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SOME RESULTS ON CDH SPACES—I

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A space is CDH is any two countable dense sets can be mapped one onto the other by an autohomeomorphism of the entire space. The CDH nature of separable manifolds and $\mathbb{R}^\kappa$ is examined.

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homogeneous manifold Martin's axiom

0. Introduction

A separable space X is said to be a CDH (countable dense homogeneous) space, if for any two countable dense subsets A and B in X, there is an autohomeomorphism h of X such that $h(A)=B$. In this paper, for any space X, $H(X)$ is always the set of all autohomeomorphisms of X. The rationals, the reals, the Cantor set and the closed unit interval are denoted by $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ and I respectively. κ, λ are cardinals and α, β, $\gamma, \ldots$ are ordinals. $\mathfrak{c}$ will be the size of $\mathbb{R}$. $\mathfrak{b}$ is the least cardinality of an unbounded family in ${}^\omega\omega$. The authors would like to thank A. Dow, S. Watson and the referee for suggesting various improvements, corrections and better proofs.

1. Main results

It is well known that any n-Euclidian space $\mathbb{R}^n$ is CDH (see [5]). There are two natural questions.

(A) Is every separable (topological) manifold CDH? A connected space X is called an n-manifold for a non-negative integer n, if for any point $x\in X$ there is a neighbourhood U at x such that U is homeomorphic with $\mathbb{R}^n$. Note that it is true that $w(X)\leqslant\mathfrak{c}$ for any separable manifold X, where $w(X)$ means the weight of X.

(B) Is every product $\mathbb{R}^\kappa(\kappa\leqslant\mathfrak{c})$ CDH? Similar questions are whether I^κ or $\mathbb{C}^\kappa$ are CDH for $\omega\leqslant\kappa\leqslant\mathfrak{c}$. By the Hewitt-Marczewski-Pondiczery Theorem, $\mathbb{R}^\kappa$ is separable if and only if $\kappa\leqslant\mathfrak{c}$.

When $\kappa=\omega$, the answers to both questions are known, i.e. Bennett [1] (and independently Bessaga and Pelczynski, see [2]) showed if the manifold X has weight $w(X)\leqslant\omega$, then X is CDH and Fort [4] showed that the Hilbert cube I^ω (and $\mathbb{R}^\omega$) is CDH. The present paper deals with the case that $\kappa>\omega$; the following theorems are established.

Theorem 1. *There is a separable, non-CDH manifold with weight* $\mathfrak{c}$.

Theorem 2. *Every separable manifold of weight* $<\mathfrak{b}$ *is CDH.*

Problem 1. Is it consistent that there is a separable, non-CDH manifold of weight $<\mathfrak{c}$?

For products, there are:

Theorem 3. *The space* $\mathbb{R}^{\mathfrak{c}}$ *(or* $\mathbb{C}^{\mathfrak{c}}$, $I^{\mathfrak{c}}$*) is not CDH.*

Theorem 4. *It is independent with ZFC that* $\mathbb{R}^{\omega_1}$ *(or* $\mathbb{C}^{\omega_1}$, I^{ω_1}*) is CDH, i.e.* MA_κ *implies* $\mathbb{R}^\kappa$ *(or* $\mathbb{C}^\kappa$, I^κ*) is CDH.*

All these results answer questions raised by B. Fitzpatrick or the second author. Since Bennett [1] proved every strongly locally homogeneous, separable, locally compact metric space is CDH, Theorem 1 and 3 give examples which are strongly locally homogeneous, connected and locally connected, compact (or locally Euclidean), non-CDH spaces. In this respect, Van Mill [9] got a strongly locally homogeneous, connected and locally connected, Baire, non-CDH subspace of $\mathbb{R}^2$.

In Section 4, the CDH number $\mathfrak{h}$ is discussed.

