J. Math. Anal. Appl. 376 (2011) 494–505

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Some properties of a generalized Hamy symmetric function and its applications

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ARTICLE INFO

Article history:
Received 28 January 2010
Available online 8 October 2010
Submitted by B.S. Thomson

Keywords:
Convexity
Schur-convexity
Hamy symmetric function
Monotonicity
Inequality

ABSTRACT

This paper is concerned with the generalized Hamy symmetric function

$$\sum_n(x,r;f)=\sum_{1\leqslant i_1<i_2<\cdots<i_r\leqslant n} f\left(\prod_{j=1}^r x_{i_j}^{\frac{1}{r}}\right),$$

where f is a positive function defined in a subinterval of $(0,+\infty)$. Some properties, including Schur-convexity, geometric Schur-convexity and harmonic Schur-convexity are investigated. As applications, several inequalities are obtained, some of which extend the known ones.

1. Introduction

Throughout this paper, let R_+ denote the set of all positive real numbers and R_+^n its n-product. For $\Omega\subseteq R_+^n$, let

$$\ln\Omega=\{(\ln x_1,\ln x_2,\dots,\ln x_n)\mid x=(x_1,x_2,\dots,x_n)\in\Omega\}$$

and

$$1/\Omega=\{(1/x_1,1/x_2,\dots,1/x_n)\mid x=(x_1,x_2,\dots,x_n)\in\Omega\}.$$

For a positive n-tuple $x=(x_1,x_2,\dots,x_n)\in R_+^n$, Hamy [12] introduced the symmetric function

$$F_n(x,r)=F_n(x_1,\dots,x_n;r)=\sum_{1\leqslant i_1<i_2<\cdots<i_r\leqslant n}\left(\prod_{j=1}^r x_{i_j}\right)^{\frac{1}{r}},\quad r=1,2,\dots,n. \tag{1.1}$$

In Hamy's honor, the above function is called Hamy symmetric function. Corresponding to this function is the r-th order Hamy mean

$$\sigma_n(x,r)=\frac{1}{\binom{n}{r}}\sum_{1\leqslant i_1<i_2<\cdots<i_r\leqslant n}\left(\prod_{j=1}^r x_{i_j}\right)^{\frac{1}{r}},\quad r=1,2,\dots,n, \tag{1.2}$$

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2010.10.014

where $\binom{n}{r} = \frac{n!}{(n-r)!r!}$. It is obvious that $\sigma_n(x,1)$ is the arithmetic mean

$$A_n(x) = A_n(x_1, x_2, \ldots, x_n) = \frac{x_1 + x_2 + \cdots + x_n}{n},$$

and $\sigma_n(x,n)$ is the geometric mean

$$G_n(x) = G_n(x_1, x_2, \ldots, x_n) = \sqrt[n]{x_1 x_2 \ldots x_n}.$$

There are some papers on Hamy symmetric function and its mean. For example, Hara et al. [13] established the following refinement of the classical arithmetic and geometric means inequality:

$$G_n(x) = \sigma_n(x,n) \leqslant \sigma_n(x,n-1) \leqslant \cdots \leqslant \sigma_n(x,2) \leqslant \sigma_n(x,1) = A_n(x). \tag{1.3}$$

The paper [17] by Ku et al. contains some interesting inequalities including the fact that $(\sigma_n(x,r))^r$ is log-concave. For more details, please refer to the book [6] by Bullen. In 2006, Guan [8] investigated Schur-convexity of Hamy symmetric function $F_n(x,r)$ and some inequalities were also obtained by use of the theory of majorization.

Recently, Guan [9] defined a generalized Hamy symmetric function of the form

$$\sum\nolimits_n(x,r;f) = \sum_{1 \leqslant i_1 < i_2 < \cdots < i_r \leqslant n} f\left(\prod_{j=1}^{r} x_{i_j}^{\frac{1}{r}}\right), \quad r = 1, 2, \ldots, n, \tag{1.4}$$

where f is a positive function defined in a subinterval of $(0, +\infty)$. The author investigated the geometric Schur-convexity of $\sum_n(x,r;f)$ when f is a multiplicatively convex function, i.e., GG-convex function.

The main purpose of this paper is to investigate further Schur-convexity, geometric Schur-convexity, and harmonic Schur-convexity of $\sum_n(x,r;f)$. As applications, some inequalities are established by use of the theory of majorization. Our results improve the known ones.

The notation of Schur-convex function was introduced by I. Schur in 1923 [24]. It has many important applications in analytic inequalities [4,8,10,14,19,25], combinatorial optimization [15], isoperimetric problem for polytopes [27], gamma and digamma functions [20], and other related fields. For a historical development of this kind of functions and the fruitful applications to statistics, economics and other applied fields, refer to the popular book by Marshall and Olkin [19].

Definition 1.1. (See [10,19,24–26].) A real-valued function ϕ defined on a set $\Omega \subseteq R^n$ $(n \geqslant 2)$ is said to be a Schur-convex function on Ω if

$$\phi(x_1, x_2, \ldots, x_n) \leqslant \phi(y_1, y_2, \ldots, y_n)$$

for each pair of n-tuples $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ on Ω, such that x is majorized by y (in symbols $x \prec y$), that is,

$$\sum_{i=1}^{m} x_{[i]} \leqslant \sum_{i=1}^{m} y_{[i]}, \quad m = 1, 2, \ldots, n-1, \quad \text{and} \quad \sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]},$$

where $x_{[i]}$ denotes the ith largest component in x. ϕ is called Schur-concave if $-\phi$ is Schur-convex.

The notation of geometric convexity was introduced by Montel [22] and investigated by Anderson et al. [3], Guan [11] and Niculescu [23]. The geometric Schur-convexity was investigated by Chu et al. [7], Guan [9], and Niculescu [23]. We also note that the authors use the term "Schur-multiplicative (geometric) convexity". However, we here point out that the term "geometric Schur-convexity" is more appropriate. As a matter of fact, from [7,9,11,22,23], we have no difficulty to find that f being geometric convex in convexity theory means that the function $x \mapsto \log f(e^x)$ is convex and that "Schur-multiplicative (geometric) convexity" of ϕ is equivalent to Schur-convexity of the function $x \mapsto \phi(e^x)$, which in turn, for positive functions, is equivalent to Schur-convexity of the function $x \mapsto \log \phi(e^x)$. Thus, we give an alternative definition of geometric Schur-convexity.

