Topology and its Applications 157 (2010) 1157–1162

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Connected generalised Sierpiński carpets ☆

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ARTICLE INFO

Article history:
Received 25 November 2008
Accepted 5 February 2010

MSC:
28A80
05C10
54H05

Keywords:
Fractal
Sierpiński carpet
Connected set
Graph

ABSTRACT

Generalised Sierpiński carpets are planar sets that generalise the well-known Sierpiński carpet and are defined by means of sequences of patterns. We present necessary and sufficient conditions, under which generalised Sierpiński carpets are connected, with respect to Euclidean topology.

1. Introduction

The Sierpiński carpet and the Sierpiński gasket are well-known fractals, which were originally studied by Sierpiński [6]. In this article, we take a look at generalised Sierpinski carpets and analyse under which conditions they are connected with respect to the topology induced by the Euclidean metric. We give conditions that are sufficient and conditions that are necessary and sufficient.

Sierpiński carpets are self-similar fractals in the plane that originate from the Sierpiński carpet [2,5,6]. They are constructed as follows. The unit square is divided into $n \times n$ congruent smaller subsquares of which m squares, corresponding to a given $n \times n$ pattern (called the generator of the Sierpiński carpet), are cut out together with their boundary, and then the closure (with respect to the topology induced by the Euclidean metric in the plane) is taken. At each step of the iterative construction this procedure is applied to all remaining squares, and, repeating this construction ad infinitum, the resulting object is a fractal of Hausdorff and box-counting dimension $\log(n^2 - m)/\log(n)$, called a *Sierpiński carpet* [3]. Sierpiński carpets have been used, e.g., as models for porous materials [3,7]. The Vicsek fractal [8], which is also called Vicsek snowflake, and the Cantor dust [5] are further examples of Sierpiński carpets.

In the present paper, we study sets that generalise the Sierpiński carpets mentioned before, i.e., *generalised Sierpiński carpets*, which differ from a Sierpiński carpet defined as above slightly in the construction and in the following aspects. At step k of the construction, for any $k \geqslant 1$, we apply an $m_k \times m_k$ pattern, where $m_k \geqslant 2$, for all $k \geqslant 1$, and, at any two steps $k_1 \neq k_2$ we may have distinct patterns, with $m_{k_1} \neq m_{k_2}$. Thus, generalised Sierpiński carpets need not be self-similar.

☆ This research was supported by the Austrian Science Fund (FWF), Project P20412-N18, at the Technical University of Graz, Institute for Mathematics A.
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0166-8641/$ – see front matter
doi:10.1016/j.topol.2010.02.005

In a recent paper, Cristea [1] studied connectedness properties of fractals, that are, under certain conditions, a special case of the generalised Sierpiński carpets analysed in the present paper, and Hata [4] studied connectedness properties of self-similar fractals.

The construction of generalised Sierpiński carpets is discussed in Section 2. For technical reasons, we introduce in Section 3 the notion of *exit pairs*. Our main results are situated in Sections 4 and 5, where we present necessary and sufficient conditions, under which generalised Sierpiński carpets are connected. We deal with the case where the limit set arises from sequences of finitely many different subsets of the unit square in Section 5, while in Section 4 the sequences are allowed to attain infinitely many distinct values. An example, where we apply one of our results, is given in Section 5.

2. Construction

Let $m \geqslant 1$. $S^m_{i,j} = \{(x, y) \mid \frac{i}{m} \leqslant x \leqslant \frac{i+1}{m}$ and $\frac{j}{m} \leqslant y \leqslant \frac{j+1}{m}\}$ and $\mathcal{S}_m = \{S^m_{i,j} \mid 0 \leqslant i \leqslant m-1$ and $0 \leqslant j \leqslant m-1\}$. We call the elements of $\mathcal{S}_m$ *squares* and any nonempty $\mathcal{A} \subseteq \mathcal{S}_m$ an *m-pattern*.

