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An asymptotic formula for the Koornwinder polynomials☆

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Abstract

A formula for the large-degree asymptotics of Koornwinder's multivariate Askey–Wilson polynomials is presented. In the special case of a single variable, this asymptotic formula agrees with the known leading asymptotics of the Askey–Wilson polynomials determined by Ismail and Wilson.

Keywords: (Multivariate) Orthogonal Polynomials; Asymptotics

MSC: 33D45; 33D52; 41A60

1. Introduction

The Koornwinder polynomials [12] are a family of basic hypergeometric orthogonal polynomials in several variables, unifying the (univariate) Askey–Wilson polynomials [1] and the (multivariate) Macdonald polynomials associated with the classical root systems [13]. The polynomials in question form the top of a hierarchy of classical orthogonal polynomials in several variables [4,5,20]; this hierarchy should be looked upon as a multivariate generalization of the celebrated Askey-scheme [1,11].

In recent years, a significant part of the theory surrounding the Askey–Wilson polynomials has been extended to the multivariate level of the Koornwinder polynomials [2,3,8,12–17,19,21]. This note aims to add further onto the current body of knowledge concerning the Koornwinder polynomials, by providing a formula describing their leading asymptotics as the degree tends to infinity. For the Askey–Wilson

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polynomials, such large-degree asymptotics was computed some time ago in [10] (leading asymptotics) and in [9] (full asymptotic expansion); for the Macdonald polynomials, the leading term of the asymptotics was determined recently in [18] (for type A root systems) and in [6] (for arbitrary *reduced* root systems). The asymptotic formula presented below follows by specialization of a more general result describing the leading asymptotics of orthogonal polynomials in several variables with hyperoctahedral symmetry (associated with the *nonreduced* root systems) [7].

In the one-variable case, our asymptotic formula coincides formally with the expression for the leading asymptotics of the Askey–Wilson polynomials due to Ismail and Wilson [10]. However, while Ismail and Wilson considered pointwise convergence, here we rather study the strong convergence of the polynomials in a Hilbert space sense. The proof of our asymptotic formula becomes particularly simple in the one-variable context and will be treated here in further detail.

2. Koornwinder polynomials

The hyperoctahedral group is given by the semidirect product $\Sigma_N \ltimes (\mathbb{Z}_2)^N$ of the symmetric group of N letters Σ_N and the N-fold product of the cyclic group of order two $\mathbb{Z}_2$. The hyperoctahedral monomial symmetric functions

$$m_\lambda(x_1, \ldots, x_N) = \sum_{\substack{\sigma\in\Sigma_N \\ \varepsilon_j\in\{1,-1\}}} \exp(\mathrm{i}\varepsilon_1\lambda_1 x_{\sigma_1} + \cdots + \mathrm{i}\varepsilon_N\lambda_N x_{\sigma_N}), \quad \lambda \in \Lambda, \tag{1}$$

indexed by the partitions

$$\Lambda = \{\lambda \in \mathbb{Z}^N \mid \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_N \geqslant 0\}, \tag{2}$$

form a basis for the space of even- and permutation-invariant Fourier polynomials on the torus

$$\mathbb{T}_N = \frac{\mathbb{R}^N}{(2\pi\mathbb{Z})^N}. \tag{3}$$

This monomial basis inherits a partial order from the following *hyperoctahedral dominance ordering* of the partitions:

$$\forall \lambda, \mu \in \Lambda: \ \lambda \succeq \mu \Longleftrightarrow \lambda_1 + \cdots + \lambda_k \geqslant \mu_1 + \cdots + \mu_k, \quad \text{for } k = 1, \ldots, N. \tag{4}$$

The Koornwinder polynomials arise by applying a Gram–Schmidt type procedure to the partially ordered monomial basis m_λ, $\lambda \in \Lambda$ with respect to a suitable orthogonality measure Δ on $\mathbb{T}_N$.

