The abstract equation of evolution for a Banach-space-valued function $u(t)$ may be written

$$
\frac{du(t)}{dt} = C(t) u(t)
$$

where for each $t$, $C(t)$ is a linear operator acting in the Banach space. The abstract Cauchy problem is to establish existence and uniqueness of solutions with prescribed initial data $u(s) = \Psi$, which depend continuously on the data. If $C(t) = C$ is independent of the time $t$, the solution should be $u(t) = \exp ((t - s) C) \Psi$. The Hille-Yosida theorem of semigroup theory may be used to give sense to this exponential. In general the solution is not given by such a simple formal expression. There should exist, however, a family of continuous linear operators $R(t, s)$ such that $u(t) = R(t, s) \Psi$ is the solution with initial condition $u(s) = \Psi$. Such a family is called a propagator or a Green's operator. Conditions on $C(t)$ which guarantee the existence of the family $R(t, s)$ were first given by Kato [6]. The problem has been further studied by Yosida [15; Chap. XIV], Nelson [11], Kisynski [7], and Lions [8].

Now assume that $C(t) = A(t) + B(t)$. The problem of perturbation theory considered here is to express $R(t, s)$ explicitly in terms of $A(t)$ and $B(t)$ and solutions of equations involving these operators separately. Such general formulas were obtained by Phillips [12] and Segal [13]. Their method was essentially iteration of the integral equation

$$
R(t, s) = P(t, s) + \int_s^t P(t, t') B(t') R(t', s) \, dt'.
$$
Here $P(t, s)$ is the propagator corresponding to the equation involving $A(t)$. They derived a series expression for $R(t, s)$ the terms of which are multiple integrals involving products of the $P(t, s)$ and the (bounded) operators $B(t)$. Conditions for the use of a perturbation formula which directly involves one of the operators $A(t)$ or $B(t)$ are likely to be delicate if the operator is not bounded.

Here we give a product formula which generalizes a result for the semigroup situation due to Trotter [14]. If we for convenience take $s = 0$, we obtain the representation

$$R(t, 0) = \lim_{n \to \infty} \prod_{k=0}^{n-1} \exp \left( A \left( k \frac{t}{n} \right) \frac{t}{n} \right) \exp \left( B \left( k \frac{t}{n} \right) \frac{t}{n} \right).$$

A related formula, valid in circumstances requiring that the $B(t)$ are bounded, is

$$R(t, 0) = \lim_{n \to \infty} \prod_{k=0}^{n-1} \exp \left( A \left( k \frac{t}{n} \right) \frac{t}{n} \right) \left( I + B \left( k \frac{t}{n} \right) \right).$$

This theory may be applied to partial differential equations if the Banach space is taken to be a space of functions and if for each $t$, $A(t)$ and $B(t)$ are differential operators acting in that space. Typically, $B(t)$ is a lower-order perturbation of $A(t)$. The equations encountered may be of parabolic or hyperbolic type, or, as in the case of the Schrödinger equation, neither. The theory may also be used to answer a point raised by Babbit [1] about the Feynman path integral solution of the Schrödinger equation. Nelson [10] used the Trotter formula to give a mathematical realization of this integral in the case of a time-independent equation. The present work shows that this technique extends to the case of time-dependent coefficients. The class of allowable perturbing operators $B(t)$ discussed in the related work of Daletzski [2] seems to be more restrictive in the application to the Schrödinger equation. Here the zero order coefficient need not be bounded.

1. **The Trotter Product Formula for Propagators**

Let $X$ and $Y$ be Banach spaces. We write $L(X, Y)$ for the space of continuous linear transformations on $X$ to $Y$. We abbreviate $L(X, X)$ by $L(X)$. In general, an operator from $X$ to $Y$ will be a linear transformation from a linear subspace $D$ of $X$ to $Y$, and an operator in $X$ is an operator from $X$ to $X$. 
Proposition 1. Let $X$ and $Y$ be Banach spaces. Let $D$ be a linear subspace of $X$ which is the domain of some closed operator from $X$ to $Y$. Then $D$ has a unique Banach-space structure (up to equivalence) with the following property. Let $i$ be the injection of $D$ into $X$. For each closed operator $C$ from $X$ to $Y$ with domain $D$, the Banach space structure of $D$ is the unique one (up to equivalence) such that $i \in L(D, X)$ and $C \in L(D, Y)$.

Proof. Choose any closed operator $C$ from $X$ to $Y$ with domain $D$. Give $D$ the Banach-space structure inherited from the bijection of $D$ onto the graph of $C$ in $X \times Y$. Then clearly $i \in L(D, X)$ and $C \in L(D, Y)$. Now consider any other Banach-space structure $D'$ on the domain of $C$ such that $i \in L(D', X)$ and $C \in L(D', Y)$. Then the identity map of $D'$ onto $D$ is continuous. By the open-mapping theorem, its inverse is also continuous, so the structures are the same.