2. Proofs for manifolds

Before proving Theorem 1, some geometric notions and constructions are needed. First, let H be a fixed open disk on the plane and S be its boundary. Let us call an (open) arc ab a nice arc if

(i) $a\in S$ and $b\in H$;

(ii) there is an isotopy $f_\ell:\bar{H}\times[0,1]\to\bar{H}$ such that $f_\ell(a,t)=a$ for all t and $f_\ell(x,0)=x$ for $x\in ab$, and $\{f_\ell(x,1);\ x\in ab\}$ is an arc on S at the left of a. There is an isotopy f_r with the similar properties and $\{f_r(x,1);\ x\in ab\}$ is an arc on S at the right of a; and

(iii) if ab is expressed as the topological image of $(0,1)$ under the mapping l, then $r_1<r_2$ if and only if $d(a,l(r_1))<d(a,l(r_2))$. If $t\in(0,\frac{1}{2})$ and $0<\varepsilon_1<\varepsilon_2$, then the fan-shaped area ('angle' area) $F_{t,1}$ (or $F_{-t,1}$) sided by $J_{t-\varepsilon_1}=\{f_r(x,t-\varepsilon_1);\ x\in ab\}$

and $J_{t+\varepsilon_1}=\{f_r(x, t+\varepsilon); x\in ab\}$ (or by $J_{-t-\varepsilon_1}$ and $J_{-t+\varepsilon_1}$) is contained in the interior of the corresponding fan-shaped area $F_{t,2}$ sided by $J_{t-\varepsilon_2}$ and $J_{t+\varepsilon_2}$ (or $F_{-t,2}$). A similar requirement is needed in the case of $t=0$.

Now consider Moore's example B [10, p. 376]. We have a new version of B (which is a 2-manifold). Instead of straight line rays, we use $\{ab\}=\{J_0\}$ and $\{J_{-t}\cup J_t\}$ $(0\leq t<\frac{1}{2})$ to introduce the new points to H. The topology at $x\in H$ remains untouched. For the new point $y_t=\{J_{-t}\cup J_t\}$ $(0\leq t<\frac{1}{2})$, $R_{t,\varepsilon}$ will be its neighbourhood, where $R_{0,\varepsilon}=\{y_{t'}; t'<\varepsilon\}\cup\{x\in H; x\in J_{t'}$ for $|t'|\cup\varepsilon$ and $d(a,x)<\varepsilon\}$ and $R_{t,\varepsilon}=\{y_{t'}; |t-t'|<\varepsilon\}\cup\{x\in H; x\in J_{t'}\cup J_{-t'}$ for $|t-t'|<\varepsilon$ and $d(a,x)<\varepsilon\}$. Clearly, such a space is obtained from the discrete union of H and an open disk $D=\{z; |z|<\frac{1}{2}\}$ on the complex plane C, by identifying a with $0\in C$ and $\{J_{-t}\cup J_t\}$ with the point $t\in C$. Of course, the arc ab is also assumed to be included in the image of D and in fact it is identified with the open segment on the negative real axis in C. Denote the space constructed above by $H(ab)$. If a space X is obtained iterating the above procedure, we get a new space $X(ab)$ if the point a did not appear before. It is interesting to observe that such a construction (based on Moore's idea) is almost the same as what Rudin and Zenor did [12].

Proof of Theorem 1. Take two disjoint countable dense subsets A, B of H. Let all homeomorphisms between the spaces A and B be listed as $\{h_\alpha; \alpha<\mathfrak{c}\}$. Now for $\alpha<\mathfrak{c}$ inductively define spaces X_α and points r_α, s_α on S with the following properties:

(1) $\forall\alpha<\mathfrak{c}$, H is an open dense subset of X_α;

(2) for $\alpha_1<\alpha_2<\mathfrak{c}X_{\alpha_1}\subset X_{\alpha_1}$ and X_{α_1} is open in X_{α_2};

(3) r_α, s_α, are distinct and $\forall\beta<\alpha\{r_\alpha, s_\alpha\}\cap\bigcup_{\beta<\alpha}\{r_\beta, s_\beta\}=\emptyset$.

Suppose X_β, t_β and s_β have been defined for all $\beta<\alpha$. Let r_α be any point of S differing from r_β, s_β, $(\beta<\alpha)$. Clearly, there is a sequence $\{a_n; n<\omega\}$ in A converging to r_α such that the broken line L connecting points a_n $(n<\omega)$ is a nice arc.

