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Stationary Solutions for Generalized Boussinesq Models

SEBASTIÁN A. LORCA AND JOSÉ LUIZ BOLDRINI

UNICAMP—IMECC, C.P. 6065, 13081-970, Campinas, São Paulo, Brazil

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We study the existence, regularity, and conditions for uniqueness of solutions of a generalized Boussinesq model for thermally driven convection. The model allows temperature dependent viscosity and thermal conductivity.

1. INTRODUCTION

We study the stationary problem for the equations governing the coupled mass and heat flow of a viscous incompressible fluid in a generalized Boussinesq approximation by assuming that viscosity and heat conductivity are temperature dependent. The equations are

$$\begin{aligned} -\operatorname{div}(\nu(T)\,\nabla u) + u.\nabla u - \alpha T g + \nabla p &= 0, \\ \operatorname{div} u &= 0 \\ -\operatorname{div}(k(T)\,\nabla T) + u.\nabla T &= 0 \qquad \text{in } \Omega, \end{aligned} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N=2$ or 3 throughout the paper.

Here $u(x) \in \mathbb{R}^N$ denotes the velocity of the fluid at a point $x \in \Omega$; $p(x) \in \mathbb{R}$ is the hydrostatic pressure; $T(x) \in \mathbb{R}$ is the temperature; $g(x)$ is the external force by unit of mass; $\nu(\cdot) > 0$ and $k(\cdot) > 0$ are kinematic viscosity and thermal conductivity, respectively; and α is a positive constant associated to the coefficient of volume expansion. Without loss of generality, we have taken the reference temperature as zero. For a derivation of the above equations, see, for instance, Drazin and Reid [6].

The expressions ∇, Δ, and div denote the gradient, Laplace, and divergence operators, respectively (we also denote the gradient by grad); the ith component of $u.\nabla u$ is given by $(u.\nabla u)_i = \sum_{j=1}^N u_j(\partial u_i/\partial x_j)$; $u.\nabla T = \sum_{j=1}^N u_j(\partial T/\partial x_j)$.

The boundary conditions are

$$u = 0 \qquad \text{and} \qquad T = T_0 \quad \text{on } \partial\Omega, \tag{1.2}$$

where T_0 is a given function on $\partial\Omega$ (the boundary of Ω).

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The classical Boussinesq equations correspond to the special case where ν and k are positive constants. This case has been much studied (see, for instance, Morimoto [9, 10] and Rabinowitz [13] and the references therein).

Equations (1.1) are much less studied, and they correspond to the following physical situation. For certain fluids we cannot disregard the variation of viscosity (and thermal conductivity) with temperature, this being important in the determination of the details of the flow. It is found, for example, that a liquid usually rises in the middle of a polygonal convection cell, while a gas falls. Graham [7] suggested that this is because the viscosity of a typical liquid decreases with temperature whereas that of a typical gas increases. This suggestion was subsequently confirmed by Tippelkirch's experiments (see [15]) on convection of liquid sulphur, for which the viscosity has a minimun near 153° C. He found that the direction of the flow depended on whether the temperature was above or below 153° C. Palm [11] was the first to analyse the effect of the variation of viscosity with temperature; other papers on the subject are, for instance, Busse [4] and Palm *et al.* [12]. All these papers have the Theoretical Fluid Dynamics flavour. A rigorous mathematical analysis is more difficult in this case than in the case of the classical Boussinesq equations due to the stronger nonlinear coupling between the equations.

As a step in this direction, in this paper we will study the existence and regularity of solutions of (1.1) using a spectral Galerkin method combined with fixed point arguments; we will need more estimates than the ones required in the classical case in order to handle the nonlinearity in the higher order terms of the equations.

We will show the existence of weak and strong solutions of problem (1.1), (1.2) under certain conditions of the temperature dependency of the viscosity and thermal conductivity; we allow more general external forces than the usual one (constant gravitational field) because this could be useful in certain geophysical models. Properties of regularity and uniqueness are also studied. Questions concerning stability and bifurcation are left for future work.

We observe that if we take $u \equiv 0$ in (1.1) and $g = (0, 0, 1)$, the usual approximation for the gravitational acceleration, we are left with $\operatorname{grad} p = \alpha T y = (0, 0, \alpha T)$. Consequently, $\operatorname{curl}(0, 0, \alpha T) = 0$ in Ω, and *so* $\partial T/\partial x = \partial T/\partial y = 0$ in Ω. Therefore, we see that an arbitrary temperature on the boundary will in general require motion. That is, in general the solution of (1.1), (1.2) is not trivial.

Finally, we would like to comment that analogous questions can be considered for the corresponding evolution problem. Results along these lines will appear elsewhere. Also, concerning numerical results we would like to mention the interesting paper by Bernardi *et al.* [3] that treats a related problem.

This paper is organized as follows: in Section 2 we describe the notation and the basic facts to be used later on: we also state the our main results. In Section 3, we make a technical preparation by proving certain a priori estimates that will be useful for the proofs of the main results. The proofs of Theorem 2.1 (existence of a weak solution), Theorem 2.2 (existence of strong solution), Theorem 2.3 (regularity), and Theorem 2.4 (uniqueness) are done in Sections 4, 5, 6, and 7, respectively.

2. Preliminaries and Results

In this article the functions are either R or R^N valued ($N=2$ or 3), and we will not distinguish them in our notations; this being clear from the context. The $L^2(\Omega)$-product and norm are denoted by $(\ ,\ )$ and $|\ |$, respectively, the $L^p(\Omega)$ norm by $|\ |_p$, $1 \leqslant p \leqslant \infty$; the $H^m(\Omega)$ norm is denoted by $\|\ \|_m$ and the $W^{k,p}(\Omega)$ norm by $|\ |_{k,p}$. Here $H^m(\Omega)=W^{m,2}(\Omega)$ and $W^{k,p}(\Omega)$ are the usual Sobolev spaces (see Adams [1] for their properties).

Let $D(\Omega)=\{v \in (C_0^\infty(\Omega))^N \text{ div } v=0 \text{ in } \Omega\}$, $H=$completion of $D(\Omega)$ in $L^2(\Omega)$, and $V=$completion of $D(\Omega)$ in $H^1(\Omega)$. We note that, when Ω is Lipschitz-continuous, the spaces H and V can be characterized as

$$H=\{v \in L^2(\Omega); \text{div } v=0 \text{ in } \Omega,\ v.n=0 \text{ on } \partial\Omega\},$$

$$V=\{v \in H_0^1(\Omega); \text{div } v=0 \text{ in } \Omega\},$$

where n is the external orthonormal field to $\partial\Omega$.

Let P be the orthogonal projection of $L^2(\Omega)$ onto H obtained by the usual Helmholtz decomposition. We shall denote by v^k and α^k respectively the eigenfunctions and eigenvalues of the Stokes operator $\tilde{\Delta}=-P\Delta$: $\text{Dom}(\tilde{\Delta}) \subset L^2(\Omega) \to L^2(\Omega)$, where $\text{Dom}(\tilde{\Delta})=V \cap H^2(\Omega)$ is the domain of $\tilde{\Delta}$. If Ω is Lipschitz continuous, it is known that v^k are orthogonal in the inner product $(\cdot,\cdot)$ and $(\nabla\cdot, \nabla\cdot)$ and are complete in the spaces H and V; when Ω is of class C^2 then the eigenfunctions v^k are also complete in $V \cap H^2(\Omega)$ and orthogonal with respect to the inner product $(\tilde{\Delta}\cdot, \tilde{\Delta}\cdot)$ (see, for example, Temam [14].

Similar considerations are true for the Laplace operator Δ; we will denote by ψ^k and λ_k respectively the eigenfunctions and eigenvalues of the operator $-\Delta$.