Let $C'$ be any other operator on $D$ with graph closed in $X \times Y$. Since $i \in L(D, X)$, $C'$ has graph closed in $D \times Y$. Hence by the closed graph theorem $C'$ is in $L(D, Y)$.

Note. Let $A$ be an operator from $X$ to $Y$ with domain $D$ which has an operator $\bar{A}$ as its closure. Then the inverse image under the injection of $D \times Y$ into $X \times Y$ of the graph of $\bar{A}$ is the graph of $A$. Thus by the closed-graph theorem, $A \in L(D, Y)$.

Definition. Let $X$ be a Banach space. A propagator is a family of operators $R(t, s)$ in $L(X)$, defined for $a \leq s \leq t \leq b$, such that

(i) $R(s, s) = I$,
(ii) $R(t, s) R(s, r) = R(t, r)$,
(iii) $R(t, s)$ is strongly continuous in $(s, t)$.

Theorem 1. Let $X$ be a Banach space. Assume that for each $t \in [0, b]$, $A(t)$ and $B(t)$ are the infinitesimal generators of strongly continuous semigroups of contractions $\exp(A(t) s)$ and $\exp(B(t) s)$ on $X$. Require that $C(t) = A(t) + B(t)$ is a closed operator for each $t$ and that the domain of $C(t)$ is a dense linear subspace $D$ of $X$ which is independent of $t$. Assume that for $u$ in $D$, $A(t) u$ and $B(t) u$ are $C^0$ in $t$ on $[0, b]$.

Assume that there is a propagator $R(t, s)$ consisting of contractions on $X$ such that $D$ is invariant under each $R(t, s)$. For each $\Psi$ in $D$, let $u(t) = R(t, 0) \Psi$ and further assume that $C(t) u(t)$ is $C^0$ on $[0, b]$ and that $u(t)$ is $C^1$ on $(0, b)$ and there satisfies

$$\frac{du(t)}{dt} = C(t) u(t).$$
Require also that \( u(t) \) is \( C^0 \) from \([0, b]\) to \( D \) considered as a Banach space (Proposition I).

Then, for \( t \in [0, b] \),

\[
R(t, 0) = \lim_{n \to \infty} \prod_{k=n-1}^{0} \exp \left( A \left( k \frac{t}{n} \right) \frac{t}{n} \right) \exp \left( B \left( k \frac{t}{n} \right) \frac{t}{n} \right)
\]

in the strong operator topology of \( \mathcal{L}(X) \).

Remark. Let \( C \) be any closed operator in \( X \) with domain \( D \) and with nonempty resolvent set. In order to show that \( u(t) \) is continuous from \([0, b]\) to \( D \), it would be enough to show that \( Cu(t) \) is continuous from \([0, b]\) to \( X \). For if \( \lambda \) is in the resolvent set of \( C \), then \( \lambda - C \in L(D, X) \) and \( (\lambda - C)^{-1} \) is defined on all of \( Y \). The open-mapping theorem implies that \( (\lambda - C)^{-1} \in L(X, D) \). Then

\[
u(t) = (\lambda - C)^{-1} (\lambda - C) u(t).
\]

Proof. For convenience abbreviate \( t/n \) by \( h \). We define

\[
P_k = \exp (hA((k - 1) h)),
Q_k = \exp (hB((k - 1) h)),
R_k = R(kh, (k - 1) h),
\]

\[
S_n = \prod_{k=n}^{1} P_k Q_k R(t, 0).
\]

Our task is to show that \( S_n \to 0 \) strongly as \( n \to \infty \). Now

\[
S_n = \prod_{k=n}^{1} P_k Q_k - \prod_{k=n}^{1} R_k
\]

\[
= \sum_{j=1}^{n} \prod_{k=n}^{j+1} P_k Q_k [P_j Q_j - R_j] \prod_{l=j+1}^{1} R_l
\]

\[
= \sum_{j=1}^{n} \prod_{k=n}^{j+1} P_k Q_k [P_j Q_j - R_j] R((j - 1) h, 0).
\]

Let \( u \) be in \( D \). We will show that \( S_n u \to 0 \) as \( n \to \infty \). We have

\[
\| S_n u \| \leq \sum_{j=1}^{n} \| [P_j Q_j - R_j] R((j - 1) h, 0) u \|
\]

\[
\leq n \sup_{0 \leq s \leq t} \| [\exp (hA(s)) \exp (hB(s)) - R(s | h, s)] R(s, 0) u \|.
\]
(In case \( t = b \), we need only take the supremum over the interval on which the propagator is defined.)

