Note

A note on polyharmonic functions

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Abstract

In this note we prove the following theorem:
Suppose $0 < p < \infty$ and $\alpha > -1$. Then there is a constant $C = C(p, m, n, \alpha)$ such that
\[
\int_B |u(x)|^p (1 - |x|)^{\alpha} \ dV(x) \leq C \left( |u(0)|^p + \int_B |\nabla u(x)|^p (1 - |x|)^{p+\alpha} \ dV(x) \right),
\]
for all polyharmonic functions $u$ of order $m$, on the unit ball $B \subset \mathbb{R}^n$.

Keywords: Polyharmonic functions; Weight function; Bergman space; Distortion

1. Introduction

Throughout this note $n$ is an integer greater than or equal to $2$, $D$ is a domain in the Euclidean space $\mathbb{R}^n$, $B(a, r) = \{ x \in \mathbb{R}^n \mid |x - a| < r \}$ denotes the open ball centered at $a$ of radius $r$, where $|x|$ denotes the norm of $x \in \mathbb{R}^n$ and $B$ is the open unit ball in the $n$-dimensional Euclidean space $\mathbb{R}^n$. $S = \partial B = \{ x \in \mathbb{R}^n \mid |x| = 1 \}$ is the Euclidean boundary of $B$. By $dV(x)$ we denote the Lebesgue volume measure on $B$, $d\sigma$ the surface measure on $S$ and $d\sigma_N$ the normalized surface measure on $S$. Let $\omega(r)$, $0 \leq r < 1$, be a positive weight function which is integrable on $(0, 1)$.

For $0 < p < \infty$ and $\omega(x) = \omega(|x|)$, the weighted Bergman space $b_\omega^p(B)$ is the space of all real harmonic functions $u$ on $B$ such that

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\[ \|u\|_{\omega,p} = \left( \int_B |u(x)|^p \omega(x) dV(x) \right)^{1/p} < +\infty. \]

If \( \omega(x) = (1-|x|)^\alpha \), we use the notation \( \|u\|_{\alpha,p} \).

For weighted Bergman spaces of analytic functions see, for example, in [1,5] and the references therein. Basic facts about unweighted harmonic Bergman spaces can be found in [2].

For \( u \in C(B) \) we usually write \( M^p_u(r) = \int_0^r |u(r \xi)|^p d\sigma_N(\xi), \) \( 0 \leq r < 1 \), for the integral means of \( u \). The norm in \( b^p_\omega(B) \) can then be written
\[ \|u\|_{\omega,p} = \left( \frac{1}{\sigma_n} \int_0^1 M^p_u(r, \omega(r)r^{n-1} dr \right)^{1/p}, \]

where \( \sigma_n = \sigma(S) \).

For a given weight \( \omega \) we define the function
\[ \psi(r) = \psi_\omega(r) \equiv \frac{1}{\omega(r)} \int_r^1 \omega(u) du, \quad 0 \leq r < 1, \]

and we call it the distortion function of \( \omega \). We put \( \psi(x) = \psi(|x|) \) for \( x \in B \).

**Definition 1** [5]. We say that a weight \( \omega \) is admissible if it satisfies the following conditions:

(a) There is a positive constant \( A = A(\omega) \) such that
\[ \omega(r) \geq \frac{A}{1-r} \int_r^1 \omega(u) du, \quad \text{for } 0 \leq r < 1; \]

(b) There is a positive constant \( B = B(\omega) \) such that
\[ \omega'(r) \leq \frac{B}{1-r} \omega(r), \quad \text{for } 0 \leq r < 1; \]

(c) For each sufficiently small positive \( \delta \) there is a positive constant \( C = C(\delta, \omega) \) such that
\[ \sup_{0 \leq r < 1} \frac{\omega(r)}{\omega(r + \delta \psi(r))} \leq C. \]

Observe that (a) implies \( A \psi(r) \leq 1 - r \) thus for sufficiently small positive \( \delta \) we have \( r + \delta \psi(r) < 1 \) and the quantity in the denominator of the fraction in (c) is well defined.

The following theorem was established in [5]:

**Theorem A.** Suppose \( 1 \leq p < \infty \) and \( \omega \) is an admissible weight with distortion function \( \psi \).

Let
\[ L(f) = \int_U |f(z)|^p \omega(z) dm(z) \]

and
R(f) = |f(0)|^p + \int_U |f'(z)|^p \psi(z) \omega(z) \, dm(z),

then there are finite positive constants C and C' independent of f such that

CR(f) \leq L(f) \leq C'R(f)

(1)

for all analytic functions f on the unit disc U, where dm(z) = r dr d\theta/\pi denotes the normalized Lebesgue area measure on U.