To be more specific, let us consider the following factorized weight function on the torus $\mathbb{T}_N$

$$\Delta(x_1, \ldots, x_N) = \frac{1}{2^N N!\, \mathscr{C}(x_1, \ldots, x_N)\, \mathscr{C}(-x_1, \ldots, -x_N)}, \tag{5}$$

where

$$\mathscr{C}(x_1,\ldots,x_N)=\prod_{1\leqslant j<k\leqslant N}\frac{(t\mathrm{e}^{-\mathrm{i}(x_j+x_k)},t\mathrm{e}^{-\mathrm{i}(x_j-x_k)};q)_\infty}{(\mathrm{e}^{-\mathrm{i}(x_j+x_k)},\mathrm{e}^{-\mathrm{i}(x_j-x_k)};q)_\infty}\times\prod_{1\leqslant j\leqslant N}\frac{\prod_{r=1}^4(t_r\mathrm{e}^{-\mathrm{i}x_j};q)_\infty}{(\mathrm{e}^{-2\mathrm{i}x_j};q)_\infty} \tag{6}$$

(with the standard conventions for the q-shifted factorials $(z;q)_\infty:=\prod_{m=0}^\infty(1-zq^m)$ and $(z_1,z_2,\ldots,z_k;q)_\infty:=(z_1;q)_\infty(z_2,q)_\infty\cdots(z_k;q)_\infty$). Here and below it is always assumed that the nome q and the parameters t and t_r, $r=1,\ldots,4$ lie in the domain

$$0<q<1,\quad -1<t,t_r<1. \tag{7}$$

These parameter restrictions ensure in particular that the weight function Δ is positive and smooth on $\mathbb{T}_N$. The standard inner product of the Hilbert space $L^2(\mathbb{T}_N,(2\pi)^{-N}\Delta\mathrm{d}\mathbf{x})$ is given by

$$\langle f,g\rangle_\Delta=\frac{1}{(2\pi)^N}\int_{\mathbb{T}_N}f(x_1,\ldots,x_N)\overline{g(x_1,\ldots,x_N)}\,\Delta(x_1,\ldots,x_N)\,\mathrm{d}x_1\cdots\mathrm{d}x_N \tag{8}$$

(where $\overline{g(x_1,\ldots,x_N)}$ denotes the complex conjugate of $g(x_1,\ldots,x_N)$).

The (normalized) *Koornwinder polynomials* are now defined as the polynomials of the form [12]

$$P_\lambda(x_1,\ldots,x_N)=\sum_{\mu\in\Lambda,\mu\preceq\lambda}a_{\lambda\mu}m_\mu(x_1,\ldots,x_N),\quad \lambda\in\Lambda \tag{9}$$

with coefficients $a_{\lambda\mu}\in\mathbb{C}$ such that

$$\langle P_\lambda,m_\mu\rangle_\Delta=0\quad\text{if } \mu\prec\lambda\quad\text{and}\quad\langle P_\lambda,P_\lambda\rangle_\Delta=1, \tag{10}$$

(where $a_{\lambda\lambda}$ is chosen positive by convention).

It is obvious from this definition that the polynomials P_λ are orthogonal with respect to the inner product $\langle\cdot,\cdot\rangle_\Delta$ when comparable in the partial order. A fundamental result of Koornwinder states that in fact $\langle P_\lambda,P_\mu\rangle_\Delta=0$ for all partitions $\lambda\neq\mu$ [12]. In other words, the Koornwinder polynomials P_λ, $\lambda\in\Lambda$ constitute an orthonormal basis for the hyperoctahedral-symmetric sector of the Hilbert space $L^2(\mathbb{T}_N,(2\pi)^{-N}\Delta\mathrm{d}\mathbf{x})$.

3. Asymptotic formula

The leading asymptotics of the Koornwinder polynomial $P_\lambda(x_1,\ldots,x_N)$, as the partition $\lambda\in\Lambda$ grows to infinity, turns out to be governed by the functions

$$P_\lambda^\infty(x_1,\ldots,x_N)=\sum_{\substack{\sigma\in\Sigma_N\\ \varepsilon_j\in\{1,-1\}}}\mathscr{C}(\varepsilon_1\lambda_1x_{\sigma_1},\ldots,\varepsilon_N\lambda_Nx_{\sigma_N})\exp(\mathrm{i}\varepsilon_1\lambda_1x_{\sigma_1}+\cdots+\mathrm{i}\varepsilon_N\lambda_Nx_{\sigma_N}) \tag{11}$$

with $\mathscr{C}(x_1, \ldots, x_N)$ taken from Eq. (6). To formulate the precise result, we define

$$m(\lambda) := \min_{j=1,\ldots,N} (\lambda_j - \lambda_{j+1}) \tag{12}$$

(with the convention that $\lambda_{N+1} := 0$) and, for a function f in the Hilbert space $L^2(\mathbb{T}_N, (2\pi)^{-N}\Delta \mathrm{d}\mathbf{x})$ we denote its norm by $\|f\|_\Delta := \sqrt{\langle f, f\rangle_\Delta}$.