Definition 1.2. Let $\Omega \subseteq R_+^n$ $(n \geqslant 2)$ be a set. A real-valued function $\phi : \Omega \to R$ is called a geometrically Schur-convex function on Ω if the function $x \mapsto \phi(e^x)$ is Schur-convex on $\ln \Omega$. ϕ is called geometrically Schur-concave if $-\phi$ is geometrically Schur-convex.

Recently, Xia et al. [26] introduced the notion of harmonically Schur-convex function and some interesting inequalities were obtained.

Definition 1.3. (See [26].) Let $\Omega \subseteq R_+^n$ $(n \geqslant 2)$ be a set. A real-valued function ϕ defined on Ω is called a harmonically Schur-convex function if the function $x \mapsto \phi(1/x)$ is Schur-convex on $1/\Omega$. ϕ is called a harmonically Schur-concave function on Ω if $-\phi$ is harmonically Schur-convex.

Let $M(x, y)$ and $N(x, y)$ be any two mean functions of two positive numbers $x, y \in R_+$. Anderson et al. [3] introduced the definition of MN-convex function as follows.

Definition 1.4. (See [3].) Let $f : I \to R_+$ be continuous, where I is a subinterval of $(0, \infty)$. The function f is called MN-convex (concave) if $f(M(x, y)) \leqslant (\geqslant) N(f(x), f(y))$, for all $x, y \in I$.

Remark 1.5. Let $A(x, y)$ $(G(x, y), H(x, y))$ denote the arithmetic (geometric, harmonic) mean of two positive numbers x, y, it follows from [3] that

(1) f is GG-convex $\Rightarrow$ GA-convex;
(2) f is HH-convex $\Rightarrow$ HG-convex $\Rightarrow$ HA-convex. For concavity, the implications in (1) and (2) are reversed.

2. Lemmas

In order to establish our main results we need several lemmas, which we present in this section. The following lemma is so-called Schur's condition which is very useful for determining whether or not a given function is Schur-convex or Schur-concave.

Lemma 2.1. *(See [8,10,19].) Let $\Omega \subseteq R^n$ be symmetric and convex set with nonempty interior, and let $f : \Omega \to R$ be differentiable in the interior of Ω and continuous on Ω. Then f is Schur-convex on Ω if and only if f is symmetric on Ω and*

$$(x_1 - x_2)\left(\frac{\partial f}{\partial x_1} - \frac{\partial f}{\partial x_2}\right) \geqslant 0 \tag{2.1}$$

for all $x \in \Omega^0$, where Ω^0 is the interior of Ω.

Schur's condition that guarantees a symmetric function being Schur-concave is the same as (2.1) except the direction of the inequality.

Remark 2.2. From Definitions 1.2 and 1.3, Lemma 2.1 implies the following conclusion (also see [7,9,10,26]).

Let $f(x) = f(x_1, x_2, \ldots, x_n)$ be symmetric and have continuous partial derivatives on I^n, where I is a subinterval of $(0, \infty)$. Then

(1) $f : I^n \to R$ is a geometrically Schur-convex function if and only if

$$(\ln x_1 - \ln x_2)\left(x_1 \frac{\partial f(x)}{\partial x_1} - x_2 \frac{\partial f(x)}{\partial x_2}\right) \geqslant 0 \tag{2.2}$$

for all $x \in I^n$, f is geometrically Schur-concave if and only if (2.2) is reversed.

(2) $f : I^n \to R$ is a harmonically Schur-convex function if and only if

$$(x_1 - x_2)\left(x_1^2 \frac{\partial f(x)}{\partial x_1} - x_2^2 \frac{\partial f(x)}{\partial x_2}\right) \geqslant 0 \tag{2.3}$$

for all $x \in I^n$, f is harmonically Schur-concave if and only if (2.3) is reversed.

Lemma 2.3. *(See [19, p. 89].) Let Ω be a symmetric and convex subset of R^l, and let ϕ be a Schur-convex function defined on Ω with the property for each fixed $x_2, \ldots, x_l$, the function*

$$\phi(z, x_2, \ldots, x_l) \text{ is convex in } z \text{ on } \left\{z \mid (z, x_2, \ldots, x_l) \in \Omega\right\}.$$

Then for any $n > l$,

$$\psi(x_1, \ldots, x_n) = \sum_{\pi} \phi(x_{\pi(1)}, \ldots, x_{\pi(l)})$$

is Schur-convex on $\Omega^* = \{(x_1, \ldots, x_n) \mid (x_{\pi(1)}, \ldots, x_{\pi(l)}) \in \Omega$ for all permutations $\pi\}$.

Recall that Anderson et al. [3] investigated the relation between convexity and GA (HA, GG)-convexity for a function defined in $(0, b)$, $0 < b < \infty$. We can establish a more general lemma in a similar way as follows.

Lemma 2.4. *Let $I \subseteq R_+$ be an interval and $f : I \to R_+$ be continuous in I. Then*

(1) *f is GA-convex (concave) in I if and only if $f(e^x)$ is convex (concave) in $\ln I = \{\ln x \mid x \in I\}$;*
(2) *f is HA-convex (concave) in I if and only if $f(1/x)$ is convex (concave) in $1/I = \{\frac{1}{x} \mid x \in I\}$;*
(3) *f is GG-convex (concave) in I if and only if $\ln f(e^x)$ is convex (concave) in $\ln I = \{\ln x \mid x \in I\}$.*

Proof. We only consider the case $I = (a, b)$ $(0 < a < b < \infty)$, since the case where I is another interval is similar if we define $\ln 0 = -\infty$, $\ln(+\infty) = +\infty$, $1/0 = +\infty$ and $1/\infty = 0$.

