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Nearly trivial homotopy classes between finite complexes

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Abstract

We construct examples of essential maps of finite complexes $f:X\to Y$ which are trivial of order $\geqslant n$. This latter condition implies that for any space K with cone length $\leqslant n$, the induced map $f_*=0:[K,X]\to[K,Y]$. The main result establishes a connection between the skeleta of the infinite dimensional domains of essential phantom maps and the finite dimensional domains of maps which are trivial of order $\geqslant n$. In particular, there are essential maps $f:\Sigma^{2i}(\mathbb{C}P^t/S^2)\to M(\mathbb{Z}/p^s,2l+3)$ which are trivial of order $\geqslant n$.

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1. Introduction

Let $f:X\to Y$ be a map and consider the class $\mathcal{K}(f)$ of all spaces K such that $f\circ h$ is homotopic to the constant map $*$ for every map $h:K\to X$. The larger the class $\mathcal{K}(f)$, the more nearly trivial we consider the map f to be. We are interested in finding essential maps $f:X\to Y$ such that the class $\mathcal{K}(f)$ is large. For example, if $\mathcal{K}(f)$ contains all finite dimensional complexes, then f is a phantom map. However, if f is an essential phantom map, then the domain X must be infinite dimensional. In this note we study analogs of phantom maps between finite complexes—that is, we study essential maps $f:X\to Y$ of finite complexes for which $\mathcal{K}(f)$ contains a large class of finite dimensional spaces.

A natural first step is to search for maps $f:X\to Y$ for which $\mathcal{K}(f)$ contains S^n for each $n\geqslant 0$, or equivalently, such that $\pi_*(f)=0:\pi_*(X)\to\pi_*(Y)$. We say that such a map is trivial of order at least 1. There are numerous examples of maps of this kind—for example,

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the canonical quotient map $X \times X \to X \wedge X$. To extend this notion, we will define what it means for a map to be trivial of order at least n.

A (*spherical*) *cone length decomposition of length* n for a connected space K is a sequence of cofibrations

$$S_i \to L_i \to L_{i+1}, \quad 0 \leqslant i < n,$$

where S_i is a wedge of spheres, $L_0 \equiv *$ (i.e., L_0 is contractible) and $L_n \equiv K$. The (*spherical*) *cone length of* K, denoted $\mathrm{cl}(K)$, is defined as follows: if K is contractible, set $\mathrm{cl}(K) = 0$; otherwise, $\mathrm{cl}(K)$ is the smallest integer n such that there exists a cone length decomposition of K with length n. If instead we require that $L_0 \equiv K$, $L_n \equiv *$ and that each S_i be connected, then we have a (*spherical*) *killing length decomposition of length* n. The (spherical) killing length of K, written $\mathrm{kl}(K)$, is defined analogously. It is shown in [2] that $\mathrm{kl}(K) \leqslant \mathrm{cl}(K)$ for any space K. Furthermore, any cellular decomposition of an n-dimensional complex K is a cone length decomposition of length at most n. Therefore, if K is an n-dimensional complex, then $\mathrm{kl}(K) \leqslant \mathrm{cl}(K) \leqslant n$.

These definitions suggest two numerical homotopy invariants of maps $f : X \to Y$. We write $T_c(f) \geqslant n$ if $f \circ h \simeq *$ for every map $h : K \to X$ with $\mathrm{cl}(K) \leqslant n$. Similarly, we write $T_k(f) \geqslant n$ if $f \circ h \simeq *$ for every map $h : K \to X$ with $\mathrm{kl}(K) \leqslant n$, and say that f is *trivial of order at least* n. This includes the particular case mentioned above: f is trivial of order at least 1 in this latter sense if and only if $\pi_*(f) = 0$. Since $\mathrm{kl}(K) \leqslant \mathrm{cl}(K)$ for all K, it follows that $T_k(f) \leqslant T_c(f)$. Moreover, if f is trivial of order at least n, then $f \circ h \simeq *$ for any $h : K \to X$ with K an n-dimensional complex. It follows that if X is an n-dimensional complex, then no essential map $f : X \to Y$ can be trivial of order at least n.