Let $x, y, q \in [0,1]$ such that $Q = [x, x+q] \times [y, y+q] \subseteq [0,1] \times [0,1]$. Then for any point $(z_x, z_y) \in [0,1] \times [0,1]$ we define the function

$$P_Q(z_x, z_y) = (qz_x + x, qz_y + y).$$

Let $\{\mathcal{A}_k\}_{k=1}^{\infty}$ be a sequence of patterns such that for all $k \geqslant 1$ there is an $m_k \geqslant 1$ such that $\mathcal{A}_k \subseteq \mathcal{S}_{m_k}$. We let $\mathcal{W}_1 = \mathcal{A}_1$, and call it the *set of squares of order* 1. For $n \geqslant 2$ we denote the *set of squares of order n* by

$$\mathcal{W}_n = \bigcup_{W \in \mathcal{A}_n,\, W_{n-1} \in \mathcal{W}_{n-1}} \{P_{W_{n-1}}(W)\}.$$

For a sequence of patterns $\{\mathcal{A}_k\}_{k=1}^{\infty}$ we introduce the notation $m(i) := \prod_{k=1}^{i} m_k$. We note that $\mathcal{W}_n \subset \mathcal{S}_{m(n)}$. For $n \geqslant 1$, we define $L_n = \bigcup_{W \in \mathcal{W}_n} W$. Therefore, $\{L_n\}_{n=1}^{\infty}$ is a monotonically decreasing sequence of compact sets. We write $L_\infty = \bigcap_{n=1}^{\infty} L_n$ for the *limit set of the pattern sequence* $\{\mathcal{A}_k\}_{k=1}^{\infty}$.

3. Exit pairs

A *graph* G is a pair (V, E), where $V = V(G)$ is a finite set of vertices, and the set of edges $E = E(G)$ is a subset of $\{\{u, v\} \mid u, v \in V,\ u \neq v\}$. We write $u \sim v$ if $\{u, v\} \in E(G)$ and sometimes we say u is a *neighbour* of v. The sequence of vertices $\{u_i\}_{i=0}^{n}$ is a *path between* u_0 *and* u_n in a graph $G \equiv (V, E)$, if $u_0, u_1, \ldots, u_n \in V$, $u_{i-1} \sim u_i$ for $1 \leqslant i \leqslant n$, and $u_i \neq u_j$ for $0 \leqslant i < j \leqslant n$. A *connected component* is an equivalence class of the relation, where two vertices are related if there is a path between them.

Let $\mathcal{W} \subseteq \mathcal{S}_m$. We define $G(\mathcal{W}) \equiv (V(G(\mathcal{W})), E(G(\mathcal{W})))$ to be the graph of $\mathcal{W}$, i.e., the graph whose vertices $V(G(\mathcal{W}))$ are the squares in $\mathcal{W}$, and whose edges $E(G(\mathcal{W}))$ are the unordered pairs of distinct squares, that have a nonempty intersection. Let $S^{m(n)}_{x_0,y_0}$ and $S^{m(n)}_{x_1,y_1}$ be squares of order n. If $e = \{S^{m(n)}_{x_0,y_0}, S^{m(n)}_{x_1,y_1}\} \in E(G(\mathcal{W}_n))$ then e can be of four different types. If $|x_0 - x_1| = 1$ and $y_0 = y_1$ then we say e is . If $|y_0 - y_1| = 1$ and $x_0 = x_1$ then we say e is . If $(x_0 - x_1)(y_0 - y_1) = -1$ then we say e is . If $(x_0 - x_1)(y_0 - y_1) = 1$ then we say e is .

Let $\mathcal{W} \subseteq \mathcal{S}_m$. If $\{S^m_{0,0}, S^m_{m-1,m-1}\} \subseteq \mathcal{W}$ then we call $\{S^m_{0,0}, S^m_{m-1,m-1}\}$ an *exit-pair of* $\mathcal{W}$ and denote it by . If $\{S^m_{0,m-1}, S^m_{m-1,0}\} \subseteq \mathcal{W}$ then we call $\{S^m_{0,m-1}, S^m_{m-1,0}\}$ an *exit-pair of* $\mathcal{W}$ and denote it by . Let $0 \leqslant y_0, y_1 \leqslant m-1$ and $E = \{S^m_{0,y_0}, S^m_{m-1,y_1}\} \subseteq \mathcal{W}$. If $|y_0 - y_1| \leqslant 1$ then we call E an *exit pair* and denote it by

-  if $y_0 = y_1$,
-  if $y_0 = y_1 + 1$, and
-  if $y_0 = y_1 - 1$.