Let $W^{1-1/p,p}(\partial\Omega)$ be the trace space obtained as the image of $W^{1,p}(\Omega)$ by the boundary value mapping on $\partial\Omega$, equipped with the norm $|\gamma|_{1-1/p,p,\partial\Omega}=\inf\{|v|_{1,p},\ v \in W^{1,p}(\Omega),\ v=\gamma \text{ on } \partial\Omega\}$. Similarly, when $\partial\Omega$ is sufficiently smooth, we can define the trace spaces $W^{k-1/p,p}(\partial\Omega)$ with norm $|\ |_{k-1/p,p,\partial\Omega}$. When $p=2$, we denote $H^{k-1/2}(\partial\Omega)=W^{k-1/2,2}(\partial\Omega)$ and $\|\ \|_{k-1/2,\partial\Omega}=|\ |_{k-1/2,2,\partial\Omega}$ (see Adams [1]).

We assume that we can find a function S defined in Ω satisfying $S = T_0$ on $\partial\Omega$; then we can transform Eqs. (1.1), (1.2) by introducing the new variable $\varphi = T - S$ to obtain

$$\begin{aligned} -\operatorname{div}(\nu(\varphi+S)\,\nabla u) + u.\nabla u - \alpha\varphi g - \alpha S g + \nabla p = 0,& \\ \operatorname{div} u = 0,& \\ -\operatorname{div}(k(\varphi+S)\,\nabla\varphi) + u.\nabla\varphi - \operatorname{div}(k(\varphi+S)\,\nabla S) + u.\nabla S = 0 \qquad &\text{in } \Omega \\ u = 0 \qquad \text{and} \qquad \varphi = 0 \qquad &\text{on } \partial\Omega. \end{aligned} \tag{2.1}$$

Suppose that $S \in H^1(\Omega)$ (thus $T_0 \in H^{1/2}(\partial\Omega)$); then we can reformulate (2.1) in weak form as follows: to find $u \in V$ and $\varphi \in H_0^1(\Omega)$ satisfying

$$\begin{aligned} &(\nu(\varphi+S)\,\nabla u, \nabla v) + B(u, u, v) - \alpha(\varphi g, v) - \alpha(Sg, v) = 0, \\ &\qquad \text{for all } v \text{ in } V \\ &(k(\varphi+S)\,\nabla\varphi, \nabla\psi) + b(u, \varphi, \psi) + (k(\varphi+S)\,\nabla S, \nabla\psi) + b(u, S, \psi) = 0, \\ &\qquad \text{for all } \psi \text{ in } H_0^1(\Omega). \end{aligned} \tag{2.2}$$

where $B(u, v, w) = (u.\nabla v, w) = \int_\Omega \sum_{i,j=1}^N u_j(x)(\partial v_i/\partial x_j)(x)\, w_i(x)\, dx$ and $b(u, \varphi, \psi) = (u.\nabla\varphi, \psi) = \int_\Omega \sum_{j=1}^N u_j(x)(\partial\varphi/\partial x_j)(x)\, \psi(x)\, dx$.

DEFINITION. A pair of functions $(u, T) \in V \times H^1(\Omega)$ is called a *weak solution* of (1.1), (1.2) if there exists a function S in $H^1(\Omega)$ such that $\varphi = T - S \in H_0^1(\Omega)$, $S = T_0$ on $\partial\Omega$, and, (u, φ) is a solution of (2.2).

Based on physical assumptions, throughout the paper we will suppose that

$$\nu(\tau) > 0;\ k(\tau) > 0 \qquad \text{for all} \quad \tau \in \mathbb{R}. \tag{A.1}$$

We observe that assumption (A.1) allows the cases $\liminf_{T \to +\infty} \nu(T) = 0$ or $\limsup_{T \to +\infty} \nu(T) = +\infty$ (the same holds for $k(\cdot)$).

Our first result concerns the existence of weak solutions.

THEOREM 2.1. *Let Ω be a bounded domain in $\mathbb{R}^N$ ($N = 2$ or 3) with Lipschitz continuous boundary; let the functions ν, k be continuous, $g \in L^2(\Omega)$ and $T_0 \in H^{1/2}(\partial\Omega) \cap L^\infty(\partial\Omega)$. Then there exists a weak solution of* (1.1), (1.2). *In case that* $\inf\{\nu(\tau), k(\tau);\ \tau \in \mathbb{R}\} > 0$ *and* $\sup\{\nu(\tau), k(\tau);\ \tau \in \mathbb{R}\} < \infty$ *the result is true under the weaker assumption $T_0 \in H^{1/2}(\partial\Omega)$.*

We remark that in the proof of this result we will obtain approximate solutions by using the eigenfunctions of the Stokes and Laplace operators.

Actually, for this is not necessary to obtain the result in Theorem 2.1, and we could use any basis V and $H_0^1(\Omega)$.

If we have stronger assumptions, we are able to prove the following.

THEOREM 2.2. *Let Ω be a bounded domain in $\mathbb{R}^N$ ($N=2$ or 3) with $C^{1,1}$ boundary; we suppose that v, k are of class C^1, $g \in L^3(\Omega)$ and $T_0 \in H^{3/2}(\partial\Omega)$. Then, if $\|T_0\|_{3/2,\partial\Omega}$ is small enough, there exists a strong solution of* (1.1), (1.2), *that is, there exists a pair $(u, T) \in (V \cap H^2(\Omega)) \times H^2(\Omega)$ such that*

$$\begin{aligned} P(-\operatorname{div}(v(T)\,\nabla u) + u.\nabla u - \alpha T g) &= 0 \qquad && \textit{in } L^2(\Omega), \\ -\operatorname{div}(k(T)\,\nabla T) + u.\nabla T &= 0 && \textit{in } L^2(\Omega), \\ T &= T_0 && \textit{a.e on } \partial\Omega. \end{aligned}$$

We observe that there exists a unique function p (the pressure) in $H^1(\Omega) \cap L_0^2(\Omega)$, with $L_0^2(\Omega) = \{f \in L^2(\Omega)/(f, 1) = 0\}$, such that

$$-\operatorname{div}(v(T)\,\nabla u) + u.\nabla u - \alpha T g = -\operatorname{grad} p. \tag{2.3}$$

For this, see Temam [15]. We also observe that Theorem 2.2 is true if we take α small instead of $\|T_0\|_{3/2,\partial\Omega}$ small.

The next result is concerned with the regularity of (u, T, p).

THEOREM 2.3. *Let Ω be a bounded domain in $\mathbb{R}^N$ ($N=2$ or 3) with a $C^{k+1,1}$ boundary; let the functions v, k be of class C^{k+1}, $g \in W^{k,3}(\Omega)$ and $T_0 \in W^{k+7/4,4}(\partial\Omega)$. Then a strong solution (u, T) satisfies $u \in H^{k+2}(\Omega)$ and $T \in W^{k+2,4}(\Omega)$. Moreover, the associated pressure satisfies $p \in H^{k+1}(\Omega) \cap L_0^2(\Omega)$.*

The following is a uniqueness result for "small" weak solutions.

THEOREM 2.4. *Let Ω be a bounded domain in $\mathbb{R}^N$ ($N=2$ or 3), with a $C^{k+1,1}$ boundary and v, k, k' Lipschitz continuous. There exists $\varepsilon > 0$ such that, if there exists a weak solution (u, T) of* (1.1), (1.2) *satisfying $|\nabla u| + \|T\|_2 < \varepsilon$, then it is unique.*

Finally we state two lemmas for convenience of reference.