Case 1. The sequence $\{h_\alpha(a_n); n<\omega\}$ in B does not converge in $\bar{H}=H\cup S$. Thus, $\{h_\alpha(a_n); n<\omega\}$ has at least two distinct cluster points x and y. Pick a subsequence $\{a_{n_k}; k<\omega\}$ such that $\{h_\alpha(a_{n_k}); k<\omega\}$ converges to x. It reduces to the next case.

Case 2. The sequence $\{h_\alpha(a_n); n<\omega\}$ converges to a point x in $\bar{H}$.

Subcase (a). $x\in H$. Obviously, we are able to find a nice arc (broken line) J and the isotopies f_l and f_r such that $J\cap\{a_n; n<\omega\}$ is infinite and if R is the fan-shaped area sided by $J_{-1/2}$ and $J_{1/2}$, then $\{a_n; a_n\notin\bar{R}, n<\omega\}$ is infinite. Let $X=\bigcup_{\beta<\alpha}X_\beta$ with the obvious topology. Define $X_\alpha=X(J)$ and arbitrarily take $S_\alpha\in S\backslash\bigcup_{\beta<\alpha}\{r_\beta, s_\beta\}\backslash\{r_\alpha\}$.

Subcase (b). $x\in S\backslash\bigcup_{\beta<\alpha}\{r_\beta, s_\beta\}\backslash\{r_\alpha\}$. Let $s_\alpha=x$ and take a nice arc M connecting points $\{h_\alpha(a_n); n<\omega\}$. Define $Y=X(J)$ and $X_\alpha=Y(M)$.

Subcase (c). $x\in\bigcup_{\beta<\alpha}\{r_\beta, s_\beta\}$. If there is no convergent subsequence of $\{h_\alpha(a_n); n<\omega\}$ in the space $X=\bigcup_{\beta<\alpha}X_\beta$, let $X_\alpha=X(L)$ and $S_\alpha\in S\backslash\bigcup_{\beta<\alpha}\{r_\beta, s_\beta\}\backslash\{r_\alpha\}$ arbitrarily. Otherwise, take a convergent subsequence $\{h_\alpha(a_{n_k}); k<\omega\}$ and a nice arc J connecting infinitely many points of $\{a_{n_k}; k<\omega\}$

and having infinitely many points outside of $\bar{R}$ as in Subcase (a). Let $X_\alpha = X(J)$ and arbitrarily take $S_\alpha \in S \setminus \bigcup_{\beta<\alpha} \{r_\beta, s_\beta\} \setminus \{r_\alpha\}$.

Subcase (d). $x = r_\alpha$. Since $\{a_n;\ n<\omega\} \cap \{h_\alpha(a_n);\ n<\omega\} = \emptyset$, it is possible to choose a nice arc J connecting $a_n (n<\omega)$ and let the fan-shaped area R sided by $J_{-1/2}$ and $J_{1/2}$ exclude infinitely many points of $\{h_\alpha(a_n);\ n<\omega\}$. Finally, define $X_\alpha = X(J)$ and arbitrarily pick $S_\alpha \in S \setminus \bigcup_{\beta<\alpha} \{r_\beta, s_\beta\} \setminus \{r_\alpha\}$.

The induction is completed. The space $Y = \bigcup_{\alpha<\mathfrak{c}} X_\alpha$, provided with the obvious topology, is the desired 2-manifold, which does not have any autohomeomorphism h such that $h(A) = B$. The induction steps kill all possible 'candidates' by destroying their or their inverses' continuity. □

Remark. In [3], Fitzpatrick and Zhou proved that Moore's example B is CDH.