Let $0 \leqslant x_0, x_1 \leqslant m-1$ and $E = \{S^m_{x_0,0}, S^m_{x_1,m-1}\} \subseteq \mathcal{W}$. If $|x_0 - x_1| \leqslant 1$ then we call E an *exit pair* and denote it by

-  if $x_0 = x_1$,
-  if $x_0 = x_1 + 1$, and
-  if $x_0 = x_1 - 1$.

4. Results for general sequences

Proposition 1. *L_∞ is connected if and only if L_n is connected for all $n \geqslant 1$.*

Proof. 1. Let L_∞ be connected. We indirectly assume that L_n is not connected for some $n \geqslant 1$. Thus, there are disjoint nonempty open sets U and V, such that $U \cup V$ contains L_n. As L_∞ is a subset of L_n, $U \cup V$ contains L_∞, which is a contradiction to the fact that L_∞ is connected.

2. If L_n is connected for all $n \geqslant 1$ then L_∞ is connected because it is a decreasing intersection of compact connected sets. □

Proposition 2. *For $n \geqslant 1$, L_n is connected if and only if $G(\mathcal{W}_n)$ is connected.*

Proof. We note that in $G(\mathcal{W}_n)$ two distinct squares S_1 and S_2 are connected by an edge if and only if S_1 and S_2 are connected in the Euclidean plane.

1. Let L_n be connected. We indirectly assume that $G(\mathcal{W}_n)$ is not connected. This means that there are squares S_1 and S_2 such that there is no path between S_1 and S_2 in $G(\mathcal{W}_n)$. Let C_1 be the set of all squares S such that there is a path from S_1 to S and let C_2 be $V(G(\mathcal{W}_n)) \setminus S_1$. Thus, there is no edge between a square in C_1 and a square in C_2. Therefore, we can find open sets $U_1 \supseteq S_1$ and $U_2 \supseteq S_2$ such that U_1 and U_2 are disjoint, which is a contradiction to the fact that L_∞ is connected.

2. Let $G(\mathcal{W}_n)$ be connected. It follows that L_n is path-connected, which implies that L_n is connected. □

Let $G^\circ(\mathcal{W}_n)$ be the graph whose vertex set $V(G^\circ(\mathcal{W}_n)) = \mathcal{W}_n$ and whose edge set $E(G^\circ(\mathcal{W}_n))$ consists of all edges $e \in E(G(\mathcal{W}_n))$ that satisfy

Property 1.

(a) *If e is , then $\mathcal{A}_k$ contains a for all $k \geqslant n+1$.*
(b) *If e is , then $\mathcal{A}_k$ contains a for all $k \geqslant n+1$.*
(c) *If e is , then at least one of the following statements holds.*
  (1) *There is a in $\mathcal{A}_k$ for all $k \geqslant n+1$.*
  (2) *There is a $K \geqslant n+1$ such that $\mathcal{A}_k$ contains a for $n+1 \leqslant k \leqslant K-1$, $\mathcal{A}_K$ contains a , and $\mathcal{A}_k$ contains a for $k \geqslant K+1$.*
  (3) *There is a $K \geqslant n+1$ such that $\mathcal{A}_k$ contains a for $n+1 \leqslant k \leqslant K-1$, $\mathcal{A}_K$ contains a , and $\mathcal{A}_k$ contains a for $k \geqslant K+1$.*
(d) *If e is , then at least one of the following statements holds.*
  (1) *There is a in $\mathcal{A}_k$ for all $k \geqslant n+1$.*
  (2) *There is a $K \geqslant n+1$ such that $\mathcal{A}_k$ contains a for $n+1 \leqslant k \leqslant K-1$, $\mathcal{A}_K$ contains a , and $\mathcal{A}_k$ contains a for $k \geqslant K+1$.*
  (3) *There is a $K \geqslant n+1$ such that $\mathcal{A}_k$ contains a for $n+1 \leqslant k \leqslant K-1$, $\mathcal{A}_K$ contains a , and $\mathcal{A}_k$ contains a for $k \geqslant K+1$.*