By Hölder's inequality and Sobolev imbeddings, we have

LEMMA 2.5. *There exists a constant C_B depending on Ω such that $|B(u, v, w)| \leqslant C_B\, |\nabla u|\, |\nabla v|\, |\nabla w|$ for $\forall u \in H_0^1(\Omega)$, $\forall v \in H^1(\Omega)$, $\forall w \in H_0^1(\Omega)$, and $|b(u, \varphi, \psi)| \leqslant C_B\, |\nabla u|\, |\nabla \varphi|\, |\nabla \psi|$ for $\forall u \in H_0^1(\Omega)$, $\forall \varphi \in H^1(\Omega)$, $\forall \psi \in H_0^1(\Omega)$.*

By density arguments and integration by parts (see Temam [14]), we have

LEMMA 2.6. (i) *$B(u, v, w) = -B(u, w, v)$ for $\forall u \in V$, and $\forall v, w \in H^1(\Omega)$; $b(u, \varphi, \psi) = -b(u, \psi, \varphi)$ for $\forall u \in V$, and $\forall \psi, \varphi \in H^1(\Omega)$*

(ii) *$B(u, v, v) = 0$ for $\forall u \in V$, and $\forall v \in H^1(\Omega)$; $b(u, \varphi, \varphi) = 0$ for $\forall u \in V$, and $\forall \varphi \in H^1(\Omega)$.*

In what follows we will use C as a generic positive constant which depends only on Ω, through constants appearing in the Poincaré inequality and Sobolev inequalities.

3. A PRIORI ESTIMATES

In this section we will show that problem (1.1), (1.2) satisfies a weak maximum principle. Also we will obtain a priori estimate for weak solutions.

LEMMA 3.1. *Let $\{u, T\}$ be a weak solution of* (1.1), (1.2). *Then we have*

$$\inf_{\partial\Omega} T_0 \leqslant T(x) \leqslant \sup_{\partial\Omega} T_0 \qquad \text{a.e. } in\ \bar{\Omega}. \tag{3.1}$$

Proof. Assume $l = \sup_{\partial\Omega} T_0 < \infty$ (If $l = \infty$ we are done.) We take $\psi = T^+$ in (2.2), where $T^+ = \sup\{T - l, 0\}$ to obtain $(k(T)\, \nabla T, \nabla T^+) = -b(u, T, T^+)$. An easy computation shows that $(k(T)\, \nabla T^+, \nabla T^+) = (k(T)\, \nabla T, \nabla T^+) = -b(u, T, T^+) = -b(u, T^+, T^+)$. Therefore, by using Lemma 2.6(ii), we have $\int_\Omega k(T)\, |\nabla T^+|^2 = 0$. Thus, $k(T)\, |\nabla T^+|^2 = 0$ a.e. in Ω, and consequently $|\nabla T^+|^2 = 0$ a.e. in Ω. This last equality implies that $T^+ = 0$ since $T^+ \in H_0^1$; thus, the right hand side of (3.1) follows. The left hand side (3.1) is similarly obtained. ∎

An interesting consequence of the previous lemma is that we can transform problem (1.1), (1.2) into an equivalent one. Suppose that $\inf\{k(t); t \in \mathbb{R}\} = 0$ or $\sup\{k(t), t \in \mathbb{R}\} = +\infty$ and $\sup_{\partial\Omega} |T_0| < +\infty$; then, we consider the modified function $\tilde{k}$, with the same regularity as k and satisfying $\tilde{k}(\tau) = k(\tau)$ for all $|\tau| \leqslant \sup_{\partial\Omega} |T_0|$ and $\inf\{k(t); |t| \leqslant \sup_{\partial\Omega} |T_0|\}/2 \leqslant \tilde{k}(\tau) \leqslant 2 \sup\{k(t); |t| \leqslant \sup_{\partial\Omega} |T_0|\}$ for all $\tau \in \mathbb{R}$.

Analogous considerations can be made for $\nu(\cdot)$.

Clearly, a pair (u, T) is a weak solutions of (1.1), (1.2) if and only if it is a weak solution of the following problem

$$\left.\begin{aligned} -\operatorname{div}(\tilde{\nu}(T)\, \nabla u) + u.\nabla u + \nabla p &= \alpha T g, \\ \operatorname{div} u &= 0, \\ -\operatorname{div}(\tilde{k}(T)\, \nabla T) + u.\nabla T &= 0 \qquad \text{in } \Omega. \end{aligned}\right\} \tag{3.2}$$

$$T = T_0, \qquad u = 0 \qquad \text{on } \partial\Omega. \tag{3.3}$$

Therefore, hereafter we can suppose that the functions $\nu(\cdot)$ and $k(\cdot)$ satisfy

$$\left.\begin{array}{l} 0<\nu_0(T_0)\leqslant \nu(\tau)\leqslant \nu_1(T_0) \\ 0<k_0(T_0)\leqslant k(\tau)\leqslant k_1(T_0) \end{array}\right\} \quad \text{for all} \quad \tau\in\mathbb{R}, \tag{3.4}$$

where $\nu_0(T_0)=\inf\{\nu(t);\ \ |t|\leqslant \sup_{\partial\Omega}|T_0|\}/2$, $\nu_1(T_0)=2\sup\{\nu(t);\ \ |t|\leqslant \sup_{\partial\Omega}|T_0|\}$ with analogous definitions for $k_0(T_0)$ and $k_1(T_0)$.

Remark 3.2. Obviously, if we assume that

$$\inf\{\nu(t), k(t); t\in\mathbb{R}\}>0, \qquad \sup\{\nu(t), k(t), t\in\mathbb{R}\}<+\infty, \tag{3.5}$$

then the above modification is unnecessary.

Now, we prove an a priori estimate. Let $\{u, \varphi\}$ be a weak solution of (1.1), (1.2). Thus, by taking $v=u$ and $\psi=\varphi$ in (2.2), we have

$$(\nu(\varphi+S)\nabla u, \nabla u)+B(u, u, u)-\alpha(\varphi g, u)-\alpha(Sg, u)=0, \tag{3.6}$$

$$(k(\varphi+S)\nabla\varphi, \nabla\varphi)+b(u, \varphi, \varphi)+(k(\varphi+S)\nabla S, \nabla\varphi)+b(u, S, \varphi)=0. \tag{3.7}$$

From Lemma 2.6, Hölder's inequality and (3.4), we obtain $\nu_0(T_0)|\nabla u|^2\leqslant \alpha(\varphi g, u)+\alpha(Sg, u)\leqslant \alpha\,|g|\,(|\varphi|_3+|S|_3)\,|u|_6$. By Sobolev imbeddings, we find

$$|\nabla u|\leqslant \alpha\frac{C}{\nu_0(T_0)}(|g|\,|\nabla\varphi|+|g|\,\|S\|_1). \tag{3.8}$$

Similarly, we have $k_0(T_0)\,|\nabla\varphi|^2\leqslant -b(u, S, \varphi)-(k(\varphi+S)\nabla S,\ \nabla\varphi)= b(u, \varphi, S)-(k(\varphi+S)\nabla S,\ \nabla\varphi)\leqslant |u|_6\,|\nabla\varphi|\ \ |S|_3+k_1(T_0)\,|\nabla S|\ \ |\nabla\varphi|\leqslant C\,|\nabla u|\ |S|_3\,|\nabla\varphi|+k_1(T_0)\,|\nabla S|\ \ |\nabla\varphi|$.