Killing length is also related to the weak category of a space. The reduced $(n+1)$-fold diagonal map

$$d_{n+1} : X \to \overbrace{X \wedge X \wedge \cdots \wedge X}^{n+1 \text{ factors}} = X^{(n+1)}$$

is the composite of the diagonal $X \to X^{n+1}$ with the projection $X^{n+1} \to X^{(n+1)}$ onto the smash product. We say $\mathrm{wcat}(K) \leqslant n$ if $d_{n+1} \simeq *$, i.e., if d_{n+1} is homotopic to $*$. We have the following commutative diagram for any map $h : S^n \to X$

$$\begin{tikzcd} S^n \arrow[r, "h"] \arrow[d, "d_2"'] & X \arrow[d, "d_2"] \\ S^n \wedge S^n \arrow[r, "h \wedge h"] & X \wedge X \end{tikzcd}$$

Since $\mathrm{wcat}(S^n) = 1$ for each $n \geqslant 1$, it follows that $T_c(d_2) \geqslant T_k(d_2) \geqslant 1$. We will show how this example can be greatly generalized.

The inequalities $\mathrm{wcat}(K) \leqslant \mathrm{cat}(K) \leqslant \mathrm{cl}(K)$ [6] and $\mathrm{kl}(K) \leqslant \mathrm{cl}(K)$ do not directly imply any relation between killing length and weak category, but our first result shows that such a relation does exist.

Proposition 1. *If K is a space with* $\mathrm{kl}(K) = m$, *then* $\mathrm{wcat}(K) < 2^m$.

This allows us to produce a large family of examples of essential maps between finite complexes which are trivial of order at least n.

Example 2. Let X be a finite complex with $\mathrm{wcat}(X) \geqslant 2^n$. Then $d_{2^n} : X \to X^{(2^n)}$ is essential and it follows from the naturality of the reduced diagonal that it is trivial of order at least n. Since $\mathrm{wcat}(\mathbb{C}\mathrm{P}^t) = t$, we can take $X = \mathbb{C}\mathrm{P}^{2^n}$.

In these examples, $f : X \to Y$ is an essential map with $T_k(f) \geqslant n$, and $\mathrm{wcat}(X) \geqslant 2^n$. Furthermore, if $f : X \to Y$ is essential and $T_k(f) \geqslant n$, then we must have $\mathrm{kl}(X) > n$. In view of Proposition 1, it is reasonable ask whether $\mathrm{wcat}(X) \geqslant 2^n$ whenever X is a finite complex that is the domain of an essential map which is trivial of order at least n. In the special case $n = 1$, the answer is no: in [3] we constructed essential maps $f : \Sigma^2(\mathbb{C}\mathrm{P}^t/S^2) \to S^5$ which are trivial of order at least 1, and $\mathrm{wcat}(\Sigma^2(\mathbb{C}\mathrm{P}^t/S^2)) = 1$ because it is a suspension. There remains the question: is there an upper bound on the triviality of an essential map of finite complexes whose domain is a suspension?

In this note we show that the answer to this more general question is again no. We refine and expand the examples of [3] by constructing, for each n, maps f with $T_k(f) \geqslant n$ and domain a suspension of a finite complex. Since every map with $T_k \geqslant n$ is trivial of order at least 1, this provides many more examples of the type considered in [3].

These new examples are closely related to phantom maps. Our main result (Theorem 3) forges a link between the infinite dimensional domains of phantom maps and finite complexes which are domains of maps with $T_k \geqslant n$. In fact, these finite complexes are quite common—they lurk among the skeleta of most familiar infinite dimensional spaces.

Let $M(G, m)$ denote the Moore space with homology G in dimension m. We write X_k for the k-skeleton of the CW complex X and $X_{(p)}$ for the p-localization of the space X.

Theorem 3. *Let X be a 1-connected CW complex of finite type, let $p > 3$ be a prime and let $n \geqslant 1$. Assume that there is an essential phantom map $\Sigma^{2l} X \to S^{2(k+l)+1}_{(p)}$ where $k, l \geqslant 1$. Then, for each $i < l$, there are positive integers $t = t(n)$, $s = s(n)$ and a map*

$$f : X_t / X_{2k} \to M\big(\mathbb{Z}/p^s, 2k+1\big)$$

such that $\Sigma^{2i} f : \Sigma^{2i}(X_t/X_{2k}) \to M(\mathbb{Z}/p^s, 2(k+i)+1)$ is essential and trivial of order at least n.