(1) Let $g(x) = f(e^x)$, and let $x, y \in \ln I = (\ln a, \ln b)$, so $e^x, e^y \in (a, b)$. Then f is *GA*-convex (concave) in (a, b) if and only if

$$f\left(\sqrt{e^x e^y}\right) \leqslant (\geqslant) \frac{f(e^x) + f(e^y)}{2} \quad \Leftrightarrow \quad g\left(\frac{x+y}{2}\right) \leqslant (\geqslant) \frac{g(x) + g(y)}{2},$$

hence the result.

(2) Let $g(x) = f(1/x)$, and let $x, y \in 1/I = (1/b, 1/a)$, so $1/x, 1/y \in (a, b)$. Then f is *HA*-convex (concave) in (a, b) if and only if

$$f\left(\frac{2}{x+y}\right) \leqslant (\geqslant) \frac{f(1/x) + f(1/y)}{2} \quad \Leftrightarrow \quad g\left(\frac{x+y}{2}\right) \leqslant (\geqslant) \frac{g(x) + g(y)}{2},$$

hence the result.

(3) Let $g(x) = \ln f(e^x)$, and let $x, y \in \ln I = (\ln a, \ln b)$, so $e^x, e^y \in (a, b)$. Then f is *GG*-convex (concave) in (a, b) if and only if

$$\begin{aligned} f\left(\sqrt{e^x e^y}\right) \leqslant (\geqslant) \sqrt{f\left(e^x\right) f\left(e^y\right)} \quad &\Leftrightarrow \quad \ln f\left(e^{\frac{x+y}{2}}\right) \leqslant (\geqslant) \frac{\ln f(e^x) + \ln f(e^y)}{2} \\ &\Leftrightarrow \quad g\left(\frac{x+y}{2}\right) \leqslant (\geqslant) \frac{g(x) + g(y)}{2}, \end{aligned}$$

hence the result. □

The next result is an immediate consequence of Lemma 2.4.

Corollary 2.5. *Let I be a subinterval of $(0, \infty)$ and $f : I \to R_+$ have continuous derivatives in I. Then*

(1) *f is GA-convex (concave) if and only if $xf'(x)$ is increasing (decreasing);*
(2) *f is HA-convex (concave) if and only if $x^2 f'(x)$ is increasing (decreasing);*
(3) *f is GG-convex (concave) if and only if $xf'(x)/f(x)$ is increasing (decreasing).*

Lemma 2.6. *(See [10,26].) Assume that $x_i > 0$, $i = 1, 2, \ldots, n$, $\sum_{i=1}^n x_i = s$, and $c \geqslant s$. Then*

$$\frac{c - x}{\frac{nc}{s} - 1} = \left(\frac{c - x_1}{\frac{nc}{s} - 1}, \frac{c - x_1}{\frac{nc}{s} - 1}, \ldots, \frac{c - x_n}{\frac{nc}{s} - 1}\right) \prec (x_1, x_2, \ldots, x_n) = x.$$

3. Main results

In this section, we mainly investigate Schur-convexity, geometric Schur-convexity and harmonic Schur-convexity of $\sum_n(x, r; f)$. Some relevant results in the literature are generalized and improved.

Theorem 3.1. *Let I be a subinterval of $(0, +\infty)$ and $f : I \to R_+$ be a continuous function. Then*

(1) *$\sum_n(x, r; f)$ is Schur-convex in I^n if f is decreasing and AA-convex (usual convex) in I;*
(2) *$\sum_n(x, r; f)$ is Schur-concave in I^n if f is increasing and AA-concave in I.*

Proof. (1) Let $\phi(x_1, \ldots, x_r) = f(\sqrt[r]{x_1 x_2 \ldots x_r})$, $(x_1, x_2, \ldots, x_r) \in I^r$. From Lemma 2.1 (or Chapter 3, F.1 in [19, p. 83]), it follows that the function $\sqrt[r]{x_1 x_2 \ldots x_r}$ is Schur-concave in I^r, and so $\phi = f(\sqrt[r]{x_1 x_2 \ldots x_r})$ is Schur-convex for f decreasing. Since f is convex in I, one can easily see that for each fixed $x_2, \ldots, x_r$, the function ϕ is convex in z on $\{z \mid (z, x_2, \ldots, x_r) \in I^r\}$. It follows from Lemma 2.3 that the function

$$\psi(x_1, x_2, \ldots, x_n) = \sum_\pi \phi(x_{\pi(1)}, \ldots, x_{\pi(r)})$$

is Schur-convex in I^n. Then the function $\sum_n(x, r; f)$ is Schur-convex in I^n since $\sum_n(x, r; f) = \frac{1}{r!}\psi(x_1, x_2, \ldots, x_n)$.

(2) If f is increasing and *AA*-concave in I, then $-f$ is decreasing and *AA*-convex. By the part (1), the function $-\sum_n(x,r;f)$ is Schur-convex and so $\sum_n(x,r;f)$ is Schur-concave. The proof is completed. □

Corollary 3.2. *Assume that $x_i > 0$, $i = 1, 2, \dots, n$, $\alpha \in R$, and set*

$$F_n^r(x,\alpha) = \sum_{1 \leqslant i_1 < i_2 < \cdots < i_r \leqslant n} \left(\prod_{j=1}^r x_{i_j} \right)^{\frac{\alpha}{r}}, \quad r = 1, 2, \dots, n.$$

We have

(1) *if $0 < \alpha \leqslant 1$, then $F_n^r(x,\alpha)$ is Schur-concave in R_+^n;*
(2) *if $\alpha < 0$, then $F_n^r(x,\alpha)$ is Schur-convex in R_+^n.*

Proof. Let $f(x) = x^\alpha$, $x \in (0, +\infty)$, one can easily verify that $f(x)$ is increasing and *AA*-concave for $0 < \alpha \leqslant 1$, and that $f(x)$ is decreasing and *AA*-convex for $\alpha < 0$. By Theorem 3.1, we can conclude that the results hold and so the proof is completed. □

Theorem 3.3. *Let $f : I \to R_+$ be a continuous function, where I is a subinterval of $(0, \infty)$. Then*

(1) *$\sum_n(x,r;f)$ is geometrically Schur-convex in I^n if f is GA-convex in I;*
(2) *$\sum_n(x,r;f)$ is geometrically Schur-concave in I^n if f is GA-concave in I.*