Thus,

$$|\nabla\varphi|\leqslant\frac{C}{k_0(T_0)}\,|S|_3\,|\nabla u|+\frac{k_1(T_0)}{k_0(T_0)}\,\|S\|_1 \tag{3.9}$$

Substituting (3.9) into (3.8), we obtain

$$|\nabla u|\leqslant\frac{\alpha C}{\nu_0(T_0)}\,|g|\left(\frac{C}{k_0(T_0)}\,|S|_3\,|\nabla u|+\frac{k_1(T_0)}{k_0'(T_0)}\,\|S\|_1+\|S\|_1\right).$$

Thus,

$$\begin{aligned}&\left(1-\frac{\alpha C^2}{\nu_0(T_0)\,k_0(T_0)}\,|g|\,|S|_3\right)|\nabla u|\\ &\qquad\leqslant\frac{\alpha C}{\nu_0(T_0)\,k_0(T_0)}\,|g|\,\|S\|_1\,(k_1(T_0)+k_0(T_0)).\end{aligned}$$

If we assume

$$\alpha \frac{C^2}{\nu_0(T_0)\, k_0(T_0)} |g|\, |S|_3 < \frac{1}{2}, \tag{3.10}$$

then we have

$$|\nabla u| \leqslant \frac{2\alpha C}{\nu_0(T_0)\, k_0(T_0)} |g|\, \|S\|_1\, (k_1(T_0) + k_1(T_0) + k_0(T_0)).$$

Substituting this in (3.9), we are left with

$$\begin{aligned}
|\nabla \varphi| &\leqslant \frac{C}{k_0(T_0)} |S|_3 \left[\frac{2\alpha C}{\nu_0(T_0)\, k_0(T_0)} |g|\, \|S\|_1\, (k_1(T_0) + k_0(T_0)) \right] \\
&\quad + \frac{k_1(T_0)}{k_0(T_0)} \|S\|_1 \\
&= \frac{2}{k_0(T_0)} \left(\frac{\alpha C^2\, |g|\, \|S\|_3}{\nu_0(T_0)\, k_0(T_0)} \right) \|S\|_1\, (k_1(T_0) + k_0(T_0)) \\
&\quad + \frac{k_1(T_0)}{k_0(T_0)} \|S\|_1 \\
&\leqslant \frac{2}{k_0(T_0)} \frac{1}{2} \|S\|_1\, (k_1(T_0) + k_0(T_0)) + \frac{k_1(T_0)}{k_0(T_0)} \|S\|_1 \\
&\leqslant \frac{1}{k_0(T_0)} \|S\|_1\, (2k_1(T_0) + k_0(T_0)).
\end{aligned}$$

We summarize these estimates in

$$\left.\begin{aligned}
|\nabla u| &\leqslant \frac{2\alpha C}{\nu_0(T_0)\, k_0(T_0)} |g|\, \|S\|_1\, (k_1(T_0) + k_0(T_0)) \equiv F_1(\|S\|_1), \\
|\nabla \varphi| &\leqslant \left(\frac{2k_1(T_0)}{k_0(T_0)} + 1 \right) \|S\|_1 \equiv F_2(\|S\|_1).
\end{aligned}\right\} \tag{3.11}$$

4. Existence of Weak Solutions

We start by proving the following result (compare with the one in Morimoto [9]).

Lemma 4.1. *Let Ω be a bounded domain in $\mathbb{R}^N$, $N = 2$ or 3, with Lipschitz continuous boundary. If T_0 is a function in $H^{1/2}(\partial\Omega)$, then for any* *positive numbers* ε *and* $1 \leqslant p \leqslant 6$ *if* $n=3$ *or any finite* $p \geqslant 1$ *if* $n=2$, *there exists an extension* $S \in H^1(\Omega)$ *of* T_0 *such that* $|S|_p < \varepsilon$.

Proof. By definition of $H^{1/2}(\partial\Omega)$, we can obtain an extension $\tilde{T}_0 \in H^1(\Omega)$, of T_0. For any $\delta > 0$, we consider $\partial\Omega_\delta = \{x \in \mathbb{R}^N;\ d(x, \partial\Omega) < \delta\}$ and $\beta(\cdot) \in C_0^\infty(\mathbb{R}^N)$ such that $0 \leqslant \beta(\cdot) \leqslant 1$, $\beta(\cdot) \equiv 1$ in $\partial\Omega_{\delta/2}$, $\beta(\cdot) \equiv 0$ in $\mathbb{R}^N \backslash \partial\Omega_\delta$ (we can obtain such a function by applying a differential version of Urysohn's Lemma).

Define $S(x) = \beta(x)\,\tilde{T}_0(x)$. Then S is a required extension, because $S \in H^1(\Omega)$ and

$$|S|_p \leqslant \left(\int_{\Omega \cap \partial\Omega_\delta} |\tilde{T}_0(x)|^p \, dx \right)^{1/p}.$$

Since by Sobolev embedding, $H^1(\Omega) \subset L^p(\Omega)$, with p satisfying the stated conditions, $\int_\Omega |\tilde{T}_0(x)|^p \, dx < +\infty$, and therefore we can choose $\delta > 0$ sufficiently small so that the right hand side of the above inequality is less than ε. ∎

Now we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. According to Lemma 4.1, with $p = 3$, we can choose an extension S of T_0 such that $S \in H^1$ and satisfies (3.10).

As nth aproximate solution of Eq. (2.1) we choose functions $u^n(x) = \sum_{k=1}^n c_{n,k} v^k(x)$ and $\varphi^n(x) = \sum_{k=1}^n d_{n,k} \psi^k(x)$ satisfying the equations

$$(\nu(\varphi^n + S)\,\nabla u^n, \nabla v^j) + B(u^n, u^n, v^j) - \alpha(\varphi^n g, v^j) - \alpha(Sg, v^j) = 0, \tag{4.1}$$

$$\begin{aligned}(k(\varphi^n + S)\,\nabla\varphi^n, \nabla\psi^j) + b(u^n, \varphi^n, \psi^j) + (k(\varphi^n + S)\,\nabla S, \nabla\psi^j)&\\ + b(u^n, S, \psi^j) = 0,&\end{aligned} \tag{4.2}$$

for $1 \leqslant j \leqslant n$.

First we assume the existence of (u^n, φ^n) for any $n \in \mathbb{N}$. Note that solutions (u^n, φ^n) must satisfy estimate (3.11). In fact, the identity (3.6) for u^n is obtained by multiplying (4.1) by $c_{n,j}$ and summing over j from 1 to n. Similarly, we have estimate (3.7) for φ^n.

Estimates (3.11) follow from Eqs. (3.6), (3.7) as in Section 3. Therefore, the sequence (u^n, φ^n) is bounded in $V \times H_0^1$.

Since V (respectively H_0^1) is compactly imbedded in H (respectively $L^2(\Omega)$) we can choose subsequences, which we again denote by (u^n, φ^n), and elements $u \in V$, $\varphi \in H_0^1$ such that $u^n \to u$ weakly in V and strongly in H and also $\varphi^n \to \varphi$ weakly in H_0^1, strongly in L^2 and almost everywhere in Ω. Furthermore, we can suppose that $\nabla u^n \to \nabla u$, weakly in L^2 and $\nabla\varphi^n \to \nabla\varphi$, weakly in L^2.

