Available online at www.sciencedirect.com
ScienceDirect
Journal of Functional Analysis 258 (2010) 3452–3468
JOURNAL OF Functional Analysis
www.elsevier.com/locate/jfa

A renorming in some Banach spaces with applications to fixed point theory

Carlos A. Hernandez Linares, Maria A. Japon *

Departamento de Análisis Matemático, Universidad de Sevilla, Apdo. 1160, 41080 Sevilla, Spain

Received 24 September 2009; accepted 23 October 2009

Available online 5 November 2009

Communicated by N. Kalton

Abstract

We consider a Banach space X endowed with a linear topology τ and a family of seminorms $\{R_k(\cdot)\}$ which satisfy some special conditions. We define an equivalent norm $|||\cdot|||$ on X such that if C is a convex bounded closed subset of $(X, |||\cdot|||)$ which is τ-relatively sequentially compact, then every nonexpansive mapping $T : C \to C$ has a fixed point. As a consequence, we prove that, if G is a separable compact group, its Fourier–Stieltjes algebra $B(G)$ can be renormed to satisfy the FPP. In case that $G = \mathbb{T}$, we recover P.K. Lin's renorming in the sequence space ℓ_1. Moreover, we give new norms in ℓ_1 with the FPP, we find new classes of nonreflexive Banach spaces with the FPP and we give a sufficient condition so that a nonreflexive subspace of $L_1(\mu)$ can be renormed to have the FPP.

Keywords: Fixed point theory; Renorming theory; Nonexpansive mappings; Fourier algebras; Fourier–Stieltjes algebra; Topology of convergence locally in measure

1. Introduction

Let $(X, \|\cdot\|)$ be a Banach space and C a convex closed bounded subset of X. A mapping $T : C \to C$ is called *nonexpansive* if for any $x, y \in C$ we have $\|Tx - Ty\| \leqslant \|x - y\|$. A point $x \in C$ is a fixed point of T if $Tx = x$. It is clear that Banach's Contraction Principle does not

* Corresponding author.
E-mail addresses: carloshl@us.es (C.A.H. Linares), japon@us.es (M.A. Japon).

0022-1236/$ – see front matter
doi:10.1016/j.jfa.2009.10.025

extend to the setting of nonexpansive mappings. However, some positive results concerning the existence of fixed points for this class of mappings where given in 1965 by F.E. Browder [5] and D. Göhde [13] for uniformly convex Banach spaces and by W. Kirk [16] for reflexive Banach spaces with normal structure. Since then, many authors have studied the problem of the existence of fixed points for nonexpansive mappings and many positive results have been found (see for instance [12,17] and the references therein). It is usually said that a Banach space X has the fixed point property (FPP) if every nonexpansive mapping defined from a closed convex bounded subset onto itself has a fixed point. It is well known that the geometry of the Banach space plays a fundamental role to assure the FPP. In fact, Kirk's result [16] means that a reflexive Banach space with normal structure has the FPP. Many other geometric properties are known to imply the FPP for reflexive Banach spaces (uniform Kadec Klee property, uniform Opial condition, existence of a monotone unconditional basis, etc.). Moreover the classical nonreflexive Banach spaces ℓ_1, c_0, L_1 do not have the FPP (in fact L_1 does not satisfy a stronger condition called the weakly fixed point property [2]). For a long time, it was an open question whether all Banach spaces with the FPP were reflexive. In 2008, P.K. Lin [20] found the first known nonreflexive Banach space with the FPP. In fact, the Banach space given by P.K. Lin was the sequence space ℓ_1 endowed with an equivalent norm to the usual one. His result raises the question: can any Banach space be renormed to have the FPP? This is not, in general, the case because the Banach spaces $\ell_1(\Gamma)$ and $c_0(\Gamma)$, if Γ is uncountable, and the Banach space ℓ_∞ cannot be renormed to have the FPP [8]. A positive partial answer was given by T. Domínguez Benavides [6], who proved that every reflexive Banach space can be renormed to have the FPP. This leads to the following question: Which type of nonreflexive Banach spaces can be renormed to have the FPP?

In this paper we find some classes of nonreflexive Banach spaces which under an equivalent renorming satisfy the FPP. Our techniques are inspired by those of P.K. Lin's paper [20] but our applications go beyond the sequence space ℓ_1 as we will illustrate with many examples. As particular cases, we will recover P.K. Lin's result and we will find new renormings in ℓ_1 with the FPP. Moreover, we will renorm the Fourier–Stieltjes algebra of a separable compact group to have the FPP. Notice that if G is locally compact, its Fourier–Stieltjes algebra $B(G)$ has the FPP if and only if G is finite [18]. We also find new classes of nonreflexive Banach spaces with the FPP which are nonisomorphic to any subspace of ℓ_1.

Finally, we will apply our results to the particular case of subspaces of $L_1(\mu)$ for a σ-finite measure. It is known that a closed subspace X of $L_1(\mu)$ has the FPP if and only if X is reflexive [23,7]. Nevertheless, we will show that some nonreflexive subspaces of $L_1(\mu)$ can still be renormed to have the FPP.