Let

\[
E_s(h) = \frac{1}{h} \left( \exp \left( A(s) h \right) - I \right) - A(s),
\]

\[
F_s(h) = \exp \left( A(s) h \right) \frac{1}{h} \left( \exp \left( B(s) h \right) - I \right) - B(s),
\]

\[
G_s(h) = \frac{1}{h} \left( R(s + h, s) - I \right) - C(s).
\]

Then

\[
\| S_n u \| \leq t \sup \| \left( E_s(h) + F_s(h) - G_s(h) \right) R(s, 0) u \|
\]

\[
\leq t \left\{ \sup \| E_s(h) R(s, 0) u \| + \sup \| F_s(h) R(s, 0) u \| + \sup \| G_s(h) R(s, 0) u \| \right\}.
\]

Consider each term separately.

Let \( K \) be the set of all \( R(s, 0) u, 0 \leq s \leq t \). Note that \( K \subseteq D \). The first two terms of the last expression are bounded by

\[
\sup \left( 0 \leq s \leq t, w \in K \right) \| E_s(h) w \|
\]

and

\[
\sup \left( 0 \leq s \leq t, w \in K \right) \| F_s(h) w \|.
\]

It is clear that \( \exp \left( A(s) h \right) A(s) w \rightarrow A(s) w \) as \( h \rightarrow 0 \). We will now show that this convergence is uniform in \( s \). This will be useful in treating the supremum involving \( E_s(h) \).

First, note that for \( v \) in \( X \), \( \exp \left( A(s) h \right) v \rightarrow v \) as \( h \rightarrow 0 \) uniformly in \( s \).

To see this, choose \( v \) in \( D \) and write \( \exp \left( A(s) h \right) v - v = \int_0^h \exp \left( A(s) k \right) A(s) v \, dk \). In view of continuity assumptions, the integrand is bounded by \( \sup \left( 0 \leq s \leq t \right) \| A(s) v \| < \infty \). So \( \left( \exp \left( A(s) h \right) - I \right) v \rightarrow 0 \) uniformly in \( s \). Since \( D \) is dense in \( X \), the same conclusion follows for arbitrary \( v \) in \( X \).

**Lemma 1.** Let \( Z \) and \( X \) be Banach spaces and let \( P_s(h) \) be a bounded family in \( L(Z, X) \). Let \( M \) be a compact subset of \( Z \). Assume that for each \( v \) in \( M \), \( P_s(h) v \rightarrow 0 \) uniformly in \( s \) as \( h \rightarrow 0 \). Then \( P_s(h) v \rightarrow 0 \) uniformly in \( s \) and \( v \) as \( h \rightarrow 0 \).
Proof. Otherwise there are \( v_h \) in \( M \) with \( \sup_s \| P_s(h) v_h \| \geq \epsilon \), say. But by compactness, \( v_h \) has a limit point \( v_0 \), and so
\[
\| P_s(h) v_h \| \leq \| P_s(h) (v_h - v_0) \| + \| P_s(h) v_0 \| ,
\]
which provides a contradiction.

Now the \( \exp (A(s)) - I \) are bounded in \( L(X) \). The \( A(s) w, 0 \leq s \leq t \), form a compact set in \( X \) by the continuity assumption of the theorem. Lemma 1 applies to establish that
\[
\lim_{h \to 0} (\exp (A(s) h) - I) A(s) w = 0
\]
as \( h \to 0 \) uniformly in \( s \).

Thus for \( w \) in \( D \),
\[
\| E_s(h) w \| = \left\| \frac{1}{h} (\exp (A(s) h) - I) w - A(s) w \right\|
\]
\[
= \left\| \frac{1}{h} \int_0^h (\exp (A(s) k) A(s) w - A(s) w) dk \right\|
\]
\[
\leq \sup_{0 \leq k \leq h} \| \exp (A(s) k) A(s) w - A(s) w \|
\]
\[
\to 0
\]
uniformly in \( s \) as \( h \to 0 \).

The term \( F_s(h) \) may be dealt with in much the same way. We must first show that if \( k \) is a function of \( h \) which goes to zero with \( h \), then
\[
\exp (A(s) h) \exp (B(s) k) B(s) w \to B(s) w
\]
as \( h \to 0 \) uniformly in \( s \).

For the reason explained above, \( \exp (A(s) h) v \) and \( \exp (B(s) k) v \) converge to \( v \) uniformly in \( s \) as \( h \to 0 \). But
\[
\exp (A(s) h) \exp (B(s) k) v - v
\]
\[
= \exp (A(s) h) (\exp (B(s) k) v - v) + (\exp (A(s) h) v - v),
\]
so \( \exp (A(s) h) \exp (B(s) k) v \to v \) uniformly in \( s \) as \( h \to 0 \).