This is enough to take the limit as n goes to ∞ in (4.1), (4.2) and obtain

$$\begin{aligned} &(\nu(\varphi+S)\,\nabla u, \nabla v^j) + B(u, u, v^j) - \alpha(\varphi g, v^j) - \alpha(Sg, v^j) = 0, \\ &(k(\varphi+S)\,\nabla\varphi, \nabla\psi^j) + b(u, \varphi, \psi^j) + (k(\varphi+S)\,\nabla S, \nabla\psi^j) \\ &\qquad + b(u, S, \psi^j) = 0, \qquad \forall j \in \mathbb{N}. \end{aligned} \tag{4.3}$$

In fact, in taking this limit, there is no difficulty with the nonlinear term. It is easy to see that $B(u^n, u^n, v) \to B(u, u, v)$ $\forall v \in V$ and that $b(u^n, \varphi^n, \psi) \to b(u, \varphi, \psi)$, $\forall \psi \in H^1(\Omega)$ (see, for example, Temam [13]). Also, we observe that

$$\begin{aligned} (\nu(\varphi^n+S)\,\nabla u^n, \nabla v^j) &= (\nabla u^n, \nu(\varphi^n+S)\,\nabla v^j) \to (\nabla u, \nu(\varphi+S)\,\nabla v^j) \\ &= (\nu(\varphi+S)\,\nabla u, \nabla v^j) \end{aligned}$$

because $\nu(\varphi^n+S)\,\nabla v^j \to \nu(\varphi+S)\,\nabla v^j$ strongly in $L^2(\Omega)$ due to the Lebesgue Dominated Convergence Theorem. Similarly, $(k(\varphi^n+S)\,\nabla\varphi^n, \nabla\psi^j) \to (k(\varphi+S)\,\nabla\varphi, \nabla\psi^j)$.

Since the system $\{v^k\}$ (respectively $\{\psi^k\}$) is complete in V (respectively $H_0^1(\Omega)$), (4.3) implies that (u, φ) satisfies (2.1). Therefore, $(u, \varphi+S)$ is a required weak solution.

In order to prove the solvability of the system (4.1), (4.2) for any $n \in \mathbb{N}$, we follow Heywood [8] in applying Brouwer's Fixed Point Theorem.

Let V_n be the subspace of V spanned by $\{v^1, ..., v^n\}$, and let M_n be the subspace spanned by $\{\psi^1, ..., \psi^n\}$. For every $(w, \xi) \in V_n \times M_n$ we consider the unique solution $L(w, \xi) = (u, \varphi) \in V_n \times M_n$ of the linearized equations

$$(\nu(\xi+S)\,\nabla u, \nabla v^j) + B(w, u, v^j) - \alpha(\varphi g, v^j) - \alpha(Sg, v^j) = 0, \tag{4.4}$$

$$\begin{aligned} &(k(\xi+S)\,\nabla\varphi, \nabla\psi^j) + b(w, \varphi, \psi^j) \\ &\qquad + (k(\xi+S)\,\nabla S, \nabla\psi^j) + b(w, S, \psi^j) = 0, \end{aligned} \tag{4.5}$$

for $1 \leqslant j \leqslant n$. This is a system of $2n$ linear equations for the coefficients in the expansions $u = \sum_{k=1}^n c_k v^k$, $\varphi = \sum_{k=1}^n d_k \psi^k$. Equations (4.4), (4.5) have a unique solution because the associated homogeneous system ($S = 0$) has an unique solution. In fact, if (u, φ) is a solution of the homogeneous system, proceeding as before, we multiply (4.4) by c_j, (4.5) by d_j, and sum over j from 1 to n, to obtain $\nu_0(T_0)\,|\nabla u|^2 = 0$, $k_0(T_0)\,|\nabla\varphi|^2 = 0$. Hence $u = 0$, $\varphi = 0$. The continuity of L follows by applying arguments that are similar to the ones used to take the limit in (4.1), (4.2).

We also have the estimates

$$|\nabla u| \leqslant \frac{\alpha C}{v_0(T_0)} (|g|\, |\nabla \varphi| + |g|\, \|S\|_1), \tag{4.6}$$

$$|\nabla \varphi| \leqslant \frac{C}{k_0(T_0)} |S|_3\, |\nabla w| + \frac{k_1(T_0)}{k_0(T_0)} \|S\|_1, \tag{4.7}$$

which are shown in exactly the same way as was done for a solution (u^n, φ^n) of (4.1), (4.2).

Substituting (4.7) into (4.6) and proceeding as we did to obtain (3.11) we find

$$\begin{aligned} |\nabla u| \leqslant & \frac{\alpha C^2}{v_0(T_0)\, k_0(T_0)} |g|\, |S|_3\, |\nabla w| \\ & + \frac{\alpha C}{v_0(T_0)\, k_0(T_0)} |g|\, \|S\|_1\, (k_1(T_0) + k_0(T_0)). \end{aligned}$$

Since (2.10) holds, we have

$$|\nabla u| \leqslant \frac{1}{2} |\nabla w| + \alpha \frac{C}{v_0(T_0)\, k_0(T_0)} |g|\, \|S\|_1|\, (k_1(T_0) + k_0(T_0)). \tag{4.8}$$

If we assume $|\nabla w| \leqslant F_1(\|S\|_1)$ (see (3.11)), then (4.7) and (4.8) imply that (u, φ) satisfies (3.11), that is,

$$|\nabla u| \leqslant F_1(\|S\|_1) \qquad \text{and} \qquad |\nabla \varphi| \leqslant F_2(\|S\|_1). \tag{4.9}$$

Thus, (4.4), (4.5) define a continuous mapping L from the closed and convex set $M = \{(w, \xi) \in V_n \times M_n / |\nabla w| \leqslant F_1(\|S\|_1)$ and $|\nabla \xi| \leqslant F_2(\|S\|_1)\}$ into itself. Using Brower's Fixed Point Theorem, we conclude that the map L has at least one fixed point, which is a solution of (4.1), (4.2). Thus, the proof of Theorem 2.1 is complete. ∎

5. Existence of Strong Solutions

In this section we will prove Theorem 2.2; for this we follow Temam [14] in using the equivalence of the norm given by the Stokes operator (respectively Laplacian operator) and the $V \cap H^2(\Omega)$ norm (respectively $H_0^1(\Omega) \cap H^2(\Omega)$ norm) in smooth domain. The main difficult here is to estimate the nonlinearity in the higher order terms in the velocity equation;

for this we will need an estimate for the associated pressure in the Stokes' Problem.

Proof of Theorem 2.2. We choose the extension S of T_0 such that S is the solution of the problem: $-\Delta S=0$ in Ω, $S=T_0$ on $\partial\Omega$. We know that S is a function in $H^2(\Omega)$ and satisfies

$$\|S\|_2 \leqslant C\,\|T_0\|_{3/2,\,\partial\Omega}. \tag{5.1}$$

According to the proof of Theorem 2.1, we have a sequence (u^n, φ^n) satisfying (4.1), (4.2) provided (3.10) holds. Since $|S|_3 \leqslant C\,\|S\|_1 \leqslant C\,\|T_0\|_{3/2,\,\partial\Omega}$, we conclude the existence of this sequence for $\|T_0\|_{3/2,\,\partial\Omega}$ sufficiently small. We need only to show that we can take this sequence bounded in $H^2(\Omega)$.