Proof of Theorem 2. Let X be a manifold and $w(X) < \mathfrak{c}$, dim $X = n$. Suppose A and B are two countable dense subsets of X. It is not hard to find a countable family of open subsets U_n such that

(1) $A \cup N \subseteq \bigcup_{n<\omega} U_n$;

(2) for $n \neq j$, $\bar{U}_n \cap \bar{U}_j = \emptyset$;

(3) each U_j is homeomorphic with $\mathbb{R}^n$. For each i, fix a metric d_i on $\bar{U}_i$ compatible with its topology. Assume $\mathscr{B}$ is base for X with $|\mathscr{B}| < \mathfrak{c}$. Let $\mathscr{P} = \{\langle G, H\rangle;\ G, H \in \mathscr{B}, \bar{G} \subset H$ and $G, H \approx \mathbb{R}^n\}$. Clearly $|\mathscr{P}| = |\mathscr{B}| < \mathfrak{c}$. Fix a $\langle G, H\rangle \in \mathscr{P}$. For any i, find a $k<\omega$ such that $d_i(\bar{G} \cap \bar{U}_i, \bar{U}_i \setminus H) > 1/k > 0$. Define a function $f_{G,H} = \omega \to \omega$ by $f_{G,H}(i) = k$. Let $g \in {}^\omega\omega$ satisfy that for any $\langle G, H\rangle \in \mathscr{P}$, $g \geqslant f_{G,H}$ (i.e. $\{i;\ g(i) < f_{G,H}(i)\}$ is finite). Also, for every $\bar{U}_i$ let us find an $h_i \in H(\bar{U}_i)$ such that

(1) $h_i(A \cap U_i) = B \cap U_i$;

(2) for $x \in U_i$, $d_i(x, h_i(x)) < 1/g(i)$ and

(3) $h_i \restriction \text{bdry}(U_i) = \text{id}$. The existence of h_i is guaranteed in [3]. Let $h: X \to X$ be defined by $h \restriction \bar{U}_i = h_i$ and $h \restriction X \setminus \bigcup_{i<\omega} \bar{U}_i = \text{id}$. It is easy to verify that h is a homeomorphism from X to X and $h(A) = B$. □

3. Proofs for products

Proof of Theorem 3. We will prove this for $2^{\mathfrak{c}}$. Let $\{A_\alpha\colon \alpha \in c\}$ be an independent family on ω. Let $\{(X_\alpha, n_\alpha)\colon \alpha \in c\}$ enumerate $[\omega]^\omega \times \omega$. Define an increasing sequence $\{\xi_\alpha\colon \alpha \in \omega\}$ by induction as follows: If there is β such that $\xi_\alpha \in \beta$ for each $\alpha \in \gamma$ and such that $X_\gamma \subseteq^* A_\beta$ or $X_\gamma \cap A_\beta$ is finite then let $\xi_\gamma = \beta$ (otherwise ξ_γ is not defined). Now redefine A_{ξ_γ}, when ξ_γ is defined, so that $n_\gamma \notin A_{\xi_\gamma} \subseteq^* A_{\xi_\gamma}$ and $n_\gamma \in A_{\xi_\gamma}$ if $X_\gamma \cap A_{\xi_\gamma}$ is finite. The family is still independent and so if we define

$$\varphi_n(\alpha) = \begin{cases} 1 & \text{if } n \in A_\alpha, \\ 0 & \text{if } n \notin A_\alpha, \end{cases}$$

then $D=\{\varphi_n: n\in\omega\}$ is dense in 2^c and no sequence from D converges to a point in D. To see this suppose that $\{\varphi_i: i\in X\}$ converges to φ_n. Let $(X_\alpha, n_\alpha)=(X, n)$. Then either there is some $\beta\in c$ such that $|A_\beta\cap X|=|X\backslash A_\beta|=\omega$, in which case X does not converge to any point in 2^c, or for every β either $X_\alpha\subseteq^* A_\beta$ or $X_\alpha\cap A_\beta$ is finite. In the second case ξ_α is defined and $\{\psi\in 2^c: \psi(\xi_\alpha)=0\}$ and $\{\psi\in 2^c: \psi(\xi_\alpha)=1\}$ are disjoint open sets one of which contains φ_n and the other of which contains all but finitely many members of $\{\varphi_i: i\in X\}$.

Now to see that 2^c is not CDH let S be a sequence converging to σ in 2^c and consider D and $D\cup S\cup\{\sigma\}$. $\square$

In the next section, we will see that it is consistent with ZFC that $\kappa<\mathfrak{c}$ and $\mathbb{R}^\kappa$ (or $I^\kappa, 2^\kappa$) is not CDH. Before the proof of Theorem 4, we need a useful lemma, which says any countable subset of $\mathbb{R}^\kappa$ can be moved into a set with the general position (for any finite κ, it can be realized by a rotation in $\mathbb{R}^\kappa$).