Theorem 1 (*Asymptotic formula*). *Let* ε *be* (*any value*) *in the interval* $(0, \log(1/q))$. *Then*

$$\|P_\lambda - P_\lambda^\infty\|_\Delta = \begin{cases} \mathrm{O}(\mathrm{e}^{-\varepsilon\, m(\lambda)/2}) \\ \mathrm{O}(\lambda_1^N\, \mathrm{e}^{-\varepsilon\, m(\lambda)}) \end{cases} \quad \text{as } m(\lambda) \to \infty$$

(*with both error bounds holding simultaneously*).

The asymptotic formula states that the Koornwinder polynomial P_λ converges (exponentially fast) to the asymptotic function P_λ^∞ (in the strong Hilbert space sense), when the parts λ_j of the partition λ grow to infinity in such a way that $\lambda_j - \lambda_k \to \infty$ for $j < k$. This result follows by specialization of an analogous asymptotic formula, valid for more general multivariate orthogonal polynomials with hyperoctahedral symmetry associated to a rather broad class of factorized analytic weight functions on the torus $\mathbb{T}_N$ [7].

4. The case $N = 1$: Askey–Wilson polynomials

It is instructive to exhibit the contents of the theorem in further detail for $N = 1$. Koornwinder's polynomials then specialize to the Askey–Wilson polynomials [1,11]

$$P_\ell(x) = \mathscr{N}_\ell^{-1/2} p_\ell(x), \quad \ell \in \mathbb{N} \tag{13}$$

with

$$\mathscr{N}_l = \frac{(t_1t_2t_3t_4q^{2\ell}; q)_\infty}{(t_1t_2t_3t_4q^{\ell-1}; q)_\ell (q^{\ell+1}; q)_\infty \prod_{1\leqslant r<s\leqslant 4}(t_rt_sq^\ell; q)_\infty}, \tag{14}$$

$$p_\ell(x) = \frac{(t_1t_2, t_1t_3, t_1t_4; q)_\ell}{t_1^\ell (t_1t_2t_3t_4q^{\ell-1}; q)_\ell}\, {}_4\Phi_3\left(\begin{matrix} q^{-\ell}, t_1t_2t_3t_4q^\ell, t_1\mathrm{e}^{\mathrm{i}x}, t_1\mathrm{e}^{-\mathrm{i}x} \\ t_1t_2, t_1t_3, t_1t_4 \end{matrix}; q, q\right), \tag{15}$$

where

$${}_s\Phi_{s-1}\left(\begin{matrix} a_1, \ldots, a_s \\ b_1, \ldots, b_{s-1} \end{matrix}; q, z\right) := \sum_{n=0}^\infty \frac{(a_1, \ldots, a_s; q)_n}{(b_1, \ldots, b_{s-1}; q)_n} \frac{z^n}{(q;q)_n}.$$

The corresponding asymptotic functions are given by

$$P_\ell^\infty(x) = c(x)\mathrm{e}^{\mathrm{i}\ell x} + c(-x)\mathrm{e}^{-\mathrm{i}\ell x}, \tag{16}$$

$$c(x) = \frac{\prod_{r=1}^4 (t_r\mathrm{e}^{-\mathrm{i}x}; q)_\infty}{(\mathrm{e}^{-2\mathrm{i}x}; q)_\infty}. \tag{17}$$

The orthogonality relations for the Askey–Wilson polynomials become in this notation [1,11]

$$\frac{1}{4\pi}\int_0^{2\pi} P_\ell(x)\overline{P_m(x)}\,\frac{\mathrm{d}x}{c(x)c(-x)} = \begin{cases} 0 & \text{if } \ell \neq m, \\ 1 & \text{if } \ell = m. \end{cases} \tag{18}$$