For this, we multiply Eq. (4.4) by $\alpha^j c_j$, and then sum over j from 1 to n, to obtain $(\operatorname{div}(v(\xi+S)\,\nabla u), \tilde{\Delta} u) - B(w, u, \tilde{\Delta} u) + \alpha(Sg, \tilde{\Delta} u) + \alpha(\varphi g, \tilde{\Delta} u) = 0$. By using the identity $\operatorname{div}(v(\xi+S)\,\nabla v) = v(\xi+S)\,\Delta v + v'(\xi+S)\,\nabla(\xi+S)\,\nabla v$, where $\nabla(\xi+S)\,\nabla v$ denotes the vector field which ith component is given by $[\nabla(\xi+S)\,\nabla v]_i = (\nabla(\xi+S), \nabla v_i)_{\mathbb{R}^n}$, and where $(\cdot,\cdot)_{\mathbb{R}^n}$ denotes the canonical inner product in $\mathbb{R}^n$, we find

$$\begin{aligned} -(v(\xi+S)\,\Delta u, \tilde{\Delta} u) = {} & -B(w, u, \tilde{\Delta} u) + \alpha(\varphi g, \tilde{\Delta} u) + \alpha(Sg, \tilde{\Delta} u) \\ & + (v'(\xi+S)\,\nabla(\xi+S)\,\nabla u, \tilde{\Delta} u). \end{aligned} \tag{5.2}$$

Since $-\Delta u \neq \tilde{\Delta} u$, we need the following decomposition $\tilde{\Delta} u + \operatorname{grad} q = -\Delta u$. It is well known that (see Teman [14])

$$\|q\|_1 \leqslant C\,|\tilde{\Delta} u|. \tag{5.3}$$

Now, we can rewrite (5.2) as $(v(\xi+S)\,\tilde{\Delta} u, \tilde{\Delta} u) = -B(w, u, \tilde{\Delta} u) + \alpha(\varphi g, \tilde{\Delta} u) + \alpha(Sg, \tilde{\Delta} u) + (v'(\xi+S)\,\nabla(\xi+S)\,\nabla u, \tilde{\Delta} u) + (v(\xi+S)\,\nabla q, \tilde{\Delta} u)$.

By Hölder's inequality, Sobolev imbedding, and (3.4), we have

$$\begin{aligned} |\tilde{\Delta} u|^2 \leqslant {} & \frac{C}{v_0(T_0)}\,(|\nabla w| + v_1'(T_0)(|\nabla\xi| + \|S\|_2))\,|\tilde{\Delta} u|^2 \\ & + \frac{\alpha C}{v_0(T_0)}\,|g|_3\,(|\nabla\varphi| + \|S\|_1)\,|\tilde{\Delta} u| + \left|\left(\frac{v(\xi+S)}{v_0(T_0)}\,\nabla q, \tilde{\Delta} u\right)\right|, \end{aligned} \tag{5.4}$$

where $v_1'(T_0) = 2\sup\{|v'(t)|;\ |t| \leqslant \sup_{\partial\Omega} |T_0|\}$. We observe that

$$\begin{aligned} (v(\xi+S)\,\nabla q, \tilde{\Delta} u) &= -(q, \operatorname{div}(v(\xi+S)\,\tilde{\Delta} u)) \\ &= -(q, (v'(\xi+S)\,\nabla(\xi+S), \tilde{\Delta} u)_{\mathbb{R}^n}) \end{aligned}$$

because $\tilde{\Delta} u \in V_n$. Therefore,

$$\left|\left(\frac{v(\xi+S)}{v_0(T_0)}\,\nabla q,\,\tilde{\Delta} u\right)\right| \leqslant \frac{v_1'(T_0)}{v_0(T_0)}\,|q|_4\,|\nabla(\xi+S)|_4\,|\tilde{\Delta} u|$$

$$\leqslant C\,\frac{v_1'(T_0)}{v_0(T_0)}\,(|\Delta\xi|+\|S\|_2)\,\|q\|_1\,|\tilde{\Delta} u|. \tag{5.5}$$

Combining estimates (5.4), (5.5), and (5.1), we have

$$|\tilde{\Delta} u| \leqslant \frac{\bar{C}}{v_0(T_0)}\,(|\tilde{\Delta} w| + 2v_1'(T_0)(|\Delta\xi| + \|T_0\|_{3/2,\,\partial\Omega}))\,|\tilde{\Delta} u|$$

$$+\alpha\,\frac{\bar{C}}{v_0(T_0)}\,|g|_3\,(|\Delta\varphi| + \|T_0\|_{3/2,\,\partial\Omega}). \tag{5.6}$$

Similarly, the following estimate holds,

$$|\Delta\varphi| \leqslant \frac{\bar{C}}{k_0(T_0)}\,(|\tilde{\Delta} w| + k_1'(T_0)(|\Delta\xi| + \|T_0\|_{3/2,\,\partial\Omega}))(|\Delta\varphi| + \|T_0\|_{3/2,\,\partial\Omega})$$

$$+\frac{k_1(T_0)}{k_0(T_0)}\,\|T_0\|_{3/2,\,\partial\Omega}, \tag{5.7}$$

where $k_1'(T_0) = 2\sup\{|k'(t)|;\ |t| \leqslant \sup_{\partial\Omega}|T_0|\}$ and $\bar{C}$ is a positive constant. Now, we take (w, ξ) such that $|\tilde{\Delta} w| \leqslant (4\alpha\bar{C}/v_0(T_0))\,|g|_3\,(1 + k_1(T_0)/k_0(T_0))\,\|T_0\|_{3/2,\,\partial\Omega}$, $|\Delta\xi| \leqslant (1 + 2(k_1(T_0)/k_0(T_0)))\,\|T_0\|_{3/2,\,\partial\Omega}$ and we take $\|T_0\|_{3/2,\,\partial\Omega}$ sufficiently small so that

$$\frac{\bar{C}}{k_0(T_0)}\left[\,1 + 2k_1'(T_0)\left(1 + \frac{k_1(T_0)}{k_0(T_0)}\right)\right]\,\|T_0\|_{3/2,\,\partial\Omega} < \frac{1}{2} \tag{5.8}$$

and

$$\frac{\bar{C}}{v_0(T_0)}\left[\frac{4\alpha\bar{C}}{v_0(T_0)}\,|g|_3 + 4v_1'(T_0)\right]\left(1 + \frac{k_1(T_0)}{k_0(T_0)}\right)\,\|T_0\|_{3/2,\,\partial\Omega} < \frac{1}{2}. \tag{5.9}$$

Observe that this is possible because $\lim_{\delta\to 0+} k(\delta) > 0$ and $\lim_{\delta\to 0+} v(\delta) > 0$. Then, estimates (5.6), (5.7) imply

$$|\tilde{\Delta} u| \leqslant 4\,\frac{\alpha\bar{C}}{v_0(T_0)}\,|g|_3\left(1 + \frac{k_1(T_0)}{k_0(T_0)}\right)\,\|T_0\|_{3/2,\,\partial\Omega} \equiv r$$

$$|\Delta\varphi| \leqslant \left(1 + 2\,\frac{k_1(T_0)}{k_0(T_0)}\right)\,\|T_0\|_{3/2,\,\partial\Omega} \equiv s.$$

Consequently, if $\|T_0\|_{3/2,\partial\Omega}$ is small enough, L maps the closed and convex set $\{(w,\xi)\in V_n\times M_n;\ |\tilde{\Delta} w|\leqslant r \text{ and } |\Delta\xi|\leqslant s\}$ into itself. Then we can choose a sequence (u^n,φ^n) bounded in $H^2(\Omega)$ satisfying (4.1), (4.2), and Theorem 2.2 follows. ∎

6. Regularity

We first state some lemmas that will be necessary to prove Theorem 2.3.

Lemma 6.1. *Let h be any function of class C^k such that* $\sup\{|(d^ih/dt^i)(t)|, t\in\mathbb{R}\}\leqslant C$, $i=0,\ldots,k$. *Then there exists constants $C(k)$ and $C_1(k)$ such that for all $T\in W^{k+1,4}(\Omega)$,*

$$\text{(i)}\quad |h(T)|_{k,\infty}\leqslant C(k)\sum_{l=0}^{k}|T|^l_{k+1,4};\qquad \text{(ii)}\quad |h(T)|_{k,4}\leqslant C_1(k)\sum_{l=0}^{k}|T|^l_{k,4}.$$

Proof. We only prove (i); the other inequality can be proved in the same way. We proceed by induction on k.