Proof. (1) By Definition 1.2, we only need to prove that

$$\sum\nolimits_n\left(e^x, r; f\right) = \sum_{1 \leqslant i_1 < i_2 < \cdots < i_r \leqslant n} f\left(e^{\frac{x_{i_1} + \cdots + x_{i_r}}{r}}\right)$$

is Schur-convex on $\ln(I^n) = (\ln I)^n$. One can easily see that the function $\phi(x_1, \dots, x_r) = f(e^{\frac{x_1 + \cdots + x_r}{r}})$ is Schur-convex. For each fixed $x_2, \dots, x_r$, from Lemma 2.4 it follows that the function $\phi(z, x_2, \dots, x_r)$ is convex in z on $\{z \mid (z, x_2, \dots, x_r) \in \ln(I^r)\}$. It follows from Lemma 2.3 that the function

$$\psi(x_1, x_2, \dots, x_n) = \sum_\pi \phi(x_{\pi(1)}, \dots, x_{\pi(r)})$$

is Schur-convex in $\ln(I^n)$. This shows that the function $\sum_n(e^x, r; f)$ is Schur-convex in $\ln(I^n)$ since $\sum_n(e^x, r; f) = \frac{1}{r!}\psi(x_1, x_2, \dots, x_n)$.

(2) If f is *GA*-concave in I, then $-f$ is *GA*-convex. By the part (1), the function $-\sum_n(x,r;f)$ is geometric Schur-convex and so $\sum_n(x,r;f)$ is geometric Schur-concave. The proof is completed. □

Remark 3.4. When f is monotonic and *GG*-convex, Guan [9] proved that $\sum_n(x,r;f)$ is geometrically Schur-convex in I^n. By Remark 1.5, one can easily see that Theorem 3.3 generalizes Theorem 2.3 in [9].

Theorem 3.5. *Let $f : I \to R_+$ be a continuous function, where I is a subinterval of $(0, +\infty)$. Then*

(1) *$\sum_n(x,r;f)$ is harmonically Schur-convex in I^n if f is increasing and HA-convex in I;*
(2) *$\sum_n(x,r;f)$ is harmonically Schur-concave in I^n if f is decreasing and HA-concave in I.*

Proof. (1) By Definition 1.3, we need to prove that the function $\sum_n(1/x, r; f)$ is Schur-convex in $1/I^n = (1/I)^n$. Note that

$$\sum\nolimits_n(1/x, r; f) = \sum_{1 \leqslant i_1 < i_2 < \cdots < i_r \leqslant n} f\left(\prod_{j=1}^r \left(\frac{1}{x_{i_j}}\right)^{1/r} \right).$$

Put

$$\varphi(x_1, \dots, x_r) = f\left(\frac{1}{\sqrt[r]{x_1 \dots x_r}}\right).$$

Using Lemma 2.1, one can find that $\frac{1}{\sqrt[r]{x_1 \dots x_r}}$ is Schur-convex in $1/I^r = (1/I)^r$, which implies that the function $\varphi(x_1, \dots, x_r)$ is Schur-convex if f is increasing. Form Lemma 2.4, for fixed $x_2, \dots, x_r$, the function $\varphi(z, x_1, \dots, x_r)$ is convex in z. It follows from Lemma 2.3 that the function

$$\psi(x_1, x_2, \ldots, x_n) = \sum_{\pi} \varphi(x_{\pi(1)}, \ldots, x_{\pi(r)})$$

is Schur-convex in $1/I^n$. This implies that the function $\sum_n(1/x, r; f)$ is Schur-convex in $1/I^n$ since $\sum_n(1/x, r; f) = \frac{1}{r!}\psi(x_1, x_2, \ldots, x_n)$.

(2) If f is decreasing and *HA*-concave in I, then $-f$ is increasing and *HA*-convex. By the part (1), the function $-\sum_n(x, r; f)$ is harmonic Schur-convex and so $\sum_n(x, r; f)$ is harmonic Schur-concave. The proof is completed. □

Remark 3.6. From Remark 1.5, one can easily find that the conclusion of Theorem 3.5(1) holds if f is increasing and *GA*-convex. The result in Theorem 3.5(2) is true if f is decreasing and *GA*-concave.

Using Corollary 2.5, one can easily verify that $f(x) = x$ is increasing and *AA*-concave, *GA*-convex, and increasing and *HA*-convex in R_+. Using Theorems 3.1, 3.3 and 3.5, respectively, we can establish the following corollary.

Corollary 3.7. *Hamy symmetric function $F_n(x, r)$, $r \in \{1, 2, \ldots, n\}$, is Schur-concave, geometrically Schur-convex, and harmonically Schur-convex in R^n_+.*

Theorem 3.8. *Assume that $f : I \to R_+$ be a real-valued function, where I is a subinterval of $(0, \infty)$, set*

$$\sigma_n^r(x; f) = \frac{1}{\binom{n}{r}} \sum_{1 \leqslant i_1 < i_2 < \cdots < i_r \leqslant n} f\left(\prod_{j=1}^{r} x_{i_j}^{1/r}\right), \quad r = 1, 2, \ldots, n,$$

the following statements hold.