The asymptotic formula in the theorem above now states that (upon taking the square)

$$\frac{1}{4\pi}\int_0^{2\pi} |P_\ell(x) - P_\ell^\infty(x)|^2\,\frac{\mathrm{d}x}{c(x)c(-x)} = \mathrm{O}(\ell^2 \mathrm{e}^{-2\varepsilon\ell}) \quad \text{as } \ell \to \infty, \tag{19}$$

with $\varepsilon \in (0, \log(1/q))$. The fact that the normalized Askey–Wilson polynomial P_ℓ tends to the asymptotic function P_ℓ^∞ for $\ell \to \infty$ in the strong Hilbert space sense is in agreement with the formula for the pointwise asymptotics of the Askey–Wilson polynomials found in [9,10].

5. Proof of the asymptotic formula for $N = 1$

The proof of the asymptotic formula in the theorem, patterned after [7], simplifies considerably for $N = 1$. We will close by briefly detailing this simplified proof. For further simplicity, our discussion is limited to the verification of the $\mathrm{O}(\mathrm{e}^{-\varepsilon\ell})$ estimate for the error term (corresponding to the $\mathrm{O}(\mathrm{e}^{-\varepsilon\, m(\lambda)/2})$ estimate in the theorem) rather than the sharper $\mathrm{O}(\ell^2\mathrm{e}^{-2\varepsilon\ell})$ estimate given in Eq. (19) (which corresponds to the $\mathrm{O}(\lambda_1^N \mathrm{e}^{-\varepsilon\, m(\lambda)})$ estimate in the theorem).

It is clear that the desired asymptotic formula follows from the asymptotic estimates

$$\|P_\ell^\infty\|_\Delta = 1 + \mathrm{O}(\mathrm{e}^{-\varepsilon\ell}), \tag{20}$$

$$\langle P_\ell^\infty, P_\ell\rangle_\Delta = 1 + \mathrm{O}(\mathrm{e}^{-\varepsilon\ell}), \tag{21}$$

for $\ell \to \infty$. Indeed, from these estimates it is immediate that

$$\|P_\ell - P_\ell^\infty\|_\Delta^2 = \langle P_\ell, P_\ell\rangle_\Delta - \langle P_\ell, P_\ell^\infty\rangle_\Delta - \langle P_\ell^\infty, P_\ell\rangle_\Delta + \langle P_\ell^\infty, P_\ell^\infty\rangle_\Delta = \mathrm{O}(\mathrm{e}^{-\varepsilon\ell}),$$

which amounts to Eq. (19) with the $\mathrm{O}(\ell^2\mathrm{e}^{-2\varepsilon\ell})$ error bound on the right-hand side replaced by an $\mathrm{O}(\mathrm{e}^{-\varepsilon\ell})$ error bound.

To infer the estimates in Eqs. (20), (21), we first observe that for $\ell > 0$

$$\langle P_\ell^\infty, m_k\rangle_\Delta = \frac{1}{2\pi}\int_0^{2\pi} \frac{\mathrm{e}^{\mathrm{i}\ell x}}{c(-x)}\,(\mathrm{e}^{\mathrm{i}kx} + \mathrm{e}^{-\mathrm{i}kx})\,\mathrm{d}x = \begin{cases} 0 & \text{if } k < \ell, \\ 1 & \text{if } k = \ell. \end{cases} \tag{22}$$

Notice in this connection that the evaluation of the integral in Eq. (22) readily follows from the fact that $1/c(-x)$ has a uniformly converging Fourier series expansion of the form $1 + \sum_{n\geqslant 1} c_n \mathrm{e}^{\mathrm{i}nx}$ (whence the constant term in the Fourier series expansion of the integrand is equal to 0 for $k < \ell$ and equal to 1 for $k = \ell$).