The \( B(s) w \) form a compact set in \( S \), so as before Lemma 1 shows that
\[
(\exp (A(s) h) \exp (B(s) k) - I) B(s) w \to 0 \quad \text{as} \quad h \to 0
\]
uniformly in \( s \).
Finally, for \( w \) in \( D \), we conclude that

\[
\| F_s(h) w \| = \left\| \frac{1}{h} \int_0^h (\exp(A(s) h) \exp(B(s) k) B(s) w - B(s) w) \, dk \right\|
\]

\[
\leq \sup (0 < h < 1) \| \exp(A(s) h) \exp(B(s) k) B(s) w - B(s) w \| \to 0
\]

uniformly in \( s \) as \( h \to 0 \).

Since \( A(t) \) and \( B(t) \) are closed operators from \( X \) to \( X \), their restrictions to \( D \) are closed operators on \( D \) to \( X \). Thus by the closed graph theorem, their restrictions are in \( L(D, X) \). We know from what has been shown above that \( E_s(h) w \) and \( F_s(h) w \) are uniformly bounded in \( s \) and \( h \) for \( w \) in \( D \). By the principle of uniform boundedness \( E_s(h) \) and \( F_s(h) \) are uniformly bounded as mappings in \( L(D, X) \).

Next notice that \( s \to A(s, 0) u \) is \( C^0 \) from \( [0, t] \) to \( D \). Hence \( K \) is a compact subset of \( D \).

It follows from Lemma 1 that \( E_s(h) w \) and \( F_s(h) w \) go to zero uniformly in \( s \) and in \( w \in K \). This takes care of the first two terms in the bound for \( \| S_n u \| \).

Now look at the remaining term.

\[
\| G_s(h) R(s, 0) u \| = \left\| \frac{1}{h} (R(s + h, 0) u - R(s, 0) u) - C(s) R(s, 0) u \right\|
\]

\[
= \left\| \frac{1}{h} \int_0^h (C(s + k) R(s + k, 0) u - C(s) R(s, 0) u) \, dk \right\|
\]

\[
\leq \sup (0 \leq k \leq h) \| C(s + k) R(s + k, 0) u - C(s) R(s, 0) u \|.
\]

Now \( C(s) R(s, 0) u \) is \( C^0 \) in \( s \), hence uniformly continuous on \( [0, t] \). It follows that

\[
G_s(h) R(s, 0) u \to 0
\]
as \( h \to 0 \) uniformly in \( s \).

Thus \( S_n u \to 0 \). Since \( D \) is dense in \( S \) and the \( S_n \) are uniformly bounded, this shows that \( S_n \to 0 \) strongly. This completes the proof.

In order to apply the presently available theorems on solutions of time-dependent equations, it is useful to have the following criterion.

**Proposition 2.** Let \( X \) be a Banach space. Let \( D \subset X \) be a Banach space such that the inclusion is continuous. Let \( C(t) \in L(D, X) \) for each
t \in [a, b]. Assume that \( u(t) \) is a function from \([a, b]\) to \( D \) such that \( u(t) \) and \( C(t)u(t) \) are continuous from \([a, b]\) to \( X \). Assume in addition that for some fixed \( \mu \), the \( (\mu - C(t))^{-1} \) are bounded in \( L(X, D) \) uniformly for \( t \) in \([a, b]\). Then \( u(t) \) is continuous from \([a, b]\) to \( D \).

Proof.

\[ u(t) = (\mu - C(t))^{-1} (\mu - C(t)) u(t). \]

So

\[ \| u(t) - u(t') \|_D \leq M \| (\mu - C(t)) u(t) - (\mu - C(t')) u(t') \|_X. \]

2. APPLICATION TO THE FEYNMAN INTEGRAL

The preceding result will be applied to the Feynman path integral solution of the Schrödinger equation

\[ i \frac{du(t)}{dt} = -\frac{1}{2m} \Delta u(t) + V(t)u(t) \]

for \( u(t) \) in \( L^2(R^l) \) for each \( t \). For each \( t \), \( V(t) \) is a (not necessarily bounded) real-valued function on \( R^l \).

**Theorem 2.** Let \( V(t) = \sum_k V_k(t) \) be a finite sum of real-valued measurable functions \( V_k(t) \) on \( R^l \) depending on the real parameter \( t \). Assume that for each \( k \), there is a \( p \) with

\[ \begin{align*}
2 & \leq p \leq \infty, & l = 1, 2, 3, \\
2 & < p \leq \infty, & l = 4, \\
\frac{l}{2} & \leq p \leq \infty, & l \geq 5,
\end{align*} \]

such that \( V_k(t) \) is a function of \( t \) with values in \( L^p(R^l) \) which is continuous and of bounded variation.