(1) *If f is GA-convex, then*

$$\sigma_n^n(x; f) \leqslant \sigma_n^{n-1}(x; f) \leqslant \cdots \leqslant \sigma_n^2(x; f) \leqslant \sigma_n^1(x; f). \tag{3.1}$$

(2) *If f is GA-concave, then the inequality* (3.1) *is reversed, that is,*

$$\sigma_n^n(x; f) \geqslant \sigma_n^{n-1}(x; f) \geqslant \cdots \geqslant \sigma_n^2(x; f) \geqslant \sigma_n^1(x; f). \tag{3.2}$$

Proof. As the proofs are similar, here we give the proof of (1). Using the same method as in original Hamy's proof for Hamy symmetric function [6], we only need to prove that

$$\sigma_n^{k+1}(x; f) \leqslant \sigma_n^k(x; f), \quad k = 1, 2, \ldots, n-1. \tag{3.3}$$

Since f is *GA*-convex, then we have

$$\begin{aligned}
&\sum_{1 \leqslant i_1 < \cdots < i_{k+1} \leqslant n} f\big((x_{i_1} x_{i_2} \ldots x_{i_{k+1}})^{\frac{1}{k+1}}\big) \\
&= \sum_{1 \leqslant i_1 < i_2 < \cdots < i_{k+1} \leqslant n} f\Big(\big((x_{i_2} \ldots x_{i_{k+1}})^{\frac{1}{k}} (x_{i_1} x_{i_3} \ldots x_{i_{k+1}})^{\frac{1}{k}} \ldots (x_{i_1} x_{i_2} \ldots x_{i_k})^{\frac{1}{k}}\big)^{\frac{1}{k+1}}\Big) \\
&\leqslant \sum_{1 \leqslant i_1 < \cdots < i_{k+1} \leqslant n} \frac{1}{k+1} \Big\{ f\big((x_{i_2} \ldots x_{i_{k+1}})^{\frac{1}{k}}\big) + f\big((x_{i_1} x_{i_3} \ldots x_{i_{k+1}})^{\frac{1}{k}}\big) + \cdots + f\big((x_{i_1} x_{i_2} \ldots x_{i_k})^{\frac{1}{k}}\big) \Big\} \\
&= \frac{n-k}{k+1} \sum_{1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n} f\big((x_{i_1} x_{i_2} \ldots x_{i_k})^{\frac{1}{k}}\big),
\end{aligned}$$

which implies that

$$\begin{aligned}
\sigma_n^{k+1}(x; f) &= \frac{1}{\binom{n}{k+1}} \sum_{1 \leqslant i_1 < i_2 < \cdots < i_{k+1} \leqslant n} f\big((x_{i_1} x_{i_2} \ldots x_{i_{k+1}})^{\frac{1}{k+1}}\big) \\
&\leqslant \frac{1}{\binom{n}{k+1}} \frac{n-k}{k+1} \sum_{1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n} f\big((x_{i_1} x_{i_2} \ldots x_{i_k})^{\frac{1}{k}}\big) \\
&= \frac{1}{\binom{n}{k}} \sum_{1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n} f\big((x_{i_1} x_{i_2} \ldots x_{i_k})^{\frac{1}{k}}\big) \\
&= \sigma_n^k(x; f).
\end{aligned}$$

This shows that (3.3) holds and so the proof is completed. □

Remark 3.9. When f is *GG*-convex, Guan [9] also obtained the inequality (3.1). However, Remark 1.5 implies that Theorem 3.8 generalizes Theorem 2.1 in [9]. And moreover, by the definition of *GA*-convex, we can deduce the following so-called Jensen type inequality for *GA*-convex function

$$f\left(\sqrt[n]{x_1x_2\ldots x_n}\right) \leqslant \frac{f(x_1)+f(x_2)+\cdots+f(x_n)}{n}. \tag{3.4}$$

It is clear that the inequality (3.1) is a refinement of the inequality (3.4).

Since $f(x)=x$ is *GA*-convex in $(0,+\infty)$. The following conclusion immediately follows from Theorem 3.8.

Corollary 3.10. *(See [8,13].) If $x=(x_1,x_2,\ldots,x_n)\in R_+^n$, then*

$$G_n(x)=\sigma_n(x,n)\leqslant\sigma_n(x,n-1)\leqslant\cdots\leqslant\sigma_n(x,2)\leqslant\sigma_n(x,1)=A_n(x).$$

4. Some applications

In this section, taking particular function f in Theorems 3.1, 3.3, 3.5 and 3.8, we establish some interesting inequalities. Some relevant results in the literature are recovered and generalized.

Theorem 4.1. *If $0<x_i<1$, $i=1,\ldots,n$, and $k\in\{1,2,\ldots,n\}$, then the sequence*

$$\frac{k}{\binom{n-1}{k-1}}\sum_{1\leqslant i_1<\cdots<i_k\leqslant n}\frac{\sum_{j=1}^k x_{i_j}}{k-\sum_{j=1}^k x_{i_j}}$$

is non-increasing in $k=1,2,\ldots,n$.

Proof. Let $f(t)=\frac{\ln t}{1-\ln t}$, $t\in(1,e)$. Differentiating it yields

$$f'(t)=\frac{1}{t(1-\ln t)^2}\quad\text{and}\quad\left(tf'(t)\right)'=\frac{2}{t(1-\ln t)^3}.$$

This implies that $f(t)$ is *GA*-convex in $(1,e)$. Using Theorem 3.8 and noting that

$$\sigma_n^k(t;f)=\frac{1}{\binom{n}{k}}\sum_{1\leqslant i_1<\cdots<i_k\leqslant n}\frac{\sum_{j=1}^k \ln t_{i_j}}{k-\sum_{j=1}^k \ln t_{i_j}},$$

one can see that the sequence

$$\frac{1}{\binom{n}{k}}\sum_{1\leqslant i_1<\cdots<i_k\leqslant n}\frac{\sum_{j=1}^k \ln t_{i_j}}{k-\sum_{j=1}^k \ln t_{i_j}}$$

is non-increasing in $k=1,2,\ldots,n$. This and letting $t_i=e^{x_i}\in(1,e)$, $i=1,2,\ldots,n$, have completed the proof. □

Remark 4.2. With the conditions of Theorem 4.1, Shapiro's inequality [10] reads as follows

$$\sum_{i=1}^n\frac{x_i}{1-x_i}\geqslant\frac{nS_n}{n-S_n}, \tag{4.1}$$

where $S_n=\sum_{i=1}^n x_i$, $0<x_i<1$, $i=1,\ldots,n$. One can easily see that Theorem 4.1 gives a refinement of Shapiro's inequality (4.1).