If $k=0$, the result is trivial. So suppose the result is true for any $j\in\mathbb{N}$ such that $0\leqslant j<k$, and take $\beta=(\beta_1,\ldots,\beta_n)$, $\beta_i\in\mathbb{N}$, $|\beta|=\beta_1+\cdots+\beta_n=k$. Then, if $i\in\{1,\ldots,n\}$ is such that $\beta_i>0$, we have $\partial^\beta(h(T))=\partial^{\beta'}(\partial_{x_i}(h(T))=\partial^{\beta'}(h'(T).\partial_{x_i}T)=\sum_{\gamma+\delta=\beta'}c(\gamma,\delta)\,\partial^\gamma(h'(T))\,\partial^\delta(\partial_{x_i}T)$, where $\beta'=\beta-e_i$ and e_i is the ith vector in the canonical basis of $\mathbb{R}^n$, $c(\gamma,\delta)$ are positive constants, and $\gamma=(\gamma_1,\ldots,\gamma_n)$, $\delta=(\delta_1,\ldots,\delta_n)$, $\gamma_j,\delta_j\in\mathbb{N}$.

Thus, by the inductive hypothesis and the Sobolev imbeddings,

$$\begin{aligned}
|\partial^\beta(h(T))|_\infty &\leqslant \sum_{\gamma+\delta=\beta'} c(\gamma,\delta)\,|\partial^\gamma(h'(T))|_\infty\,|\partial^\delta(\partial_{x_i}T)|_\infty\\
&\leqslant \sum_{\gamma+\delta=\beta'} c(\gamma,\delta)\,M(|\gamma|)\sum_{l=1}^{|\gamma|}|T|^l_{|\gamma|+1,4}\,|T|_{|\delta|+1,\infty}\\
&\leqslant \sum_{\gamma+\delta=\beta'} c(\gamma,\delta)\,M(|\gamma|)\,C\sum_{l=1}^{|\gamma|}|T|^l_{|\gamma|+1,4}\,|T|_{|\delta|+2,4}.
\end{aligned}$$

Now, we observe that $|\gamma|\leqslant k-1$, $|\delta|\leqslant k-1$ and so $|\gamma|+1\leqslant k$ and $|\delta|+2\leqslant k+1$. Thus,

$$|\partial^\beta(h(T))|_\infty\leqslant C(k)\sum_{l=1}^{k}|T|^l_{k+1,4}$$

and the lemma is proved. ∎

LEMMA 6.2. *If h satisfies the conditions of Lemma* 6.1, *then for all T in $W^{k+1,4}(\Omega)$ and all f in $W^{k,p}(\Omega)$ there holds*

$$|h(T)\,f|_{k,p} \leqslant C_1(k)\left(\sum_{l=1}^{k} |T|_{k+1,4}^{l}\right)|f|_{k,p}.$$

Proof. Let $\beta=(\beta_1,...,\beta_n)$, $\beta_i \in \mathbb{N}$, $|\beta|=\beta_1+\cdots+\beta_n=k$. As before

$$\partial^\beta(h(T)\,f)=\sum_{\gamma+\delta=\beta} c(\gamma,\delta)\,\partial^\gamma(h(T))\,\partial^\delta f;$$

therefore, by Lemma 6.1, we have

$$\begin{aligned}
|\partial^\beta(h(T)\,f)|_p &\leqslant \sum_{\gamma+\delta=\beta} c(\gamma,\delta)|\,\partial^\gamma(h(T))|_\infty\,|\partial^\delta f|_p \\
&\leqslant \sum_{\gamma+\delta=\beta} c(\gamma,\delta)\,C(\gamma,\delta)\sum_{l=1}^{|\gamma|} |T|_{|l|+1,4}^{l}\,|\partial^\delta f|_p \\
&\leqslant C_1(k)\left(\sum_{l=1}^{k} |T|_{k+1,4}^{l}\right)|f|_{k,p}.
\end{aligned}$$

This proves the lemma. ∎

LEMMA 6.3. *Let* (u,T) *be a strong solution of* (1.1), (1.2). *Assume* $T_0 \in W^{7/4,4}(\partial\Omega)$; *then* $T\in W^{2,4}(\Omega)$.

Proof. According to Section 3, we can suppose that (3.4) holds. We observe that $T\in H^2(\Omega)$ satisfies

$$\begin{aligned}
-\Delta T+\frac{k'(T)}{k(T)}|\nabla T|^2+\frac{1}{k(T)}\,u.\nabla T=0 \qquad &\text{in } \Omega \\
T=T_0 \qquad &\text{on } \partial\Omega.
\end{aligned} \tag{6.1}$$

Since $T_0 \in W^{7/4,4}(\partial\Omega)\subseteq W^{5/3,3}(\partial\Omega)$ and $T\in H^2(\Omega)$, $u\in H'(\Omega)$ (because (u,T) is a strong solution), and $(k'(T)/k(T))\,|\nabla T|^2+(1/k(T))\,u.\nabla T\in L^3(\Omega)$. Thus we can apply the well-known L^p-regularity properties of the Laplace operator, obtaining T in $W^{2,3}(\Omega)$. By using Sobolev imbedding we have that $\nabla T\in L^q(\Omega)$, for all $1\leqslant q<\infty$. Consequently, $(k'(T)/k(T))\,|\nabla T|^2+(1/k(T))\,u.\nabla T\in L^4(\Omega)$, and by applying the L^p-regularity once again, we find $T\in W^{2,4}(\Omega)$. ∎

Proof of Theorem 2.3. We proceed inductively on k. If $k=0$ the result follows by Lemma 6.3. Now we suppose the result is true for $k-1$ (that is,

$T \in W^{k+1,4}(\Omega)$). Note that if $\beta = (\beta_1, ..., \beta_n)$, $\beta_i \in \mathbb{N}$, $|\beta| = \beta_i + \cdots + \beta_n = k$, then

$$\partial^\beta(u \nabla T) = \sum_{\gamma+\delta=\beta} c(j)\, \partial^\gamma u\, \partial^\delta(\nabla T) + u\, \partial^\beta(\nabla T),$$

where $c(j)$ are positive constants, $\gamma = (\gamma_1, ..., \gamma_n)$, $\delta = (\delta_1, ..., \delta_n)$, $\gamma_i, \delta_i \in \mathbb{N}$. Thus,

$$|\partial^\beta(u \nabla T)|_4 \leqslant \sum_{\gamma+\delta=\beta} c(j)\, |\partial^\gamma u|_4\, |\partial^\delta(\nabla T)|_\infty + |u|_\infty\, |\partial^\beta(\nabla T)|_4,$$

and Sobolev imbeddings together with the inductive hypothesis imply $|u\nabla T|_{k,4} \leqslant C_{\Omega,k}\, |u|_{k+1,2}\, |T|_{k+1,4} + C_{\Omega,k}\, \|u\|_2\, |T|_{k+1,4} < \infty$. By using Lemma 6.2, we conclude $(1/\nu(T))\, u \nabla T \in W^{k,4}(\Omega)$. Similarly as before, there holds

$$\begin{aligned} |\partial^\beta\, |\nabla T|^2|_4 &\leqslant \sum_{\gamma+\delta=\beta} c(j)\, |\partial^\gamma \nabla T|_\infty\, |\partial^\delta \nabla T|_4 + |\partial^\beta \nabla T|_4\, |\nabla T|_\infty \\ &\leqslant C_{\Omega,k}\, |T|^2_{k+1,4} + C_{\Omega,k}\, |T|_{k+1,4}\, |T|_{2,4} < \infty. \end{aligned}$$

Therefore, we have $(k'(T)/k(T))\, |\nabla T|^2 \in W^{k,4}(\Omega)$. Now, by applying the L^p-regularity for problem (6.1), we see that $T \in W^{k+2,4}(\Omega)$.