Lemma 3.1. *If $D=\{d_n;\ n<\omega\}$ is a countable subset of $\mathbb{R}^\kappa$* (or I^κ, 1^κ) *for any infinite cardinal κ, then there is an autohomeomorphism h such that for any $\alpha<\kappa$, $n_1, n_2<\omega$, if $n_1\neq n_2$ then $\pi_\alpha(h(d_{n_1}))\neq\pi_\alpha(h(d_{n_2}))$, where π_α is the αth projection.*

Lemma 3.2. *If $\{d_n;\ n<\omega\}$ is a dense subset of $X=\prod_{\alpha<\kappa}X_\alpha$, where $X_\alpha=[0,1]^\omega$, then there is $f\in H(X)$ such that $\forall\alpha, \forall n$, $\pi_\alpha(f(d_n))$ is not on the 'boundary' of $[0,1]^\omega$, i.e. if p_i is the* ith *projection, then $0<p_i\circ\pi_\alpha(f(d_n))<1$.*

Proof of Theorem 4. We only prove the theorem for $X=I^\kappa=\prod_{\alpha<\kappa}I_\alpha$, where $I_\alpha=I$. The other two cases are similar. Only one remark is needed for the case (III), i.e. $X''=\mathbb{C}^\kappa$. We can assume the two countable dense subsets A and B of X'' in context will have the property: $\forall\alpha$, $\pi_\alpha(A)$ and $\pi_\alpha(B)$ are order dense according to the order inherited from R. In fact, for any α, there is $h_\alpha\in H(\mathbb{C}_\alpha)$ such that $h_\alpha(\pi_\alpha(A))$ does not contain any 'end' points of $\mathbb{C}$. Then $h=\prod_{\alpha<\kappa}h_\alpha\in H(X'')$ and $h(A)$ is order dense.

By using Lemma 3.1 twice and Lemma 3.2, we can assume the two countable dense subsets $A=\{a_n;\ n<\omega\}$ and $B=\{b_n;\ n<\omega\}$ have the properties:

(1) $\forall\alpha$, $\{a_n(\alpha);\ n<\omega\}$ and $\{b_n(\alpha);\ n<\omega\}$ are order dense in $(0,1)$;

(2) $\forall\alpha<\kappa$, $\forall n\neq m$, $a_n(\alpha)\neq a_m(\alpha)$ and $b_n(\alpha)\neq b_m(\alpha)$, i.e. A and B have the general position.

Given $A=\{a_n: n\in\omega\}$ and $B=\{b_n: n\in\omega\}$ dense in $\mathbb{R}^\kappa$ and in general position define $\mathbb{P}$ to consist of all pairs (f, Φ) where

(1) f is a partial injection from ω to ω;

(2) Φ is a finite partial function from κ to ω;

(3) $(\forall\alpha\in\operatorname{dom}\Phi)(\forall m, n>\Phi(\alpha))((a_m(\alpha)<a_n(\alpha)$ if and only if $b_{f(m)}<b_{f(n)}(\alpha))$ and $(b_m(\alpha)<b_n(\alpha)$ if and only if $a_{f^{-1}(m)}(\alpha)<a_{f^{-1}(n)}(\alpha)))$.

Define $(f, \Phi)\leqslant(g, \Phi)$ if and only if $g\subseteq f$ and $\Psi\subseteq\Phi$. Note that if Φ and Φ' are compatible as functions and (f, Φ) and (f, Φ') are in $\mathbb{P}$ then $(f, \Phi\cup\Phi')$ extends both of them. Hence to show that $\mathbb{P}$ is σ-centred it suffices to show that the set of finite functions from κ to ω is σ-centred under inclusion. But this follows from the fact that $\kappa \leqslant 2^\omega$.