To show that the theorem is not vacuous, and to give the desired examples, we recall the following basic result from the theory of phantom maps (see [19, Theorem D] and [10, Theorems 5.2 and 5.4]). Let $\widehat{Y}_p$ denote the Sullivan p-completion of the space Y [16], and write $\mathrm{Ph}(X, Y) \subseteq [X, Y]$ for the set of phantom maps from X to Y.

Theorem. *Let X be a 1-connected CW complex of finite type. If the pointed mapping space $\mathrm{map}_*(X, \widehat{S}_p^{2(k+l)+1})$ is weakly contractible, then*

$$\mathrm{Ph}\big(X, S^{2(k+l)+1}_{(p)}\big) = \big[X, S^{2(k+l)+1}_{(p)}\big] \cong H^{2(k+l)}(X; \mathbb{R}).$$

The mapping space condition holds for any infinite loop space or for any suspension of such a space [11, Theorem 2]. Thus Theorem 3 applies, for example, when $X = \mathbb{C}\mathrm{P}^\infty$, and we conclude that there are essential maps

$$\Sigma^{2l}\bigl(\mathbb{C}\mathrm{P}^t/S^2\bigr) \to M\bigl(\mathbb{Z}/p^s, 2l+3\bigr)$$

which are trivial of order at least n. Other examples can be obtained by applying Theorem 3 to the classifying space BG, where G is any simply connected Lie group [10, Theorem 5.6]. In particular, it follows that for any $k, l, m, n \geqslant 1$ there are essential maps $f : BU(m)_t/BU(m)_{2k} \to S^{2k+1}$ such that $\Sigma^{2i} f$ is essential and trivial of order at least n for each $i < l$.

For information about the cone length and killing length of spaces in a more general context, we refer the reader to [5,2,3]. The invariants T_c and T_k introduced in this paper are closely related to the the essential category weight of a map (also known as the strict category weight), as studied in [15,13]. In fact, a map $f : X \to Y$ has essential category weight at least n, written $E(f) \geqslant n$, if $f \circ h \simeq *$ for any map $h : K \to X$ with $\mathrm{cat}(K) \leqslant n$. Since $\mathrm{cat}(K) \leqslant \mathrm{cl}(K)$, $E(f)$ is a lower bound for $T_c(f)$.

2. Proofs

In this section, we prove Proposition 1 and Theorem 3. In Section 2.1 we establish Proposition 1. We then give some definitions and lemmas which are used in Section 2.3 to prove Theorem 3.

We only consider based spaces of the homotopy type of 1-connected CW complexes. We use localization techniques, and write $\lambda : X \to X_{(p)}$ for the natural map from X to its localization at the prime p. We refer to [9] for the standard properties of localization.

2.1. Proof of Proposition 1

We proceed by induction. If $\mathrm{kl}(K) = 0$, then K is contractible and the result follows. Assume that the result is known for all spaces with killing length less than m and that $\mathrm{kl}(K) = m$. Write $S \xrightarrow{j} K \xrightarrow{k} L$ for the first step in a minimal killing length decomposition for K, so $\mathrm{kl}(L) = m - 1$. Since $d_2 \circ j \simeq * : S \to K \wedge K$, there is a map $\delta : L \to K \wedge K$ such that $d_2 \simeq \delta \circ k$. Thus we have the homotopy commutative diagram

$$\begin{tikzcd}
K \arrow[r, "d_2"] \arrow[dr, "k"'] & K \wedge K \arrow[r, "d_{2m-1}"] & K^{(2^m)} \\
& L \arrow[u, "\delta"] \arrow[r, "d_{2m-1} \simeq *"] & L^{(2^{m-1})} \arrow[u, "\delta^{(2^{m-1})}"']
\end{tikzcd}$$

Since $\mathrm{kl}(L) = m - 1$, the inductive hypothesis shows that $d_{2^{m-1}} \simeq * : L \to L^{(2^{m-1})}$, and therefore $d_{2^m} = d_{2^{m-1}} d_2 \simeq * : K \to K^{(2^m)}$. $\square$

2.2. Lemmas

We begin with a lemma which will allow us to modify a given killing length decomposition.