As in the proof of Lemma 6.3, we have that u is a solution of the Stokes problem

$$\begin{aligned} -\Delta u + \operatorname{grad}\left(\frac{p}{\nu(T)}\right) &= -\frac{\nu'(T)}{\nu(T)^2}\, p\, \nabla T - \frac{1}{\nu(T)}\, u.\nabla u \\ &\quad + \alpha\, \frac{T}{\nu(T)}\, g + \frac{\nu'(T)}{\nu(T)}\, \nabla T.\nabla u, \\ \operatorname{div} u &= 0 \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{6.2}$$

where p satisfies (2.2). As above, we show by induction that the right hand side terms in the first equation of (6.2) are in $H^k(\Omega)$. By Cattabriga's Theorem (see [5] and [2]) applied to (6.2), we find that $u \in H^{k+2}(\Omega)$, $p/\nu(T) \in H^{k+1}(\Omega)$.

Applying Lemma 6.2 we conclude that $p \in H^{k+1}(\Omega)$. This completes the proof. ∎

7. Uniqueness

In this section we will prove Theorem 2.4. Let (u_1, T_1), (u_2, T_2) be a weak solution of (1.1), (1.2) such that T_1 and T_2 are in $H^2(\Omega)$. Put $w = u_1 - u_2$, $\xi = T_1 - T_2$. Then $w \in V$, $\xi \in H_0^1(\Omega) \cap H^2(\Omega)$ and satisfy for $\forall v \in V$, $\forall \psi \in L^2(\Omega)$ $(\nu(T_1)\,\nabla w, \nabla v) + B(w, u_1, v) + B(u_2, w, v) - \alpha(\xi g, v) + ((\nu(T_1) - \nu(T_2))\,\nabla u_2, \nabla v) = 0$, $-(\operatorname{div}(k(T_1)\,\nabla \xi), \psi) + b(w, T_1, \psi) + b(u_2, \xi, \psi) - (\operatorname{div}((k(T_1) - k(T_2))\,\nabla T_2), \psi) = 0$.

We take $v = w$ and $\psi = -\Delta\xi$ in these last equalities, thus obtaining

$$|\nabla w| \leqslant \frac{\alpha C}{\nu_0(T_0)}\,|g|\,|\Delta\xi| + \frac{C_B}{\nu_0(T_0)}\,|\nabla u|\,|\nabla u_1| + \frac{C_1}{\nu_0(T_0)}\,|\xi|_\infty\,|\nabla u_2| \tag{7.1}$$

$$\begin{aligned}|\Delta\xi| \leqslant{} & \frac{C}{k_0(T_0)}\,\|T_1\|_2\,|\nabla w| + \frac{C}{k_0(T_0)}\,|\nabla u_2|\,|\Delta\xi| + \frac{C_1}{\nu_0(T_0)}\,|\xi|_\infty\,|\Delta T_2| \\ & + \frac{k_1'(T_0)}{k_0(T_0)}\,C\,\|T_2\|_2\,|\Delta\xi| + \frac{C_1}{k_0(T_0)}\,\|T_2\|_2^2\,|\xi|_\infty,\end{aligned} \tag{7.2}$$

where C_1 is a constant such that $|k'(t) - k'(s)| + |k(t) - k(s)| + |\nu(t) - \nu(s)| \leqslant C_1\,|t - s|$ for all t, s in $\mathbb{R}$. This is shown in exactly the same way as in the case of the nth aproximate solutions (u^n, φ^n) in Section 4 and 5.

We note that $|\xi|_\infty \leqslant C\,|\Delta\xi|$. Thus, (7.2) implies that

$$\begin{aligned}|\Delta\xi| \leqslant{} & \frac{C}{k_0(T_0)}\,\|T_1\|_2\,|\nabla w| \\ & + \frac{C}{k_0(T_0)}\,[\,|\nabla u_2| + (C_1 + k_1'(T_0))\,\|T_2\|_2 + C_1\,\|T_2\|_2^2]\,|\Delta\xi|.\end{aligned}$$

Assume that $(C/k_0(T_0))[\,|\nabla u_2| + (C_1 + k_1'(T_0))\,\|T_2\|_2 + C_1\,\|T_2\|_2^2)] < \frac{1}{2}$; then

$$|\Delta\xi| \leqslant \frac{2}{k_0(T_0)}\,\|T_1\|_2\,|\nabla w|. \tag{7.3}$$

Substituting (7.3) into (7.1), we obtain

$$|\nabla w| \leqslant \left[\frac{C}{\nu_0(T_0)\,k_0(T_0)}\,\|T_1\|_2\,(\alpha C\,|g| + C_1\,|\nabla u_2|) + \frac{C_B}{\nu_0(T_0)}\,|\nabla u_1|)\right]\,|\nabla w|.$$

Thus, if $(C/\nu_0(T_0)\,k_0(T_0))\,\|T_1\|_2\,(\alpha c\,|g| + C_1\,|\nabla u_2|) + (C_B/\nu_0(T_0))\,|\nabla u_1| < 1$, we have $|\nabla w| = |\Delta\xi| = 0$. Since $w \in V$ and $\xi \in H_0^1(\Omega) \cap H^2(\Omega)$, we see $w = 0$, $\xi = 0$ in Ω. Therefore, $u_1 = u_2$, $T_1 = T_2$.

References

1. R. A. Adams, "Sobolev Spaces," Academic Press, New York, 1975.
2. C. Amrouche and V. Girault, On the existence and regularity of the solution of Stokes problem in arbitrary dimension, *Proc. Japan Acad.* **67** (1991), 171–175.
3. C. Bernardi, F. Laval, B. Metivet, and B. Pernaud-Thomas, Finite element approximation of viscous flow with varying density, *SIAM J. Numer. Anal.* **29** (1992), 1203–1243.
4. F. H. Busse, The stability of finite amplitude cellular convection and its relation to an extremun principle, *J. Fluid Mech.* **30** (1967), 625–649.
5. I. Cattabriga, Su un problema al contorno relativo al sistema di equazioni di Stokes, *Rend. Sem. Univ. Padova* **31** (1961), 308–340.
6. P. G. Drazin and W. H. Reid, "Hydrodynamic Stability," Cambridge Univ. Press, 1981.
7. A. Graham, Shear patterns in a unstable layer of air, *Philos. Trans. Roy. Soc. London Ser. A* **232** (1933), 285–296.
8. J. G. Heywood, The Navier–Stokes equations: on the existence, regularity and decay of solutions, *Indiana Univ. Math. J.* **29** (1980), 639–681.
9. H. Morimoto, On the existence of weak solutions of equation of natural convection, *J. Fac. Sci. Univ. Tokyo* **36** (1989), 87–102.
10. H. Morimoto, On the existence and uniqueness of the stationary solution to the equations of natural convection, *Tokyo J. Math.* **14** (1991), 220–226.
11. E. Palm, On the tendency towards hexagonal cells in steady convection, *J. Fluid Mech.* **8** (1960), 183–192.
12. E. Palm, T. Ellingsen, and B. Gjevik, On the ocurrence of cellular motion in Bénard convection, *J. Fluid Mech.* **30** (1967), 651–661.
13. P. H. Rabinowitz, Existence and nonuniqueness of rectangular solutions of the Bénard problem, *Arch. Rational Mech. Anal.* **29** (1968), 32–57.
14. R. Temam, "Navier–Stokes Equations," North-Holland, Amsterdam, 1977.
15. H. von Tippelkirch, Über Konvektionszeller insbesondere im flüssigen Schwefel, *Beiträge Phys. Atmos.* **20** (1956), 37–54.