Theorem 1. *L_∞ is connected if and only if $G^\circ(\mathcal{W}_n)$ is connected for all $n \geqslant 1$.*

Proof. 1. Let L_∞ be connected. We indirectly assume that $G^\circ(\mathcal{W}_n)$ is not connected, for some $n \geqslant 1$. Let C be a connected component of $G^\circ(\mathcal{W}_n)$. Let S be an arbitrary square in C and S_0 be an arbitrary square in $\mathcal{W}_n \setminus C$. We will show that there is a $k = k(S, S_0) \geqslant n$ such that the Euclidean distance between $S \cap L_k$ and $S_0 \cap L_k$ is at least $1/m(k)$, i.e.,

$$\min_{\substack{x \in S \cap L_k \\ y \in S_0 \cap L_k}} |x - y| \geqslant \frac{1}{m(k)}.$$

In the first case, where S and S_0 are not neighboured in $G(\mathcal{W}_n)$, the Euclidean distance between S and S_0 is at least $1/m(n)$.

In the second case, we assume that S and S_0 are neighboured in $G(\mathcal{W}_n)$. If S and S_0 share only a corner, then we assume without loss of generality, that the edge between S and S_0 in $G(\mathcal{W}_n)$ is . Since there is no edge between S and S_0 in $G^\circ(\mathcal{W}_n)$, we obtain from (a) that there is a $k \geqslant n+1$, such that $\mathcal{A}_k$ contains no . Thus, the Euclidean distance between $S \cap L_k$ and $S_0 \cap L_k$ is at least $1/m(k)$. If S and S_0 share a side, then we assume without loss of generality that S and S_0 are horizontal neighbours. Since there is no edge between S and S_0 in $G^\circ(\mathcal{W}_n)$, none of the statements (1)–(3) in Property 1(c) holds. By Property 1(c)(1), there is a minimal $K \geqslant n+1$ such that there is no in $\mathcal{A}_K$. If $\mathcal{A}_K$ contains no , then we set $k_1 = K$. If $\mathcal{A}_K$ contains a , then we conclude with Property 1(c)(2) that there is some $k \geqslant K+1$ such that $\mathcal{A}_k$ contains no , and we set $k_1 = k$. If $\mathcal{A}_K$ contains no , then we set $k_2 = K$. If $\mathcal{A}_K$ contains a , then we conclude with Property 1(c)(3) that there is a $k \geqslant K+1$ such that $\mathcal{A}_k$ contains no , and we set $k_2 = k$. Therefore, the Euclidean distance between $S \cap L_k$ and $S_0 \cap L_k$ is at least $1/m(k)$, where $k = k(S, S_0) = \max\{k_1, k_2\}$. Let

$$k_0 = \max\{k(S, S_0) \mid S \in C,\ S_0 \in \mathcal{W}_n \setminus C\}.$$

Furthermore, let $A = L_{k_0} \cap \bigcup_{S \in C} S$ and $B = L_{k_0} \cap \bigcup_{S_0 \in \mathcal{W}_n \setminus C} S_0$. We have

$$\min_{\substack{x \in A \\ y \in B}} |x - y| \geqslant \frac{1}{m(k_0)} > 0.$$

Thus, on the one hand, $L_K = A \cup B$, and, on the other hand, it is possible to find an open set U_A that contains A and an open set U_B that contains B, such that U_A and U_B are disjoint. As $U_A \cup U_B$ contains L_∞, we have a contradiction to the fact that L_∞ is connected.

2. Let $G^\circ(\mathcal{W}_n)$ be connected, for $n \geqslant 1$. Therefore, $G(\mathcal{W}_n)$ is connected, for $n \geqslant 1$, which implies that L_∞ is connected, by Propositions 1 and 2. □

As a summary, we now have the subsequent theorem.

