewhat more direct and circumvents these intermediate theorems altogether.

**REFERENCES**