Then for \( \Psi \in L^2(R^l) \),

\[ u(x, t) = \lim_{n \to \infty} \int \cdots \int \exp \left[ i \sum_{j=0}^{n-1} \left\{ \frac{1}{2} m \left( \frac{\Delta X_j}{\Delta t} \right)^2 - V(X_j, t_j) \right\} \Delta t \right] \times \Psi(X_0) \left( \frac{2\pi i \Delta t}{m} \right)^{-\frac{1}{2}} dX_0 \cdots dX_{n-1}, \]

where \( X_n = x, \Delta t = t/n, t_j = j\Delta t, \Delta X_j = |X_{j+1} - X_j| \), is the solution to the Schrödinger equation with \( u(x, 0) = \Psi(x) \).
Remark. The expression in the theorem is formally an integral over the space of all paths \(X(\tau), 0 \leq \tau \leq t\), in \(\mathbb{R}^r\) such that \(X(t) = X\) of

\[
\exp \left[ i \int_0^t \frac{1}{2} m \left( \frac{dX(\tau)}{d\tau} \right)^2 - V(X(\tau), \tau) \right] \Psi(X(0)).
\]

This was Feynman's \([4]\) solution of the Schrödinger equation.

Proof. Let \(X\) be the Hilbert space \(H = L^2(\mathbb{R}^r)\). \(\Delta\) is the Laplacian regarded as a self-adjoint operator in \(L^2(\mathbb{R}^r)\). Let \(A(t) = A = (i/2m) \Delta\) and \(B(t)\) be multiplication by \(-iV(t)\). Let \(C(t) = A(t) + B(t)\). With these choices, we will verify the hypotheses of Theorem 1.

The result of Kato \([6]\) provides the following criterion. Let \(C(t)\) for each \(t\) in \([a, b]\) be a skew-adjoint operator in a Hilbert space \(H\). Assume that the domain \(D\) of \(C(t)\) is independent of \(t\). Give \(D\) the Banach-space structure described in Proposition 1. Assume that \(C(t)\) is continuous and of bounded variation in \(L(D, H)\) as a function of \(t\). Further assume that there exists a \(\mu > 0\) such that \((\mu - C(t))^{-1}\) is bounded in \(L(H, D)\) as a function of \(t\). We may conclude that there exists a unique propagator \(R(t, s)\) consisting of unitary operators on \(H\) such that \(D\) is invariant under each \(R(t, s)\) and such that for each \(\Psi\) in \(D\) and each \(s\), \(u(t) = R(t, s) \Psi\) is \(C^1\) in \(t\) and satisfies the equation

\[
\frac{du(t)}{dt} = C(t) u(t)
\]

with \(u(s) = \Psi\).

Now we verify these conditions for the operator \(C(t) = A + B(t)\) given above. Let \(H^2\) be the Sobolev space of all functions in \(H\) with first and second partial derivatives in \(H\). \(H^2\) may also be regarded as the domain of the Laplacian with the graph norm. Clearly \(A \in L(H^2, H)\).

For each \(a > 0\), there is an estimate of the form

\[
\| V_k(t) u \| \leq a \| \Delta u \| + b \| u \|.
\]

(See, e.g., Faris \([3, \text{Section 4}]\).) The constant \(b\) depends on \(a\) and on \(V_k(t)\). Fix \(a\). Then when \(p \neq 1/2\), \(b\) depends on \(V_k(t)\) only through \(\| V_k(t) \|_p\). By the continuity assumption these norms are bounded, so \(b\) may be chosen independently of \(t\).

Even in the borderline case \(p = 1/2\), \(l \geq 5\), a value of \(b\) may be chosen which gives the estimate for all \(t\). To see this, let \(W = \| V_k(t) \|\). For arbitrary \(b \geq 0\), write \(W = W_1 + W_2\), where \(W_1 = \min (W, b)\).
Then always \( \| W_2 u \| \leq b \| u \| \). On the other hand,
\[
\| W_2 u \| \leq \| W_2 \|_p \| u \|_p
\]
where \( \frac{1}{p} + \frac{1}{s} = \frac{1}{2} \), and \( \| u \|_p \leq C \| Au \| \) by application of Sobolev inequalities as in [3]. Now for fixed \( t \) it is clear that \( \| W_2 \|_p \downarrow 0 \) as \( b \to \infty \). But since \( V_k(t) \) is continuous in \( L^p \) as a function of \( t \), so are \( W_1 \) and \( W_2 = W - W_1 \). Hence \( \| W_2 \|_p \downarrow 0 \) uniformly in \( t \) as \( b \to \infty \).