Theorem 2. *The following conditions are equivalent.*

(a) L_∞ *is connected.*
(b) L_n *is connected for all* $n \geqslant 1$.
(c) $G(\mathcal{W}_n)$ *is connected for all* $n \geqslant 1$.
(d) $G^\circ(\mathcal{W}_n)$ *is connected for all* $n \geqslant 1$.

We define $G^\circ(\mathcal{A}_n)$ to be the graph whose vertex set $V(G^\circ(\mathcal{A}_n)) = \mathcal{A}_n$ and whose edge set $E(G^\circ(\mathcal{A}_n))$ consists of all unordered pairs e of different squares in $\mathcal{A}_n$, that have a nonempty intersection and satisfy Property 1.

Theorem 3. *If* $G^\circ(\mathcal{A}_n)$ *is connected for all* $n \geqslant 1$ *then* $G^\circ(\mathcal{W}_n)$ *is connected for all* $n \geqslant 1$.

Proof. We show by induction that $G^\circ(\mathcal{W}_n)$ is connected for $n \geqslant 1$. Since $G^\circ(\mathcal{W}_1)$ is equal to $G^\circ(A_1)$, $G^\circ(\mathcal{W}_1)$ is connected. We assume for $n \geqslant 2$ that $G^\circ(\mathcal{W}_{n-1})$ is connected. The set $\mathcal{W}_n$ is constructed by "substitution" of $\mathcal{A}_n$ for each square of $\mathcal{W}_{n-1}$. Let S_1 and S_2 be distinct squares of $\mathcal{V}(G^\circ(\mathcal{W}_n))$. We show that there is a path in $G^\circ(\mathcal{W}_n)$ between S_1 and S_2. For $k \geqslant 1$, let $\{U_1, \ldots, U_k\}$ be a path in $G^\circ(\mathcal{W}_{n-1})$, where U_1 contains S_1 and U_k contains S_2 as a subset, respectively. If $k = 1$ then there is a path in $G^\circ(\mathcal{W}_n)$ between S_1 and S_2, because $G^\circ(\mathcal{A}_n)$ is connected. Let $k \geqslant 2$ and $e = \{U_i, U_{i+1}\}$ for some i with $1 \leqslant i \leqslant k-1$. The edge e can have four types. If e is , then $\mathcal{A}_n$ contains a , by Property 1(a). Thus, the copy of $\mathcal{A}_n$ that is substituted for U_i is connected with the copy of $\mathcal{A}_n$ that is substituted for U_{i+1} via an exit pair E. The two exits of the exit pair E are connected by an edge in $G^\circ(\mathcal{W}_n)$, since Property 1(a) holds. If e is then we use analogue arguments. If e is then Property 1(c)(1), Property 1(c)(2), or Property 1(c)(3) holds. If Property 1(c)(1) holds we use similar arguments as before. If Property 1(c)(2) holds, then there is a $K \geqslant n$ such that $\mathcal{A}_k$ contains a for $n \leqslant k \leqslant K-1$, $\mathcal{A}_K$ contains a , and $\mathcal{A}_k$ contains a for $k \geqslant K+1$. If $K \geqslant n+1$ then $\mathcal{A}_n$ contains a . Thus, the copy of $\mathcal{A}_n$ that is substituted for U_i is connected with the copy of $\mathcal{A}_n$ that is substituted for U_{i+1} via an exit pair E. The two exits of the exit pair E are connected by an edge in $G^\circ(\mathcal{W}_n)$, since Property 1(c)(2) holds for $G^\circ(\mathcal{W}_n)$, also. If $K = n$ then $\mathcal{A}_n$ contains a . Thus, the copy of $\mathcal{A}_n$ that is substituted for U_i is connected with the copy of $\mathcal{A}_n$ that is substituted for U_{i+1} via an exit pair E. The two exits of the exit pair E are connected by a or a edge in $G^\circ(\mathcal{W}_n)$, since $\mathcal{A}_k$ contains a for $k \geqslant K+1 = n+1$. If Property 1(c)(3) holds, the same arguments work and if e is we also may use the previous methods. Thus, we obtain a path in $G^\circ(\mathcal{W}_n)$ between S_1 and S_2 and, therefore, $G^\circ(\mathcal{W}_n)$ is connected. □