Using the fact that A and B are in general position and dense it can be shown that for each $n \in \omega$

$$D(n) = \{(f, \Phi) \in \mathbb{P}\colon n \in \operatorname{dom}(f) \cap \operatorname{range}(f)\}$$

is dense. Also $E(\alpha) = \{(f, \Phi) \in \mathbb{P}\colon \alpha \in \operatorname{dom}(\Phi)\}$ is dense. If G is generic for the sets $D(n)$ and $E(\alpha)$ let $F = \bigcup\{f\colon (f, \Phi) \in G\}$ and define H on 2^κ by

$$H(x)(\alpha) = \operatorname{limsup}\{g_{F(n)}(\alpha)\colon a_n(\alpha) \leqslant x(\alpha)\}.$$

It is routine to check that H is the desired homeomorphism. □

4. The CDH numbers

If we define the uncountable cardinals $\mathfrak{h} = \min\{\kappa;\, 2^\kappa \text{ is not CDH}\}$, $\mathfrak{h}_1 = \min\{\kappa;\, I^\kappa \text{ is not CDH}\}$ and $\mathfrak{h}_2 = \min\{\kappa;\, \mathbb{R}^\kappa \text{ is not CDH}\}$, the following questions would be interesting:

Problem 2. How big is $\mathfrak{h}$ (or $\mathfrak{h}_1$, $\mathfrak{h}_2$)? How big is $\operatorname{cf}(\mathfrak{h})$? Or particularly, what are relationships among $\mathfrak{h}$, $\mathfrak{h}_1$, $\mathfrak{h}_2$? For $\lambda < \kappa$, does 2^κ CDH imply 2^λ is CDH?

Recall the following cardinals. $\mathfrak{p} = \min\{\kappa;\, \mathrm{MA}_\kappa$ (σ-centred) fails$\}$, $\mathfrak{q} = \min\{\kappa;\, 2^\kappa$ is not sequentially separable, i.e. 2^κ has no countable dense subset D such that any point of 2^κ can be the limit of a sequence in $D\}$ (e.g. see [14]). By a method due to Rothberger [11], one can show that $\mathfrak{q} = \min\{\kappa;\, \forall X \subseteq \mathbb{R}$ if $|X| \geqslant \kappa$, there is a non G_δ-set of the subspace X, (i.e. X is not a Q-set). It is well known that $\mathfrak{q} \geqslant \mathfrak{p}$. To the authors' knowledge, the question whether $\mathfrak{q} \leqslant \mathfrak{p}$ remains open. We will prove $\mathfrak{p} \leqslant \mathfrak{h} \leqslant \mathfrak{q}$, hence another question is raised.

Problem 3. Is it true that $\mathfrak{h} = \mathfrak{p}$?

Theorem 4.1. $\mathfrak{p} \leqslant \mathfrak{h}$ *(or* $\mathfrak{h}_1$, $\mathfrak{h}_2$*)* $\leqslant \mathfrak{q}$.

Proof. We only prove it for $\mathfrak{h}$. $\mathfrak{p} \leqslant \mathfrak{h}$ has been shown in Theorem 4. We are to prove that if $\kappa \geqslant \mathfrak{q}$, then 2^κ is not CDH. Let $X = \prod_{\alpha<\kappa} X_\alpha = \mathbb{C}$. Assume $D = \{r_i;\, i < \omega\}$ is a dense subset of $\mathbb{C}$. If we regard the index set κ as a subset of p, the irrationals in $(0, 1)$. Let $I_0 = (0, 1)$, $I_{(0)} = (0, 1/2)$, $I_{(1)} = (1/2, 1)$. For any finite sequence s of 0 or 1, if I_s has been defined, let $I_{s(0)}$ (or $I_{s(1)}$) be the left (or right) 'half' of I_s. For any function f with $\operatorname{dom} f = {}^n 2$, ran $f \subset D$, let $x_f(\alpha) = r_i$ if $\alpha \in I_s$ and $f(s) = r_i$. Then $E = \{x_f;\, f \in {}^{(n_2)}D\}$ is dense in X. It can be shown that each $x_f \notin E$ is a limit of a sequence in E. Since $2^\kappa = X$ is not sequentially separable, there is a $y \in X$ such that

y cannot be a limit of any sequence in E. Define $E' = E \cup \{y\}$. Then there is no $h \in H(X)$ with $h(E) = E'$.