In particular, \( \| W_2 \|_p \leq a \) for all \( t \) if \( b \) is sufficiently large.

Thus \( B(t) \) has a restriction in \( L(H^2, H) \) and for arbitrary \( a > 0 \), there is a \( b \) with
\[
\| B(t) u \| \leq a \| Au \| + b \| u \|
\]
\( b \) depends on \( a \) but not on \( t \). We immediately conclude that the domain of \( C(t) = A + B(t) \) is \( H^2 \) for each \( t \) and \( C(t) \in L(H^2, H) \). Further, for \( \mu \) real and nonzero, we have
\[
\| B(t)(\mu - A)^{-1} v \| \leq a \| (\mu - A)^{-1} v \| + b \| (\mu - A)^{-1} v \|
\]
\[
\leq \left( a + \frac{b}{|\mu|} \right) \| v \|.
\]
Thus for \( |\mu| \) sufficiently large, \( B(t)(\mu - A)^{-1} \in L(H) \) has arbitrarily small norm. Since it may be in particular chosen less than one, we have
\[
(\mu - C(t))^{-1} = (\mu - A)^{-1} [1 - B(t)(\mu - A)^{-1}]^{-1} \in L(H, H^2).
\]
It follows that \( C(t) \) is skew-adjoint in \( X \) and that the Banach space \( D \) is just \( H^2 \).

Now observe that \( \mu \) may actually be chosen so that the norms of \( B(t)(\mu - A)^{-1} \in L(H) \) are bounded by an arbitrarily small constant uniformly in \( t \). Choose this constant to be less than one. The above formula shows that for suitable \( \mu \), \( (\mu - C(t))^{-1} \) is bounded in \( L(H, D) \) as a function of \( t \).

The derivations of the inequalities used above provide the information that for \( p \) in the appropriate range, the norm of a multiplication operator \( V \) in \( L(H^2, H) = L(D, H) \) is bounded by a constant times the norm of \( V \) in \( L^p \). (In the special case \( p = \frac{1}{2} I \), we need only note that \( \| V u \| \leq \| V \|_p \| u \|_p \) where \( \frac{1}{2} = \frac{1}{p} + \frac{1}{s} \). Then we may use Sobolev inequalities as before to conclude that \( \| u \|_p \) is bounded in terms of the \( H^2 \) norm of \( u \).) It follows that \( V_k(t) \) is continuous and of bounded variation as a function of \( t \) with values in \( L(D, H) \). (In the terminology of Hille and Phillips [5; Section 3.2], it is of strong
bounded variation.) Thus $B(t)$ and $C(t) = A + B(t)$ are continuous and of bounded variation in $L(D, H)$.

Thus Kato's theorem provides the solution $u(t) = R(t, s)\Psi$ of the Schrödinger equation. It is a solution in the Hilbert space sense when $u(s) = \Psi \in D$, and even when $\Psi \notin D$, but $\Psi \in H$, it may be considered a solution in a generalized sense.

Next we must verify the remaining hypotheses of Theorem 1, again taking $s = 0$ for notational convenience. By Proposition 2, $u(t)$ is continuous in $t$ with values in $D$. Thus the solution $u(t)$ of the Schrödinger equation with $u(0) = \Psi$ is given by

$$u(t) = \lim_{n \to \infty} \prod_{k=0}^{n} \exp \left( \frac{it}{2n} \Delta \right) \exp \left( \frac{-it}{n} V \left( \frac{k}{n} \right) \right) \Psi.$$

Since $\exp (iux)$ is convolution by $(4\pi u)^{-3/2} \exp (i |x|^2/4u)$, this establishes the result.

**Example.** Let $V_k(x, t)$ be a real valued function which for each $t$ is in $L^p(R^l)$ as a function of $x$. Assume that $p \geq 2$, $p > \frac{l}{2}$ and that the case $p = l/2 = 2$ is excluded. Require that $V_k(x, t)$ is $C^1$ in $t$ for each $x$. Further require that there is a function $\phi_k(x)$ in $D(R^l)$ such that $|\partial V_k/\partial t(x, t)| \leq \phi_k(x)$ for all $t$. Set $V(x, t) = \sum_k V_k(x, t)$ and consider $V(t) = V(x, t)$ for each $t$ as a function of $x$. Then $V(t)$ satisfies the hypotheses of Theorem 2.