Lemma 4 [14]. *Let*

$$S_0 \xrightarrow{j_0} L_0 \xrightarrow{k_0} L_1$$

$$S_1 \xrightarrow{j_1} L_1 \xrightarrow{k_1} L_2$$

be two cofibrations in which S_0 and S_1 are wedges of spheres, S_0 is $(n-1)$-connected, L_0 and S_1 are n-connected and L_2 is $(n+1)$-connected. Then there is another pair of cofibrations

$$\overline{S}_0 \xrightarrow{\overline{j}_0} L_0 \xrightarrow{\overline{k}_0} \overline{L}_1$$

$$\overline{S}_1 \xrightarrow{\overline{j}_1} \overline{L}_1 \xrightarrow{\overline{k}_1} L_2$$

where $\overline{S}_0$ and $\overline{S}_1$ are wedges of spheres, $\overline{S}_0$ is n-connected and $\overline{S}_1$ and $\overline{L}_1$ are both $(n+1)$-connected.

Proof. Write $S_0 = T \vee U$ where T is the subwedge consisting of all n-spheres in S_0 and U is the complementary subwedge. Since $j_0|_T \simeq *$, $L_1 \simeq \Sigma T \vee C$ where C is the cofiber of the map $j_0|_U$. Notice that C is n-connected.

Write $S_1 = V \vee W$, where V is the subwedge of all $(n+1)$-spheres and W is the complementary subwedge. Applying H_{n+1} to the second cofibration, we obtain the exact sequence

$$H_{n+1}(V) \longrightarrow H_{n+1}(\Sigma T \vee C) \longrightarrow H_{n+1}(L_2) = 0$$

Thus $j = j_1|_V : V \to \Sigma T \vee C$ is surjective on H_{n+1} and hence on π_{n+1}. Let $b_1, \ldots, b_k \in \pi_{n+1}(\Sigma T)$ be the standard generators, and choose $a_1, \ldots, a_k \in \pi_{n+1}(V)$ such that $j_*(a_i) = b_i$ for each i. Then the map $s = (a_1, \ldots, a_k) : \Sigma T \to V$ satisfies $j \circ s = i_{\Sigma T}$, the inclusion of ΣT into $\Sigma T \vee C$. The long homology exact sequence of the cofibration

$$\begin{tikzcd} \Sigma T \arrow[r, "s"'] & V \arrow[l, bend right, "\overline{j}"'] \arrow[r] & A \end{tikzcd}$$

where $\overline{j} = p_{\Sigma T} \circ j$ and $p_{\Sigma T}$ projects $\Sigma T \vee C$ onto ΣT, induces the split short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & H_{n+1}(\Sigma T) \arrow[r, "s_*"'] & H_{n+1}(V) \arrow[l, bend right, "\overline{j}_*"'] \arrow[r] & H_{n+1}(A) \arrow[r] & 0 \end{tikzcd}$$

Thus A has the homotopy type of a wedge of $(n+1)$-spheres and there is a homotopy equivalence $(s,t): \Sigma T \vee A \to V$ for some map $t: A \to V$.

With these identifications, the following diagram commutes

$$\begin{tikzcd}
V \vee W \arrow[r, "j_1"] & \Sigma T \vee C \\
\Sigma T \vee A \vee W \arrow[u, "(s,t)\vee \mathrm{id}"] \arrow[r, "(i_{\Sigma T}, g)"] & \Sigma T \vee C \arrow[u, equal]
\end{tikzcd}$$

where $g: A \vee W \to \Sigma T \vee C$ is some map. Thus we have a square of cofibrations

$$\begin{tikzcd}
\Sigma T \arrow[r, equal] \arrow[d, "i"] & \Sigma T \arrow[r] \arrow[d, "i_{\Sigma T}"] & * \arrow[d] \\
\Sigma T \vee (A \vee W) \arrow[r, "(i_{\Sigma T}, g)"] \arrow[d] & \Sigma T \vee C \arrow[r] \arrow[d] & L_2 \arrow[d] \\
A \vee W \arrow[r, "p_C \circ g"] & C \arrow[r] & L_2
\end{tikzcd}$$

We have now constructed the following pair of cofibrations

$$U \xrightarrow{j_0|_U} L_0 \longrightarrow C$$

$$A \vee W \xrightarrow{p_C \circ g} C \longrightarrow L_2$$

It remains to move the A term in the second cofibration to the first cofibration. By definition, U is n-connected so the map $L_0 \to C$ is surjective in π_{n+1}. Since A is a wedge of $(n+1)$-spheres, the map $p_C \circ g|_A : A \to C$ lifts through $l: A \to L_0$.