Example. The reverse direction of Theorem 3 is not valid, as the sequence $\{\mathcal{A}_k\}_{k=1}^\infty$ provides a counter-example, with $\mathcal{A}_1 = \mathcal{B}_1$, $\mathcal{A}_2 = \mathcal{B}_2$, and $\mathcal{A}_i = \mathcal{B}_1$ for $i \geqslant 3$, where $\mathcal{B}_1$ and $\mathcal{B}_2$ are defined by the set of white squares in Fig. 1.

5. Sequences with finitely many values

Let $r \geqslant 1$ and $\mathcal{A} : \mathbb{N} \to \{\mathcal{B}_1, \ldots, \mathcal{B}_r\}$ be surjective, where for all $1 \leqslant k \leqslant r$ there is an $m_k \geqslant 1$ such that $\mathcal{B}_k \subseteq \mathcal{S}_{m_k}$. Furthermore, let $1 \leqslant n \leqslant r$ and $G^*(\mathcal{B}_n)$ be the graph whose vertex set $V(G^*(\mathcal{B}_n)) = \mathcal{B}_n$ and whose edge set $E(G^*(\mathcal{B}_n))$ consists of all edges $e \in E(G(\mathcal{B}_n))$ that satisfy

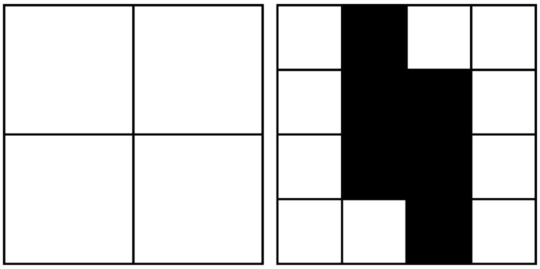

Fig. 1. $\mathcal{B}_1$ and $\mathcal{B}_2$.

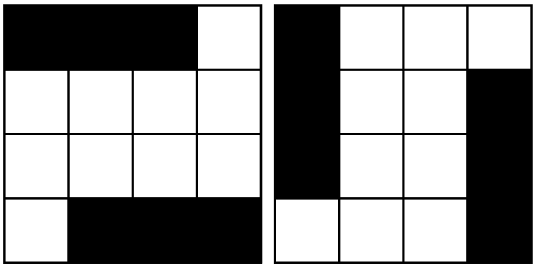

Fig. 2. $\mathcal{B}_1$ and $\mathcal{B}_2$.

Property 2.

(a) *If e is , then $\mathcal{B}_k$ contains a for all $1 \leqslant k \leqslant r$.*
(b) *If e is , then $\mathcal{B}_k$ contains a for all $1 \leqslant k \leqslant r$.*
(c) *If e is , then for each k, where $1 \leqslant k \leqslant r$, at least one of the following statements holds.*
  (1) *There is a in $\mathcal{B}_k$.*
  (2) *There is a in $\mathcal{B}_k$ and $\mathcal{B}_i$ contains a for all $1 \leqslant i \leqslant r$.*
  (3) *There is a in $\mathcal{B}_k$ and $\mathcal{B}_i$ contains a for all $1 \leqslant i \leqslant r$.*
(d) *If e is , then for each k, where $1 \leqslant k \leqslant r$, at least one of the following statements holds.*
  (1) *There is a in $\mathcal{B}_k$.*
  (2) *There is a in $\mathcal{B}_k$ and $\mathcal{B}_i$ contains a for all $1 \leqslant i \leqslant r$.*
  (3) *There is a in $\mathcal{B}_k$ and $\mathcal{B}_i$ contains a for all $1 \leqslant i \leqslant r$.*

Theorem 4. *If $G^*(\mathcal{B}_n)$ is connected for $1 \leqslant n \leqslant r$ then $G^\circ(\mathcal{A}_k)$ is connected for all $k \geqslant 1$.*