**Proof.** The difference quotients

$$\Delta_k(x, t, h) = \frac{1}{h} (V_k(x, t + h) - V_k(x, t))$$

are all dominated by $\phi_k(x)$, by the mean value theorem. Therefore

$$\int \left| \Delta_k(x, t, h) - \frac{\partial V_k(x, t)}{\partial t} \right|^p dx \to 0$$

as $h \to 0$ for each $t$, by the dominated convergence theorem. So $V_k(x, t)$ is differentiable in $L^p$ as a function of $t$. $\partial V_k/\partial t(x, t)$ is continuous in $L^p$ as a function of $t$ by a second application of the dominated convergence theorem. Now for each $t$, let $V_k(t) = V_k(x, t)$ as an $L^p$ function of $x$. Then we have just shown that $V_k(t)$ is a $C^1$ function of $t$. So it is the integral of its derivative, and hence of bounded variation.
3. A Product Formula for Bounded Perturbations

The propagator for the differential equation \( \frac{du(t)}{dt} = B(t) u(t) \) may be given by the product formula

\[
\lim_{n \to \infty} \prod_{k=0}^{n-1} \exp \left( B \left( k \frac{t}{n} \right) \frac{t}{n} \right).
\]

Only when the various \( B(t) \) commute may we replace this by

\[
\lim_{n \to \infty} \exp \left( \sum_{k=0}^{n-1} B \left( k \frac{t}{n} \right) \frac{t}{n} \right) = \exp \left( \int_0^t B(\tau) \, d\tau \right).
\]

Another expression for the propagator in the general case, however, is

\[
\lim_{n \to \infty} \prod_{k=n-1}^{n-1} \left( I + B \left( k \frac{t}{n} \right) \frac{t}{n} \right),
\]

which by definition is the product integral \( \int_0^t (I + B(s) \, ds) \) (Masani [9] treats product integrals.) This suggests a perturbation formula based on a hybrid of the Trotter formula and product integrals. Then given the propagator for the unperturbed equation, it will not be necessary to solve other equations in order to get the approximating expression for the full equation. (That is, the exponential is avoided by only considering a sufficient number of terms in its expansion.)

**Theorem 3.** Let \( X \) be a Banach space. For each \( t \in [0, b] \), let \( A(t) \) be the infinitesimal generator of a strongly continuous semigroup of contractions \( \exp (A(t) s) \) on \( X \). Assume that the domain \( D \) of \( A(t) \) is independent of \( t \) and that \( A(t) u \) is continuous in \( t \) for \( u \) in \( D \). For each \( t \in [0, b] \), let \( B(t) \) be a dissipative operator in \( L(X) \). Assume that \( B(t) \) is strongly continuous in \( t \).

Let \( C(t) = A(t) + B(t) \). Assume that there is a propagator \( R(t, s) \) consisting of contractions on \( X \) such that \( D \) is invariant under each \( R(t, s) \). For each \( \Psi \in D \), let \( u(t) = R(t, 0) \Psi \) and further assume that \( C(t) u(t) \) is \( C^0 \) on \( [0, b] \), and that \( u(t) \) is \( C^1 \) on \( (0, b) \) and there satisfies

\[
\frac{du(t)}{dt} = C(t) u(t).
\]

Require also that \( u(t) \) is \( C^0 \) from \( [0, b] \) to the Banach space \( D \).
Then

\[ R(t, 0) = \lim_{n \to \infty} \prod_{k=n-1}^{0} \exp \left( A \left( k \frac{t}{n} \right) \frac{t}{n} \right) \left( I + B \left( k \frac{t}{n} \right) \frac{t}{n} \right) \]

in the strong operator topology of \( L(X) \).

Note. Assume that \( A(t) = A \) is independent of the time parameter \( t \) and that \( B(t) \psi \) is a \( C^1 \) function of \( t \) for each \( \psi \) in \( X \). Then a theorem of Phillips [12] (See also Segal [13, Lemma 3.1]) shows that this is one situation when the conditions of Theorem 3 are satisfied. (The condition that \( u(t) \) is continuous from \([0, b]\) to \( D \) may be verified by noting that the principle of uniform boundedness shows that the norms of the \( B(t) \in L(X) \) are uniformly bounded in \( t \) by some constant \( M \). It easily follows that for \( \mu > M \),

\[ (\mu - C(t))^{-1} = (\mu - A - B(t))^{-1} \in L(X, D) \]

is bounded uniformly in \( t \). Application of Proposition 2 completes the argument.]