The p-th power mean of a positive n-tuple $x=(x_1,x_2,\ldots,x_n)$ is

$$M_n^p(x)=\begin{cases}\left(\frac{x_1^p+x_2^p+\cdots+x_n^p}{n}\right)^{1/p}, & p\neq 0,\\ G_n(x), & p=0.\end{cases}$$

For $0<x_i\leqslant 1/2$, $i=1,2,\ldots,n$, the well-known Ky Fan inequality [5, p. 5] reads as follows

$$\frac{G_n(x)}{G_n(1-x)}\leqslant\frac{A_n(x)}{A_n(1-x)}, \tag{4.2}$$

where $1-x=(1-x_1,1-x_2,\ldots,1-x_n)$. The inequality (4.2) has stimulated many researchers to give new proofs, improvements and generalizations of it. See, for example [1,2,18] and the references cited therein. Now we establish the following Ky Fan type inequalities.

Theorem 4.3. *Let $\sum_{i=1}^n x_i = s \leqslant 1$, $x = (x_1, x_2, \dots, x_n) \in (0,1)^n$ $(n \geqslant 2)$. If $\alpha \in (-\infty, 0) \cup (0, 1]$, then*

$$\left(\frac{F_n^r(x,\alpha)}{F_n^r(1-x,\alpha)}\right)^{1/\alpha} \leqslant \frac{A_n(x)}{A_n(1-x)}, \quad r = 1, 2, \dots, n, \tag{4.3}$$

where $F_n^r(x,\alpha)$ is defined as Corollary 3.2. *In particular,*

$$\frac{M_n^\alpha(x)}{M_n^\alpha(1-x)} \leqslant \frac{A_n(x)}{A_n(1-x)}. \tag{4.4}$$

Proof. From Lemma 2.6, it follows that

$$\frac{1-x}{n/s-1} = \left(\frac{1-x_1}{n/s-1}, \dots, \frac{1-x_n}{n/s-1}\right) \prec (x_1, x_2, \dots, x_n) = x. \tag{4.5}$$

(i) When $\alpha < 0$, by Corollary 3.2, $F_n^r(x,\alpha)$ is Schur-convex. This and (4.5) lead to

$$\frac{F_n^r(1-x,\alpha)}{F_n^r(x,\alpha)} \leqslant \left(\frac{n}{s}-1\right)^\alpha.$$

This implies (4.3).

(ii) When $0 < \alpha \leqslant 1$, it follows from Corollary 3.2 that $F_n^r(x,\alpha)$ is Schur-concave. This and (4.5) lead to

$$\frac{F_n^r(1-x,\alpha)}{F_n^r(x,\alpha)} \geqslant \left(\frac{n}{s}-1\right)^\alpha.$$

This also implies (4.3).

The cases (i) and (ii) show that (4.3) holds. Taking $r = 1$ in (4.3), we can obtain (4.4). The proof is completed. □

Remark 4.4. Taking limits in (4.4) as $\alpha \to 0$ yields

$$\frac{G_n(x)}{G_n(1-x)} \leqslant \frac{A_n(x)}{A_n(1-x)},$$

where $0 < x_i < 1$, $i = 1, 2, \dots, n$, and $\sum_{i=1}^n x_i \leqslant 1$.

Using Corollary 2.5(1), one can easily verify that the function $f(x) = 1/x$ is *GA*-convex in R_+. Theorem 3.8 immediately gives us the following result which was established by Hara et al. [13].

Theorem 4.5. *(See [13].) If $x = (x_1, x_2, \dots, x_n) \in R_+^n$, $H_n(x) = n/\sum_{i=1}^n (1/x_i)$, and $r \in \{1, 2, \dots, n\}$, then the sequence*

$$u(H,G,x;r) = \left(\frac{1}{\binom{n}{r}} \sum_{1 \leqslant i_1 < i_2 < \dots < i_r \leqslant n} \frac{1}{(\prod_{j=1}^r x_{i_j})^{1/r}}\right)^{-1},$$

is non-decreasing with respect to k $(1 \leqslant k \leqslant n)$, that is,

$$H_n(x) = u(H,G,x;1) \leqslant u(H,G,x;2) \leqslant \dots \leqslant u(H,G,x;n-1) \leqslant u(H,G,x;n) = G_n(x). \tag{4.6}$$

Hara et al. [13] also established a more general result than those of Corollary 3.10 and Theorem 4.5 by use of the p-th power mean $M_n^p(x)$. Fix $n \in N$, $x = (x_1, x_2, \dots, x_n) \in R_+^n$ and choose k with $1 \leqslant k \leqslant n$. For any $s, t \in R$, let

$$\begin{aligned} u(s,t,x;k) &= M_{\binom{n}{k}}^s\left(M_k^t(x_1, \dots, x_k), \dots, M_k^t(x_{n-k+1}, \dots, x_n)\right) \\ &= \left\{\frac{1}{\binom{n}{k}} \sum_{1 \leqslant i_1 < i_2 < \dots < i_k \leqslant n} \left(\frac{x_{i_1}^t + x_{i_2}^t + \dots + x_{i_k}^t}{k}\right)^{s/t}\right\}^{1/s}, \end{aligned}$$

the authors investigated the monotonicity of $u(s,t,x;k)$ with respect to k. Here we give an alternative proof.

Theorem 4.6. *(See [13].) If $s \leqslant t$, then the sequence $u(s,t,x;k)$ is non-decreasing with respect to k with $1 \leqslant k \leqslant n$.*

Proof. If $s=0$ and $t=0$, then $u(s,t,x;k)=(\prod_{i=1}^n x_i)^{1/n}$ for $k=1,2,\ldots,n$. The result is obvious. If $s\neq 0$ and $t=0$, one can prove the result as Corollary 3.10 does by taking $f(x)=x^s$, $x\in R_+$. Now we consider the case $s\neq 0$ and $t\neq 0$. To this end, let $f(y)=(\ln y)^{s/t}$, $y\in(1,\infty)$. Differentiating it yields

$$\big(yf'(y)\big)'=\frac{s}{t}\left(\frac{s}{t}-1\right)(\ln y)^{\frac{s}{t}-1}.$$

Consider the following three possible cases.