Finally, the desired pair of cofibrations is obtained as follows: let $\overline{S}_0 = U \vee A$, $\overline{j}_0 = (j_0|_U, l)$ and let $\overline{L}_1$ be the cofiber of $\overline{j}_0$; let $\overline{S}_1 = W$ and let $\overline{j}_1$ be the composite $W \xrightarrow{p_C \circ g|_W} C \hookrightarrow \overline{L}_1$. It is a simple matter to verify that the cofiber of $\overline{j}_1$ is homotopy equivalent to L_2. $\square$

Proposition 5. *Let $c \geqslant 2$ and let K be a $(c-1)$-connected CW complex with* $\mathrm{kl}(K) = m$. *Then*

(a) *K has a minimal killing length decomposition in which each S_i and L_i is $(c-1+i)$-connected*;
(b) *for each $q \geqslant 0$,* $\mathrm{kl}(K_q) \leqslant m+2$;
(c) *for each $q \geqslant 0$,* $\mathrm{kl}(K/K_q) \leqslant 2m+2$.

Proof. Let $S_i \xrightarrow{j_i} L_i \xrightarrow{k_i} L_{i+1}$, $0 \leqslant i < m$, be a minimal killing length decomposition for K—that is, $L_0 \equiv K$, $L_m \equiv *$, and S_i is a wedge of spheres for each i.

We observe that, to prove (a), it is enough to show that K has a minimal killing length decomposition in which S_0 is $(c-1)$-connected and S_i and L_i are c-connected for $i > 0$. Then (a) follows on applying this to the resulting L_1 and its minimal killing length decomposition of length $m-1$, and then applying it to the resulting L_2 and its minimal killing length decomposition of length $m-2$, and so on. Let k be the greatest integer for which S_0 is $(k-1)$-connected and S_i and L_i are k-connected for $i>0$. We want to show there is a killing length decomposition of K with $k \geqslant c$. Assume that $k<c$. Let i denote the greatest index for which S_i or L_i is not $(k+1)$-connected. If $i>0$ then we have a pair of cofibrations

$$S_{i-1} \longrightarrow L_{i-1} \longrightarrow L_i$$

$$S_i \longrightarrow L_i \longrightarrow L_{i+1}$$

in which S_{i-1} is $(k-1)$-connected, S_i and L_{i-1} are k-connected, and L_{i+1} is $(k+1)$-connected. By Lemma 4, these cofibrations can be replaced by the cofibrations

$$\overline{S}_{i-1} \longrightarrow L_{i-1} \longrightarrow \overline{L}_i$$

$$\overline{S}_i \longrightarrow \overline{L}_i \longrightarrow L_2$$

in which $\overline{S}_{i-1}$ is k-connected and $\overline{L}_i$ and $\overline{S}_i$ are $(k+1)$-connected. Continuing in this way, we eventually obtain a spherical killing length decomposition in which each S_i and L_i, $i>0$, is $(k+1)$-connected. Applying Lemma 4 to the first two cofibrations in this decomposition shows that there is a killing length decomposition of K with S_0 k-connected and S_i and L_i $(k+1)$-connected for all $i>0$. This shows that there is a minimal killing length decomposition for K in which S_0 is $(c-1)$-connected and S_i and L_i are c-connected for $i>0$.

Now we prove (b). Let $\overline{S}_i$ be the subwedge of S_i consisting of the spheres with dimension at most q, let $\overline{L}_0 = K_q$ and let $\overline{j}_0 : \overline{S}_0 \to \overline{L}_0$ be a lift of $j_0|_{\overline{S}_0}$, which exists by cellular approximation. Define $\overline{L}_1$ to be the cofiber of $\overline{j}_0$. Thus, we have a diagram of cofibration sequences

$$\begin{tikzcd}
\overline{S}_0 \arrow[r] \arrow[d] & \overline{L}_0 \arrow[r] \arrow[d] & \overline{L}_1 \arrow[d] \\
S_0 \arrow[r] & L_0 \arrow[r] & L_1
\end{tikzcd}$$