Since $\mathfrak{q}$ could be less than $\mathfrak{c}$ in some model of ZFC, by combining Theorem 5.1 with Theorem 4 we get

Corollary 4.2. *It is independent that for all* $\kappa < \mathfrak{c}$ 2^κ *is CDH.*

Remark 4.3. If we define $\mathfrak{q}_1 = \min\{|X|;\ X$ is not a Q-set in $\mathbb{R}\}$, then $\mathfrak{q}_1 \leqslant \mathfrak{q}$ and actually we have $\mathfrak{h} \leqslant \mathfrak{q}_1$.

We will list another result to end this section.

Theorem 4.4. *Let* $m = \min\{|\mathcal{F}|;\ \mathcal{F}$ *is a maximal independent family, where* $\mathcal{F} \subset p(\omega)$ *is called an independent family if for any* $E_0, \ldots, E_m$, $F_0, \ldots, F_n$ *in* $\mathcal{F}$, $\bigcap_{i \leqslant m} E_i \cap \bigcap_{j \leqslant n} (\omega \backslash F_j)$ *is infinite*$\}$, *then* $\mathfrak{h} \leqslant \mathfrak{m}$, and it is consistent with ZFC that $\mathfrak{h} < \mathfrak{m}$.

Proof. Note that if $D = \{d_n;\ n < \omega\}$ is dense in 2^κ, then $\{F_\alpha;\ \alpha < \kappa,\ F_\alpha = \{n;\ d_n(\alpha) = 0\}\}$ is an independent family. On the other hand, if $\mathcal{F} = \{F_\alpha;\ \alpha < \kappa\}$ is an independent family, then $D = \{d_n;\ n < \omega$, where for $\alpha < \kappa$, $d_n(\alpha) = 0$ iff $n \in F_\alpha\}$ is dense in 2^κ. Besides, $\mathcal{F}$ is maximal iff for any $E \subset D$, if E is dense, then $D \backslash E$ is not dense. Now, assume $\kappa = \mathfrak{m}$. Let D be the corresponding dense subset given from $\mathcal{F}$ and E be the dense subset consisting of 'step functions' in the proof of the previous theorem. Clearly, E does not have the property which D has. So D and E are not homeomorphic.

Price [7] showed $\mathfrak{p} \leqslant \mathfrak{m}$, but they are not equal, which can be seen in the following model. Let M be a model of GCH and N be the $\mathbb{P}$-generic extension, where $\mathbb{P} = F(\omega_2, 2)$ (see [7]). Let $\{r_\alpha;\ \alpha < \omega_2\}$ be the Cohen reals. It is well-known that $Y = \{r_\alpha;\ \alpha < \omega_2\}$ is a Lusin set (e.g. see [8]). Let $Z = \{r_\alpha;\ \alpha < \omega\}$. Z is not a Q-set. In fact, take a countable dense (in Z) set $A \subset Z$. If A is G_δ in Z, then $A = \bigcap_n U_n \cap Z$, where U_n is an open dense set of $\mathbb{R}$. Then $Z = A \cup \bigcup_{n<\omega} (\mathbb{R} \backslash U_n) \cap Z \subseteq (A \cup \bigcup_{n<\omega} \mathbb{R} \backslash U_n) \cap Y$. But the latter set is countable. Now, we assert that every set of size ω_1 in $\mathbb{R}$ is not a Q-set. It would follow that $\mathfrak{q} = \omega_1$ and $\mathfrak{h} \leqslant \mathfrak{q}$ by Theorem 4.1. But $\mathfrak{m} = \omega_2$ in N. Suppose S is a Q-set and $|S| = \omega_1$. By a result due to Rothberger [11], there is a denumerable base D, i.e. $|D| \leqslant \omega$ and every point $x \in 2^{\omega_1}$ is the limit of a sequence in D. But the corresponding base $\Phi(Z)$ (for notations, see [11]) is not a denumerable base. Hence 2^{ω_1} is not CDH. Hence $\mathfrak{h} \leqslant \omega_1$. □

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