Proof. Set \( h = t/n \). Define

\[ \begin{align*}
    P_k &= \exp \left( A((k - 1) h) h \right), \\
    Q_k &= I + B((k - 1) h) h, \\
    R_k &= R(kh, (k - 1) h), \\
    S_n &= \prod_{k=n}^{1} P_k Q_k - R(t, 0).
\end{align*} \]

As before, we have

\[ S_n = \sum_{j=1}^{f+1} \prod_{k=n}^{j-1} P_k Q_k \left[ P_j Q_j - R_j \right] R((j - 1) h, 0). \]

In order to show that \( S_n \to 0 \) strongly as \( n \to \infty \), we will choose \( u \) in \( D \) and show \( S_n u \to 0 \). But

\[ \left\| S_n u \right\| \leq n \max (1 \leq j \leq n) \left\| \prod_{k=n}^{j-1} P_k Q_k \right\| \left\| (P_j Q_j - R_j) R((j - 1) h, 0) u \right\| \]

\[ \leq t \max (1 \leq j \leq n) \prod_{k=n}^{j+1} \left\| Q_k \right\| \]

\[ \times \frac{1}{n} \sup (0 \leq s \leq t) \left\| \left[ \exp(A(s) h) (I + B(s) h) - R(s + h, s) \right] R(s, 0) u \right\|. \]
For fixed \( v \) in \( X \), \( B(s) v \) is continuous in \( s \), \( 0 \leq s \leq t \), and therefore the \( B(s) v \) are bounded in \( X \). By the principle of uniform boundedness, the \( B(s) \) are bounded in \( L(X) \). Thus \( \| B(s) \| \leq M \).

So

\[
\max (1 \leq j \leq n) \prod_{k=n}^{j+1} \| Q_k \| \leq \prod_{k=n}^{1} (1 + \| B((k - 1) h) \| h) \\
\leq (1 + Mh)^n = \left(1 + \frac{Mt}{n}\right)^n \leq e^{Mt}
\]

Thus the first term in the bound for \( \| S_n u \| \) is bounded independently of \( n \).

The second term contains the expression

\[
\frac{1}{h} \left[ \exp (A(s) h) (1 + B(s) h) - R(s + h, s) \right] \\
= \frac{1}{h} \left[ \exp (A(s) h) - I \right] + \exp (A(s) h) B(s) - \frac{1}{h} \left[ R(s + h, s) - I \right]
\]

applied to vectors of the form \( w = R(s, 0) u \).

Choose \( w \in D \). Then \( \frac{1}{h} \left[ \exp (A(s) h) - I \right] w \rightarrow A(s) w \) as \( h \rightarrow 0 \) uniformly in \( s \). This is proved exactly as is the corresponding assertion in the proof of Theorem 1. [In case \( A(s) = A \) is independent of time, as in the situation described in the note, there is no uniformity to prove and the result is immediate.]

Also, \( \exp (A(s) h) B(s) w \rightarrow B(s) w \) as \( h \rightarrow 0 \) uniformly in \( s \). A quick way to see this is to consider the family of functions \( f_h \), where \( f_h(s) = \exp (A(s) h) B(s) w \). This family is equicontinuous, since

\[
\| f_h(s) - f_h(s_0) \| \leq \| B(s) w - B(s_0) w \|.
\]

Since \( f_h(s) \) obviously converges to \( B(s) w \) for each \( s \) in the compact interval \([0, t]\), it converges uniformly in \( s \).

It is clear that

\[
\frac{1}{h} \left[ \exp (A(s) h) - I \right] - A(s) \quad \text{and} \quad \exp (A(s) h) B(s) - B(s)
\]

have restrictions in \( L(D, X) \). Since when these operators are applied to \( w \) in \( D \), the resulting vectors in \( X \) are bounded in \( s \) and \( h \), the operators in \( L(D, X) \) are themselves bounded in \( s \) and \( h \).

Let \( K \) be the set of all \( R(s, 0) u \), \( 0 \leq s \leq t \). Then \( K \) is a compact subset of \( D \).
Thus Lemma 1 applies to show that

$$\frac{1}{h} \left[ \exp(A(s)h) - I \right] R(s, 0) u \to A(s) R(s, 0) u$$

and \( \exp(A(s)h) B(s) R(s, 0) u \to B(s) R(s, 0) u \) uniformly in \( s \).

Finally, \( \frac{1}{h} [R(s + h, 0) - R(s, 0)] u \to C(s) R(s, 0) u \) uniformly in \( s \). A suitable argument is contained in the proof of Theorem 1.

We conclude that

$$\frac{1}{h} \left[ \exp(A(s)h)(I + B(s)h) - R(s + h, s) \right] R(s, 0) u$$

$$\to [A(s) + B(s) - C(s)] R(s, 0) u = 0$$

uniformly in \( s \) as \( h \to 0 \).

Thus \( S_n u \to 0 \) as \( n \to \infty \) for \( u \) in the dense set \( D \). This implies that \( S_n \to 0 \) strongly.

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