In this diagram, $\overline{L}_1$ is a subcomplex of L_1 which contains the q-skeleton of L_1. By cellular approximation, we may lift $j_1|_{\overline{S}_1} : \overline{S}_1 \to L_1$ to a map $\overline{j}_1 : \overline{S}_1 \to \overline{L}_1$. Continuing in this way, we obtain cofibration sequences $\overline{S}_i \to \overline{L}_i \to \overline{L}_{i+1}$ for each $0 \leqslant i < m$ in which each inclusion $\overline{L}_i \to L_i$ is a $(q-1)$-equivalence. Since $L_m \equiv *$, $\overline{L}_m$ is $(q-1)$-connected. Now $\overline{L}_m$ is $(q+1)$-dimensional by construction, which shows that L_m has killing length at most 2. Append the two cofibrations of a killing length decomposition of L_m to the previously constructed sequence of m cofibrations to obtain a killing length decomposition for K_q with length $m+2$.

Finally, we prove (c). The cofibration $K \to K/K_q \to \Sigma K_q$ yields the inequality $\mathrm{kl}(K/K_q) \leqslant \mathrm{kl}(K) + \mathrm{kl}(\Sigma K_q)$ by [3, Theorem 3.4]. Since $\mathrm{kl}(\Sigma K_q) \leqslant \mathrm{kl}(K_q)$ the result follows from part (b). □

If $p > 3$ is an odd prime, then $S^{2k+1}_{(p)}$ is a homotopy commutative, homotopy associative H-space [1]. Hence $[K, S^{2k+1}_{(p)}]$ is a p-local Abelian group for any finite complex K. Our next lemma provides an upper bound on the exponent of this group, and may be interesting in its own right.

Lemma 6. *Let K be a $(2k+1)$-connected finite complex $(k \geqslant 1)$ with $\mathrm{kl}(K) = m$ and let $p > 3$ be a prime. Then $[K, S^{2k+1}_{(p)}]$ is a finite Abelian group with exponent dividing p^{mk}.*

Proof. We work by induction on $\mathrm{kl}(K)$. If $\mathrm{kl}(K) = 0$, then K is contractible so the conclusion is obvious. Now assume the result is known for any space with killing length less than m. By Lemma 4 we may find a minimal killing length decomposition for K in which all terms are $(2k+1)$-connected. If $\bigvee S^{n_i} \to K \to L$ is the first step in such a decomposition, then $\mathrm{kl}(L) < m$. From the exact sequence

$$[\bigvee S^{n_i}, S^{2k+1}_{(p)}] \longleftarrow [K, S^{2k+1}_{(p)}] \longleftarrow [L, S^{2k+1}_{(p)}]$$

we see that $[K, S^{2k+1}_{(p)}]$ is a finite group with exponent at most the product of the exponents of $[\bigvee S^{n_i}, S^{2k+1}_{(p)}]$ and $[L, S^{2k+1}_{(p)}]$. By the inductive hypothesis applied to L and a result of Cohen et al. [7], this product is at most $p^k p^{k(m-1)} = p^{mk}$. $\square$

The following well-known lemma will be used in the proof of Lemma 8.

Lemma 7. *Let $A \xrightarrow{i} B \xrightarrow{j} C$ be a cofibration and let $f : X \to B$ be any map. If $j \circ f \simeq *$, then there is a map $s : \Sigma X \to \Sigma A$ such that $\Sigma i \circ s \simeq \Sigma f$.*

Proof. Since $j \circ f \simeq *$, the composite factors through CX, the cone on X. Thus we may construct a homotopy commutative ladder of cofibrations

$$\begin{array}{ccccccc} X & \longrightarrow & CX & \longrightarrow & \Sigma X & \xrightarrow{\mathrm{id}} & \Sigma X \\ \downarrow{\scriptstyle f} & & \downarrow & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle \Sigma f} \\ B & \xrightarrow{j} & C & \longrightarrow & \Sigma A & \xrightarrow{\Sigma i} & \Sigma B \end{array} \qquad \square$$

Armed with this lemma, we give a criterion which guarantees that certain maps $f : X \to S^{2k+1}$ remain essential when composed with the standard inclusion map $\iota_s : S^{2k+1} \hookrightarrow M(\mathbb{Z}/p^s, 2k+1)$.

Lemma 8. *Let X be a finite complex and let $h : X \to S^{2k+1}$ be a map such that for some odd prime p, $\lambda \circ \Sigma^2 h \in [\Sigma^2 X, S^{2k+3}_{(p)}]$ is nontrivial and has finite order divisible by p. Then the composite*

$$X \xrightarrow{h} S^{2k+1} \xrightarrow{\iota_s} M(\mathbb{Z}/p^s, 2k+1)$$

is essential for sufficiently large s.