Case 1. If $0<s\leqslant t$, then the function f is *GA*-concave from Corollary 2.5(1). It follows from Theorem 3.8 that the sequence

$$\sigma_n^k(y;f)=\frac{1}{\binom{n}{k}}\sum_{1\leqslant i_1<i_2<\cdots<i_k\leqslant n} f\left(\prod_{j=1}^k y_{i_j}^{1/k}\right)=\frac{1}{\binom{n}{k}}\sum_{1\leqslant i_1<i_2<\cdots<i_k\leqslant n}\left(\frac{\ln y_{i_1}+\cdots+\ln y_{i_k}}{k}\right)^{s/t}$$

is non-decreasing in k. For $x=(x_1,x_2,\ldots,x_n)\in R_+^n$, letting $y_i=\exp(x_i^t)$, $i=1,2,\ldots,n$, and noticing that $s>0$, we conclude that $u(s,t,x;k)$ is also non-decreasing with respect to k.

Case 2. If $s\leqslant t<0$, then the function f is *GA*-convex by Corollary 2.5(1). It follows from Theorem 3.8 that the sequence

$$\sigma_n^k(y;f)=\frac{1}{\binom{n}{k}}\sum_{1\leqslant i_1<i_2<\cdots<i_k\leqslant n} f\left(\prod_{j=1}^k y_{i_j}^{1/k}\right)=\frac{1}{\binom{n}{k}}\sum_{1\leqslant i_1<i_2<\cdots<i_k\leqslant n}\left(\frac{\ln y_{i_1}+\cdots+\ln y_{i_k}}{k}\right)^{s/t}$$

is non-increasing in k. Taking $y_i=\exp(x_i^t)$, $i=1,2,\ldots,n$, and noticing that $s<0$, we conclude that $u(s,t,x;k)$ is non-decreasing with respect to k.

Case 3. If $s<0<t$, then the function f is *GA*-convex from Corollary 2.5(1). Using the same method as Case 2 does, we can also conclude that $u(s,t,x;k)$ is non-decreasing with respect to k. The proof is completed. □

Theorem 4.7.

(1) *If* $x=(x_1,x_2,\ldots,x_n)\in(0,1)^n$ *and* $r\in\{1,2,\ldots,n\}$, *then*

$$\frac{1}{\binom{n}{r}}\sum_{1\leqslant i_1<i_2<\cdots<i_r\leqslant n}\frac{\prod_{j=1}^r x_{i_j}^{\frac{1}{r}}}{1+\prod_{j=1}^r x_{i_j}^{\frac{1}{r}}}\geqslant\frac{G_n(x)}{1+G_n(x)},\tag{4.7}$$

$$\frac{1}{\binom{n}{r}}\sum_{1\leqslant i_1<i_2<\cdots<i_r\leqslant n}\frac{1}{1+\prod_{j=1}^r x_{i_j}^{\frac{1}{r}}}\leqslant\frac{1}{1+G_n(x)}.\tag{4.8}$$

(2) *If* $x=(x_1,x_2,\ldots,x_n)\in[1,\infty)^n$ *and* $r\in\{1,2,\ldots,n\}$, *then*

$$\frac{1}{\binom{n}{r}}\sum_{1\leqslant i_1<i_2<\cdots<i_r\leqslant n}\frac{\prod_{j=1}^r x_{i_j}^{\frac{1}{r}}}{1+\prod_{j=1}^r x_{i_j}^{\frac{1}{r}}}\leqslant\frac{G_n(x)}{1+G_n(x)},\tag{4.9}$$

$$\frac{1}{\binom{n}{r}}\sum_{1\leqslant i_1<i_2<\cdots<i_r\leqslant n}\frac{1}{1+\prod_{j=1}^r x_{i_j}^{\frac{1}{r}}}\geqslant\frac{1}{1+G_n(x)}.\tag{4.10}$$

Proof. We clearly see that

$$\big(\ln G_n(x),\ln G_n(x),\ldots,\ln G_n(x)\big)\prec(\ln x_1,\ln x_2,\ldots,\ln x_n).\tag{4.11}$$

Let $f(x)=\frac{x}{1+x}$ and $g(x)=\frac{1}{1+x}$, $x\in(0,\infty)$. Directly computing gives us

$$\big(xf'(x)\big)'=\frac{1-x}{(1+x)^3},\tag{4.12}$$

and

$$\big(xg'(x)\big)'=\frac{x-1}{(1+x)^3}.\tag{4.13}$$

Thus, (4.12) and Corollary 2.5(1) show that $f(x)$ is GA-convex for $x \in (0,1)$ and GA-concave for $x \in [1,\infty)$. Therefore, Theorem 3.3 and (4.11) lead to (4.7) and (4.9).

On the other hand, (4.13) and Corollary 2.5(1) show that $g(x)$ is GA-concave for $x \in (0,1)$ and GA-convex for $x \in [1,\infty)$. Thus, Theorem 3.3 and (4.11) lead to (4.8) and (4.10). The proof is completed. □

If we take $r = 1$ in Theorem 4.7, then we get the following corollary.

Corollary 4.8. *If $x = (x_1, x_2, \dots, x_n) \in [1,\infty)^n$, then*

(1) $$A_n\left(\frac{x}{1+x}\right) \leqslant \frac{G_n(x)}{1+G_n(x)}, \tag{4.14}$$

(2) $$A_n\left(\frac{1}{1+x}\right) \geqslant \frac{1}{1+G_n(x)}. \tag{4.15}$$

Both (4.14) *and* (4.15) *are reversed if $x = (x_1, x_2, \dots, x_n) \in (0,1)^n$.*

Theorem 4.9. *If $x = (x_1, \dots, x_n) \in R_+^n$, $H_n(x) = n/\sum_{i=1}^n (1/x_i)$, and $k \in \{1, 2, \dots, n\}$, then*

$$\frac{1}{1+A_n(x)} \leqslant \frac{1}{\binom{n}{k}} \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} \frac{1}{1 + (\prod_{j=1}^k x_{i_j})^{1/k}} \leqslant \frac{1}{1+H_n(x)}, \quad k = 1, 2, \dots, n. \tag{4.16}$$