Proof. Let $n \geqslant 1$ and $\mathcal{A}_n = \mathcal{B}_N$, where $1 \leqslant N \leqslant r$. We will show that $G^*(\mathcal{B}_N)$ is a subgraph of $G^\circ(\mathcal{A}_n)$ with the same vertex set, which implies that $G^\circ(\mathcal{A}_n)$ is connected. Let e be an edge in $E(G^*(\mathcal{B}_N))$. If e is , then $\mathcal{B}_k$ contains a for all $1 \leqslant k \leqslant r$. Thus, $\mathcal{A}_k$ contains a for all $k \geqslant n+1$, such that $e \in E(G^\circ(\mathcal{A}_n))$, by Property 1(a). If e is , then we obtain in the same way that $e \in E(G^\circ(\mathcal{A}_n))$. Let e be . If there is a in $\mathcal{B}_k$ for all k with $1 \leqslant k \leqslant r$ then there is a in $\mathcal{A}_k$ for all $k \geqslant n+1$, such that $e \in E(G^\circ(\mathcal{A}_n))$, by Property 1(c)(1). Otherwise, we assume, without loss of generality, that Property 2(c)(2) holds such that $\mathcal{B}_i$ contains a for all $1 \leqslant i \leqslant r$. Furthermore, for all $1 \leqslant k \leqslant r$, there is either a , a , or a in $\mathcal{B}_k$. Thus, Property 1(c)(2) or Property 1(c)(3) of the definition of $G^\circ(\mathcal{A}_n)$ holds and therefore $e \in E(G^\circ(\mathcal{A}_n))$. If e is then we use the same arguments and so we conclude that $G^*(\mathcal{B}_N)$ is a subgraph of $G^\circ(\mathcal{A}_n)$. □

The reciprocal of Theorem 4 does not hold, as the sequence $\{\mathcal{A}_k\}_{k=1}^{\infty}$ provides a counter-example, with $\mathcal{A}_{2i-1} = \mathcal{B}_1$ and $\mathcal{A}_{2i} = \mathcal{B}_2$, for $i \geqslant 1$, where $\mathcal{B}_1$ and $\mathcal{B}_2$ are as in Fig. 2. An application of Theorem 4 is the following example.

Example. Let $\mathcal{B}_1$, $\mathcal{B}_2$, and $\mathcal{B}_3$ be as in Fig. 3 and $\mathcal{A} : \mathbb{N} \to \{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3\}$. Since $G^*(\mathcal{B}_n)$ is connected for $1 \leqslant n \leqslant 3$, Theorem 4 yields that $G^\circ(\mathcal{A}_n)$ is connected for all $n \geqslant 1$. By Theorem 3, $G^\circ(\mathcal{W}_n)$ is connected for all $n \geqslant 1$. With Theorem 1 we obtain that L_∞ is connected.

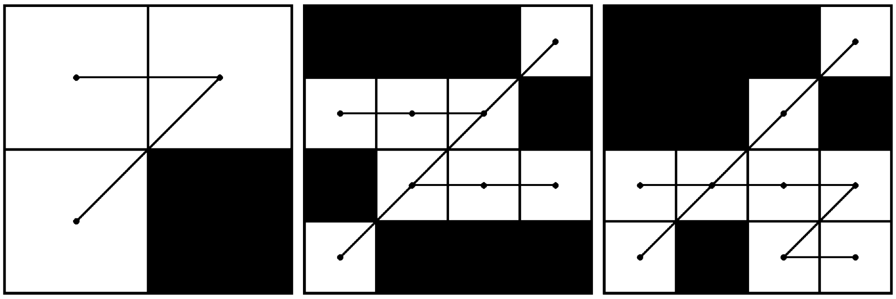

Fig. 3. $\mathcal{B}_1$, $\mathcal{B}_2$, and $\mathcal{B}_3$ together with $G^*(\mathcal{B}_1)$, $G^*(\mathcal{B}_2)$, and $G^*(\mathcal{B}_3)$.

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