Proof. Consider the diagram

$$\begin{tikzcd}
 & S^{2k+1} \arrow[r, "\lambda"] \arrow[d, "p^s"'] & S^{2k+1}_{(p)} \arrow[d, "p^s"] \\
X \arrow[r, "h"] \arrow[dr, "\iota_s\circ h"'] & S^{2k+1} \arrow[r, "\lambda"] \arrow[d, "\iota_s"'] & S^{2k+1}_{(p)} \arrow[d, "(\iota_s)_{(p)}"] \\
 & M(\mathbb{Z}/p^s, 2k+1) \arrow[r, "="] & M(\mathbb{Z}/p^s, 2k+1)
\end{tikzcd}$$

in which the vertical sequences are cofibrations and p^s denotes the map with degree p^s. If $\iota_s \circ h \simeq *$, then $(\iota_s)_{(p)} \circ \lambda \circ h \simeq *$, and so $\Sigma(\lambda \circ h)$ lifts through the map $p^s : S^{2k+2}_{(p)} \to S^{2k+2}_{(p)}$ by Lemma 7. Suspending once more, we obtain the lift indicated by the dashed line in the diagram

$$\begin{tikzcd}
 & & S^{2k+3}_{(p)} \arrow[d, "p^s"] \\
\Sigma^2 X \arrow[r, "\Sigma^2 h"'] \arrow[urr, dashed] & S^{2k+3} \arrow[r, "\lambda"'] & S^{2k+3}_{(p)}.
\end{tikzcd}$$

The torsion subgroup of $[\Sigma^2 X, S^{2k+3}_{(p)}]$ is a finite Abelian p-group, and so it has an exponent p^e. If $s \geqslant e$, then $\lambda \circ \Sigma^2 h \simeq *$, which is a contradiction, and so $\iota_s \circ h$ is essential. $\square$

2.3. *Proof of Theorem* 3

Let $\mathcal{G}$ denote the tower $\{[\Sigma(\Sigma^{2l} X_t), S^{2(k+l)+1}_{(p)}]\}$. Since $\mathrm{Ph}(\Sigma^{2l} X, S^{2(k+l)+1}_{(p)})$ is naturally isomorphic to $\lim^1 \mathcal{G}$, the tower $\mathcal{G}$ cannot be Mittag-Leffler [4]. This means that the index of $\mathrm{Im}([\Sigma(\Sigma^{2l} X_t), S^{2(k+l)+1}_{(p)}]) \subseteq [\Sigma(\Sigma^{2l} X_{2k}), S^{2(k+l)+1}_{(p)}]$ (which is finite by [12, Proposition 0]) is unbounded as t increases. Let r be the rank of the group $[\Sigma(\Sigma^{2l} X_{2k}), S^{2(k+l)+1}_{(p)}]$, let T be its torsion subgroup, and write

$$A_t = \mathrm{Im}\big([\Sigma^{2l+1} X_t, S^{2(k+l)+1}_{(p)}] \to [\Sigma^{2l+1} X_{2k}, S^{2(k+l)+1}_{(p)}]\big)$$

and

$$Z_t = \mathrm{Im}\big([\Sigma^{2l+1} X_{2k}, S^{2(k+l)+1}_{(p)}] \to [\Sigma^{2l}(X_t/X_{2k}), S^{2(k+l)+1}_{(p)}]\big).$$

Choose t large enough that the index of $A_t \subseteq [\Sigma^{2l+1} X_{2k}, S^{2(k+l)+1}_{(p)}]$ is divisible by $p^{r((2n+2)(k+l)+1)}|T|$. Then the quotient $[\Sigma^{2l+1} X_{2k}, S^{2(k+l)+1}_{(p)}]/A_t T$ is an Abelian group which is generated by a set of at most r elements and which has order divisible by $p^{r((2n+2)(k+l)+1)}$. Since there is a surjection of finite groups

$$Z_t \cong [\Sigma^{2l+1} X_{2k}, S^{2(k+l)+1}_{(p)}]/A_t \to [\Sigma^{2l+1} X_{2k}, S^{2(k+l)+1}_{(p)}]/A_t T,$$

it follows that Z_t also contains elements of order divisible by $p^{(2n+2)(k+l)+1}$.