Proof. One can easily verify that the function $f(x) = \frac{1}{1+x}$, $x \in (0,\infty)$, is decreasing and AA-convex. Therefore, from Theorem 3.1, we can see that

$$\sum\nolimits_n (x, k; f) = \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} \frac{1}{1 + (\prod_{j=1}^k x_{i_j})^{1/k}}$$

is Schur-convex. This and the expression $(A_n(x), A_n(x), \dots, A_n(x)) \prec (x_1, \dots, x_n)$ imply the left inequality in (4.16). On the other hand, straightforward calculation gives us

$$f'(x) = -\frac{1}{(1+x)^2} \quad \text{and} \quad \left(x^2 f'(x)\right)' = -\frac{2x}{(1+x)^3}.$$

This together with Corollary 2.5(2) shows that $f(x)$ is decreasing and HA-concave in R_+^n. One can easily see that

$$\left(\frac{1}{H_n(x)}, \frac{1}{H_n(x)}, \dots, \frac{1}{H_n(x)}\right) \prec \left(\frac{1}{x_1}, \frac{1}{x_2}, \dots, \frac{1}{x_n}\right) = \frac{1}{x}. \tag{4.17}$$

Thus, the right inequality in (4.16) immediately follows from Theorem 3.5 and (4.17). The proof is completed. □

Remark 4.10.

(1) Taking $k = 1$ in (4.16), we obtain

$$\frac{1}{1+A_n(x)} \leqslant \frac{1}{n} \sum_{i=1}^n \frac{1}{1+x_i} \leqslant \frac{1}{1+H_n(x)}. \tag{4.18}$$

This inequality is also produced from the last formula by the end of [21].

(2) Using the expression $(x_1, x_2, \dots, x_n) \prec (S_n, 0, \dots, 0)$ (see [19, p. 133].) and Theorem 3.1, we also obtain

$$\frac{1}{n} \sum_{i=1}^n \frac{1}{1+x_i} \leqslant 1 - \frac{S_n}{n(1+S_n)}, \tag{4.19}$$

where $S_n = \sum_{i=1}^n x_i$, $x_i > 0$, $i = 1, \dots, n$. This inequality was proposed by Janous [16].

(3) It is natural to ask which is sharper between the inequality (4.19) and the right inequality in (4.18). It is uncertain. As a matter of fact, if $x = (1/2, 2)$, then we have $\frac{1}{1+H_n(x)} = \frac{5}{9} < 1 - \frac{S_n}{n(1+S_n)} = \frac{9}{14}$. If $x = (1/4, 1/10)$, we obtain $\frac{1}{1+H_n(x)} = \frac{7}{8} > 1 - \frac{S_n}{n(1+S_n)} = \frac{47}{54}$.

Theorem 4.11. *If $x = (x_1, x_2, \ldots, x_n) \in R_+^n$ and $r \in \{1, 2, \ldots, n\}$, then*

$$\frac{H_n(x)}{1+H_n(x)} \leqslant \frac{1}{\binom{n}{r}} \sum_{1\leqslant i_1<i_2<\cdots<i_r\leqslant n} \frac{\prod_{j=1}^r x_{i_j}^{\frac{1}{r}}}{1+\prod_{j=1}^r x_{i_j}^{\frac{1}{r}}} \leqslant \frac{A_n(x)}{1+A_n(x)}. \tag{4.20}$$

In particular,

$$\frac{H_n(x)}{1+H_n(x)} \leqslant A_n\left(\frac{x}{1+x}\right) \leqslant \frac{A_n(x)}{1+A_n(x)}. \tag{4.21}$$

Proof. Let $f(x) = \frac{x}{1+x}$, directly computing yields

$$f'(x) = \frac{1}{(1+x)^2}, \qquad f''(x) = -\frac{2}{(1+x)^3}, \qquad \left(x^2 f'(x)\right)' = \frac{2x}{(1+x)^3}. \tag{4.22}$$

This together with Corollary 2.5(2) implies that $f(x)$ is increasing and *HA*-convex in R_+. Using (4.17) and Theorem 3.5, we arrive at the left inequality in (4.20). From (4.22), one can easily see that $f(x)$ is also increasing and *AA*-concave. Thus, Theorem 3.1 and the expression $(A_n(x), \ldots, A_n(x)) \prec (x_1, \ldots, x_n)$ implies the right inequality in (4.20). Taking $r = 1$ in (4.20) leads to (4.21) and so the proof is completed. □

Using Theorems 4.7 and 4.11, we obtain the following results.

Corollary 4.12. *If $x = (x_1, x_2, \ldots, x_n) \in (0, 1)^n$ and $r \in \{1, 2, \ldots, n\}$, then*

$$\frac{G_n(x)}{1+G_n(x)} \leqslant \frac{1}{\binom{n}{r}} \sum_{1\leqslant i_1<i_2<\cdots<i_r\leqslant n} \frac{\prod_{j=1}^r x_{i_j}^{\frac{1}{r}}}{1+\prod_{j=1}^r x_{i_j}^{\frac{1}{r}}} \leqslant \frac{A_n(x)}{1+A_n(x)}. \tag{4.23}$$

In particular,

$$\frac{G_n(x)}{1+G_n(x)} \leqslant A_n\left(\frac{x}{1+x}\right) \leqslant \frac{A_n(x)}{1+A_n(x)}. \tag{4.24}$$

Corollary 4.13. *If $x = (x_1, x_2, \ldots, x_n) \in [1, \infty)^n$ and $r \in \{1, 2, \ldots, n\}$, then*

$$\frac{H_n(x)}{1+H_n(x)} \leqslant \frac{1}{\binom{n}{r}} \sum_{1\leqslant i_1<i_2<\cdots<i_r\leqslant n} \frac{\prod_{j=1}^r x_{i_j}^{\frac{1}{r}}}{1+\prod_{j=1}^r x_{i_j}^{\frac{1}{r}}} \leqslant \frac{G_n(x)}{1+G_n(x)}. \tag{4.25}$$

In particular,

$$\frac{H_n(x)}{1+H_n(x)} \leqslant A_n\left(\frac{x}{1+x}\right) \leqslant \frac{G_n(x)}{1+G_n(x)}. \tag{4.26}$$

Acknowledgment

The authors are very grateful to the referee for his (her) suggestions for the improvement of this paper.

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