In [6] we showed that the first inequality in (1) holds also when p \in (0, 1). In [7] we generalized Theorem A in the case of harmonic functions on the unit ball B \subset \mathbb{R}^n. It is an open problem whether the second inequality in (1) holds when p \in (0, 1). In order to solve the open problem we proved the following theorem, which is a special case of Theorem 2 in [4].

Theorem B. Suppose 0 < p < \infty and \alpha > -1. Then there is a constant C = C(p, \alpha) such that

\int_U |f(z)|^p (1 - |z|)^\alpha dm(z) \leq C \left( |f(0)|^p + \int_U |f'(z)|^p (1 - |z|)^{p+\alpha} dm(z) \right),

for all f \in H(U).

A real valued function u is called polyharmonic of order m on a domain D if u \in C^\infty(D) and \Delta^m u \equiv 0, where m is a positive integer, \Delta denotes the Laplacian and \Delta^m u = \Delta^{m-1}(\Delta u). We denote by \mathcal{H}_m(B) the space of polyharmonic functions of order m on B. In particular, \mathcal{H}_1(B) is the class of all harmonic functions on B.

The purpose of the note is to prove an analogous theorem to Theorem B in the case of polyharmonic functions on the unit ball. We were motivated by [3]. We prove the following theorem.

Theorem 1. Suppose 0 < p < \infty, \alpha > -1 and u \in \mathcal{H}_m(B). Then there is a constant C = C(p, \alpha) such that

\int_B |u(x)|^p (1 - |x|)^\alpha dV(x) \leq C \left( |u(0)|^p + \int_B |\nabla u(x)|^p (1 - |x|)^{p+\alpha} dV(x) \right).

2. Auxiliary results

In order to prove the main result we need several auxiliary results. Throughout the following proofs C denotes a positive constant which may change from line to line.

Lemma 1 was essentially proved in Lemma 5 in [3], because u \in \mathcal{H}_m(B) implies D^\beta u \in \mathcal{H}_m(B) for every multi-index \beta.
Lemma 1. Let $0 < p < \infty$. Then for every multi-index $\beta$,
\[ |D^\beta u(a)|^p \leq \frac{C}{r^p} \int_{B(a,r)} |D^\beta u|^p \, dV \quad \text{whenever } B(a, r) \subset B, \]
for all $u \in \mathcal{H}_m(B)$ and some constant $C$ depending only on $\beta, p, m$ and $n$.

Lemma 2 [3, Lemma 7]. Let $g(r)$ be a nonnegative continuous function on the interval $[0, 1)$, $\lambda > 0$ and let $\alpha > -1$. Then there is a constant $C = C(\alpha, \lambda)$ such that
\[ \int_0^1 g^{1/\alpha}(r)(1-r)^{\alpha} \, dr \leq C \left( \max_{r \in [0,1/2]} g^{1/\alpha}(r) + \int_0^1 g^{1/\alpha} \left( \frac{1+r}{2} \right) - g(r) \, \frac{1}{\alpha} \right) \]

One can easily prove the following lemma.

Lemma 3. Let $p > 0$, $u \in C^1(B)$, $0 \leq r < 1$. Then
\[ |M^p_p(\rho, u) - M^p_p(r, u)| \leq (\rho - r)^p \int_S \sup_{r < t < \rho} |\nabla u(t\zeta)|^p \, d\sigma(\zeta) \quad \text{for } p \in (0, 1] \]
and
\[ |M_p(\rho, u) - M_p(r, u)| \leq (\rho - r)^{1/p} \left( \int_S \sup_{r < t < \rho} |\nabla u(t\zeta)|^p \, d\sigma(\zeta) \right) \quad \text{for } p \geq 1, \]
for every $\rho$ and $r$, such that $0 \leq r < \rho < 1$.

Lemma 4. Suppose $0 < p < \infty$ and $\alpha > -1$. Then there is a constant $C = C(p, \alpha, m, n)$ such that
\[ M^p_p(u, 1/2) = \max_{|x| \leq 1/2} |u(x)|^p \leq C \left( |u(0)|^p + \int_B |\nabla u(x)|^p (1-|x|)^{p+\alpha} \, dV(x) \right) \]
for all $u \in \mathcal{H}_m(B)$.

Proof. Since
\[ u(x_0) - u(0) = \int_0^1 u'(tx_0) \, dt = \int_0^1 \langle \nabla u(tx_0), x_0 \rangle \, dt, \]
by elementary inequalities we obtain
\[ |u(x_0)|^p \leq c_p \left( |u(0)|^p + |x_0|^p \max_{|x| \leq 1/2} |\nabla u(x)|^p \right), \quad \text{(2)} \]
for each $x_0 \in B(0, 1/2)$, where $c_p = 1$ for $0 < p < 1$ and $c_p = 2^{p-1}$ for $p \geq 1$. 