The commutativity of the diagram

$$\begin{tikzcd}
{[\Sigma X_t, S^{2k+1}_{(p)}]} \arrow[r] \arrow[d, "\Sigma^{2l}"] & {[\Sigma X_{2k}, S^{2k+1}_{(p)}]} \arrow[r] \arrow[d, "\Sigma^{2l}", "\cong"'] & {[X_t/X_{2k}, S^{2k+1}_{(p)}]} \arrow[d, "\Sigma^{2l}"] \\
{[\Sigma^{2l+1} X_t, S^{2(k+l)+1}_{(p)}]} \arrow[r] & {[\Sigma^{2l+1} X_{2k}, S^{2(k+l)+1}_{(p)}]} \arrow[r] & {[\Sigma^{2l}(X_t/X_{2k}), S^{2(k+l)+1}_{(p)}]}
\end{tikzcd}$$

clearly shows that $Z_t \subseteq \mathrm{Im}(\Sigma^{2l}) \subseteq [\Sigma^{2l}(X_t/X_{2k}), S^{2(k+l)+1}_{(p)}]$. Thus there is a map $g : X_t/X_{2k} \to S^{2k+1}_{(p)}$ such that $\Sigma^{2l} g$ has finite order divisible by $p^{(2n+2)(k+l)+1}$. Notice that g itself also must have finite order since it is an element of the finite group Z_t, and so $\Sigma^{2i} g$ has finite order divisible by $p^{(2n+2)(k+l)+1}$ for $0 \leqslant i \leqslant l$.

Since the composition $\lambda \circ \Sigma^{2l}(p^{(2n+2)(k+l)} \circ g)$ has finite order divisible by p, Lemma 8, applied to $\Sigma^{2(l-1)} g$, shows that $\iota_s \circ p^{(2n+2)(k+l)} \circ \Sigma^{2(l-1)} g$ is essential if s is large enough. Fix such an s and define $f = \iota_s \circ p^{(2n+2)(k+l)} \circ g$. Thus $\Sigma^{2i} f$ is essential for each $0 \leqslant i < l$.

Finally, we demonstrate that, for $0 \leqslant i < l-1$, the essential map $\Sigma^{2i} f$ is trivial of order at least n. Let $\mathrm{kl}(K) \leqslant n$ and let $h : K \to \Sigma^{2i}(X_t/X_{2k})$ be any map. Since $\Sigma^{2i}(X_t/X_{2k})$ is $2(k+i)$-connected, h factors through $\overline{h} : K/K_{2(k+i)} \to \Sigma^{2i} X_t/X_{2k}$. Since $\Sigma^{2i} g$ has finite order, the induced homomorphism $\pi_{2(k+i)+1}(\Sigma^{2i} g) = 0$. Therefore $\Sigma^{2i} g \circ \overline{h}$ can be extended to a map $\widetilde{h} : K/K_{2(k+i)+1} \to S^{2(k+i)+1}_{(p)}$ as in the diagram

$$\begin{tikzcd}
K \arrow[r, "h"] \arrow[d] & \Sigma^{2i}(X_t/X_{2k}) \arrow[r, "\Sigma^{2i} g"] \arrow[rrr, bend left, "\Sigma^{2i} f"] & S^{2(k+i)+1}_{(p)} \arrow[r, "p^{(2n+2)(k+l)}"] & S^{2(k+i)+1}_{(p)} \arrow[r, "\iota_s"'] & M \\
K/K_{2(k+i)} \arrow[r] \arrow[ur, "\overline{h}"] & K/K_{2(k+i)+1} \arrow[ur, "\widetilde{h}"] \arrow[urr, bend right, "*"'] & & &
\end{tikzcd}$$

where we have abbreviated $M = M(\mathbb{Z}/p^s, 2(k+i)+1)$. By Proposition 5(c) we have $\mathrm{kl}(K/K_{2(k+i)+1}) \leqslant 2n+2$, so $p^{(2n+2)(k+l)} \circ \widetilde{h} \simeq *$ by Lemma 6. Thus $\Sigma^{2i} f \circ h \simeq *$, which shows that $\Sigma^{2i} f$ is trivial of order at least n and completes the proof. $\square$

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