Next, we rewrite the asymptotic function as

$$P_\ell^\infty(x) = \frac{\hat{c}(x)\mathrm{e}^{\mathrm{i}(l+1)x} - \hat{c}(-x)\mathrm{e}^{-\mathrm{i}(l+1)x}}{\mathrm{e}^{\mathrm{i}x} - \mathrm{e}^{-\mathrm{i}x}} \tag{23}$$

with

$$\hat{c}(x) = \frac{\prod_{r=1}^{4}(t_r \mathrm{e}^{-\mathrm{i}x}; q)_\infty}{(q\mathrm{e}^{-2\mathrm{i}x}; q)_\infty}. \tag{24}$$

It is not difficult to see that the function $\hat{c}(x)$ has a Fourier series expansion of the form

$$\hat{c}(x) = 1 + \sum_{n \geqslant 1} \hat{c}_n \mathrm{e}^{-\mathrm{i}nx}$$

with coefficients $\hat{c}_n$ that are $\mathrm{O}(\mathrm{e}^{-\varepsilon n/2})$ as $n \to \infty$ (since the function in question is holomorphic in the lower half-plane $\mathrm{Im}(x) < \log(q^{-1/2})$ and it converges to 1 for $\mathrm{Im}(x) \to -\infty$). Let $P_\ell^{\mathrm{tr}}(x)$ denote the truncated asymptotic function obtained by replacing $\hat{c}(x)$ (24) in $P_\ell^\infty(x)$ (23) by its Fourier polynomial of degree 2ℓ

$$P_\ell^{\mathrm{tr}}(x) = \frac{\hat{c}^{\mathrm{tr}}(x)\mathrm{e}^{\mathrm{i}(l+1)x} - \hat{c}^{\mathrm{tr}}(-x)\mathrm{e}^{-\mathrm{i}(l+1)x}}{\mathrm{e}^{\mathrm{i}x} - \mathrm{e}^{-\mathrm{i}x}}, \quad \hat{c}^{\mathrm{tr}}(x) = 1 + \sum_{n=1}^{2\ell} \hat{c}_n \mathrm{e}^{-\mathrm{i}nx}.$$

Since $|\hat{c}(x) - \hat{c}^{\mathrm{tr}}(x)| = \mathrm{O}(\mathrm{e}^{-\varepsilon\ell})$, it readily follows that

$$\| P_\ell^\infty - P_\ell^{\mathrm{tr}} \|_\Delta = \mathrm{O}(\mathrm{e}^{-\varepsilon\ell}) \quad \text{as } \ell \to \infty. \tag{25}$$

We are now in the position to derive the estimates in Eqs. (20) and (21). Indeed, from Eq. (22) and the fact that the truncated asymptotic function $P_\ell^{\mathrm{tr}}(x)$ is a monic polynomial of degree ℓ (in $\mathrm{e}^{\mathrm{i}x} + \mathrm{e}^{-\mathrm{i}x}$), it is clear that $\langle P_\ell^\infty, P_\ell^{\mathrm{tr}} \rangle_\Delta = 1$. Hence, we have that

$$\begin{aligned} \| P_\ell^\infty \|_\Delta - \frac{1}{\| P_\ell^\infty \|_\Delta} &= \frac{1}{\| P_\ell^\infty \|_\Delta} \langle P_\ell^\infty, P_\ell^\infty - P_\ell^{\mathrm{tr}} \rangle_\Delta \\ &\leqslant \| P_\ell^\infty - P_\ell^{\mathrm{tr}} \|_\Delta \overset{\text{Eq. (25)}}{=} \mathrm{O}(\mathrm{e}^{-\varepsilon\ell}), \end{aligned}$$

which implies Eq. (20). Furthermore, once more invoking of Eq. (22) combined with the orthonormality of the normalized Askey–Wilson polynomials $P_\ell(x)$ (13)–(15) reveals that $\langle P_\ell^\infty, P_\ell \rangle_\Delta = 1/\langle P_\ell^{\mathrm{tr}}, P_\ell \rangle_\Delta$ $(= a_{\ell\,\ell} > 0$, cf. Eqs. (9), (10)). Hence, we have that

$$\begin{aligned} \langle P_\ell^\infty, P_\ell \rangle_\Delta - \frac{1}{\langle P_\ell^\infty, P_\ell \rangle_\Delta} &= \langle P_\ell^\infty - P_\ell^{\mathrm{tr}}, P_\ell \rangle_\Delta \\ &\leqslant \| P_\ell^\infty - P_\ell^{\mathrm{tr}} \|_\Delta \overset{\text{Eq. (25)}}{=} \mathrm{O}(\mathrm{e}^{-\varepsilon\ell}), \end{aligned}$$

which implies Eq. (21).

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