On the other hand by Lemma 1 we obtain
\[ |\nabla u(x)|^p \leq C \int_{B(x,1/4)} |\nabla u(y)|^p dV(y) \]
for each \( x \in B(0,1/2) \) and consequently
\[ \max_{|x| \leq 1/2} |\nabla u(x)|^p \leq \max_{B(0,3/4)} \left\{ C4^{p+\alpha}, C \right\} \int_{B(0,3/4)} |\nabla u(y)|^p \left(1 - |y|\right)^{p+\alpha} dV(y). \tag{3} \]
From (2) and (3) the result follows. \( \square \)

Similarly we can prove the following lemma.

**Lemma 5.** Suppose \( 0 < p < \infty \) and \( \alpha > -1 \). Then there is a constant \( C = C(p,\alpha,m,n) \) such that
\[ M^p_\infty(\nabla u,3/4) \leq C \int |\nabla u(x)|^p \left(1 - |x|\right)^{p+\alpha} dV(x), \]
for all \( u \in \mathcal{H}_m(B) \).

The proof of the following lemma is analogous to the proof of Lemma 6 in [3].

**Lemma 6.** Let \( u \in \mathcal{H}_m(B) \), \( 0 < p < \infty \), \( \alpha > -1 \) and \( f^+(r\xi) = \sup\{|f(t\xi)| | r < t < (1+r)/2\}, 0 \leq r < 1, \xi \in S. \) Then there is a constant \( C = C(p,m,n,\alpha) \) such that
\[ \int_0^1 M^p_p(\nabla u^+,r)(1-r)^{p+\alpha}r^{n-1} dr \leq C \int_0^1 M^p_p(\nabla u,r)(1-r)^{p+\alpha}r^{n-1} dr. \]

### 3. Proof of the theorem

In this section we prove the main result in this paper.

**Proof of Theorem 1.** Let \( p \in (0,1] \). Then by Lemma 2 (\( \lambda = 1 \)), Lemmas 3, 4, 5 and 6 we obtain
\[ \|u\|^{p}_{p,\alpha} = \sigma_n \int_0^1 M^p_p(u,r)(1-r)^\alpha r^{n-1} dr \leq \sigma_n \int_0^1 M^p_p(u,r)(1-r)^\alpha dr \leq C \left( M^p_p(u,1/2) + \int_0^1 |M^p_p(u,1+r/2) - M^p_p(u,r)| (1-r)^\alpha dr \right) \]
\[ \begin{align*} &\leq C \left( \max_{|x| \leq 1/2} |u(x)|^p + \int_0^{1/2} M_p^p (|\nabla u|^+, r)(1-r)^{p+\alpha} dr \right) \\
&= C \left( \max_{|x| \leq 1/2} |u(x)|^p + \int_0^{1/2} M_p^p (|\nabla u|^+, r)(1-r)^{p+\alpha} dr \right) \\
&\leq C \left( \max_{|x| \leq 1/2} |u(x)|^p + C \max_{|x| \leq 3/4} |\nabla u(x)|^p \\
&\quad + 2^{n-1} \int_{1/2}^1 M_p^p (|\nabla u|^+, r)(1-r)^{p+\alpha} r^{n-1} dr \right) \\
&\leq C \left( \max_{|x| \leq 1/2} |u(x)|^p + \int_0^1 M_p^p (\nabla u, \rho)(1-\rho)^{p+\alpha} \rho^{n-1} d\rho \right) \\
&\leq C \left( |u(0)|^p + \int_B |\nabla u(x)|^p (1-|x|)^{p+\alpha} dV(x) \right). \end{align*} \]

In the case \( p > 1 \) we apply Lemma 2 for \( \lambda = p \) and Lemma 3 for \( p > 1 \). \( \Box \)

Let \( |\nabla^k u| \) denote the norm of the \( k \)th gradient of \( u \) which is given by

\[ |\nabla^2 u| = \sqrt{u_1^2 x_1 + \cdots + u_n^2 x_n + 2u_1^2 x_1 x_2 + \cdots + 2u_{n-1}^2 x_n} \]

and \( |\nabla^k u| \) in a similar way.

By Theorem 1 and some simple calculations we obtain.

**Corollary 1.** Suppose \( 0 < p < \infty, \alpha > -1, k \in \mathbb{N} \) and \( u \in \mathcal{H}_m(B) \). Then there is a constant \( C = C(p, m, n, k, \alpha) \) such that

\[ \int_B |u(x)|^p (1-|x|)^{p+\alpha} dV(x) \leq C \left( \sum_{i=1}^{k-1} |\nabla^k u(0)|^p + \int_B |\nabla^k u(x)|^p (1-|x|)^{p+\alpha} dV(x) \right). \]

**References**