This paper is organized as follows: Section 2 is dedicated to the necessary fixed point background and we establish a technical lemma which is basic in our proofs. In Section 3 we will state our main Theorem and we will introduce the first applications: we are able to find new renormings in ℓ_1 with the FPP, we renorm $B(G)$ with the FPP if G is a separable compact group and we give new examples of nonreflexive Banach spaces with the FPP that are nonisomorphic to any subspace of ℓ_1. Section 4 is dedicated to the proof of the main Theorem. Finally, in Section 5, we will apply the main Theorem to closed subspaces of $L_1(\mu)$ when (Ω, Σ, μ) is a σ-finite measure space. We obtain a sufficient condition to assure that a nonreflexive subspace X of $L_1(\mu)$ can be renormed to have the FPP (it is known that, with the usual norm, X fails to have this property [7]). We finish the paper by introducing some examples of nonreflexive subspaces of $L_1(\mu)$ which can be renormed to have the FPP.

2. Fixed point background

Let C be a closed convex bounded subset of a Banach space $(X, \|\cdot\|)$ and $T : C \to C$ a nonexpansive mapping. Fix any $x_0 \in C$. A direct application of the Banach's Contraction Principle to the sequence of mappings $T_n : C \to C$ defined by

$$T_n x = \frac{1}{n}x_0 + \left(1 - \frac{1}{n}\right)Tx;$$

provides a sequence $\{x_n\}_n \subset C$, where x_n is the unique point of T_n, such that

$$\lim_n \|x_n - Tx_n\| = 0.$$

Such sequences are called approximated fixed point sequences (a.f.p.s.).

Moreover, if $d > 0$ and the set

$$D = \left\{x \in C\colon \limsup_n \|x_n - x\| \leqslant d\right\}$$

is nonempty, it is easy to check that D is convex, closed and a T-invariant subset of C. Hence, we can find another a.f.p.s. in D. As an application of Cantor's Intersection Theorem, we can prove the following:

Lemma 1. *Let $(X, \|\cdot\|)$ be a Banach space and C a convex, closed, bounded subset of X. Let $T : C \to C$ be a nonexpansive mapping and suppose that T is fixed point free. Then there exist some $a > 0$ and a convex closed T-invariant subset D of C such that for each approximated fixed point sequence (x_n) in D and for any $z \in D$*

$$\limsup_n \|x_n - z\| \geqslant a.$$

Proof. If the statement is false there exists an a.f.p.s. (x_n^1) in C and $z_1 \in C$ such that

$$\limsup_n \left\|x_n^1 - z_1\right\| < \frac{1}{2}.$$

Hence

$$D_1 = \left\{z \in C\colon \limsup_n \left\|x_n^1 - z\right\| \leqslant \frac{1}{2}\right\}$$

is a nonempty, convex, closed, T-invariant subset of C.

With the same argument, we deduce the existence of an approximated fixed point sequence (x_n^2) in D_1 and $z_2 \in D_1$ such that

$$\limsup_n \left\|x_n^2 - z_2\right\| < \frac{1}{2^2}.$$

Hence the set

$$D_2 = \left\{ z \in D_1\colon \limsup_n \left\| x_n^2 - z \right\| \leqslant \frac{1}{2^2} \right\}$$

is again a nonempty, convex, closed, T-invariant subset of D_1.

In this way we construct a decreasing sequence (D_n) of convex closed bounded T-invariant subsets of C such that $\operatorname{diam}(D_n) \leqslant \frac{1}{2^{n-1}}$. By the Cantor's Intersection Theorem, $\bigcap_n D_n$ is a singleton. Since each D_n is T-invariant this point has to be a fixed point of T. Thus, we have obtained a contradiction since T is fixed point free. □

Remark. Notice that if X is endowed with a topology τ such that every bounded sequence has a τ-convergent subsequence, then the conditions of Lemma 1 also imply that

$$\inf\left\{ \limsup_n \|x_n - x\|\colon\ (x_n) \subset D,\ (x_n) \text{ a.f.p.s. } x_n \to x \text{ in } \tau \right\} > 0.$$

Indeed, applying the triangular inequality, for every $(x_n) \subset D$ a.f.p.s. such that (x_n) converges to x in the topology τ, we have

$$\limsup_n \|x_n - x\| \geqslant \frac{1}{2} \limsup_m \limsup_n \|x_n - x_m\| \geqslant \frac{a}{2}.$$

3. Main result and first examples

In this section we state the main result of this paper. As a consequence, we obtain the renorming given in [20] in the sequence space ℓ_1, which provided the first known nonreflexive Banach space with the FPP. Also we will give new equivalent norms on ℓ_1 with the FPP and we will obtain new classes of nonreflexive Banach spaces with the FPP. In particular, we will prove that the Fourier–Stieltjes algebra $B(G)$ of a separable compact group can be renormed to have the FPP. Notice that $B(G)$ itself has the FPP if and only if G is finite [18] (Theorem 5.8). More applications of the main Theorem will be studied in the last section.

Let $(X, \|\cdot\|)$ be a Banach space endowed with a linear topology τ. Assume that there exists a family of seminorms $R_k : X \to [0, +\infty)$ $(k \geqslant 1)$ that satisfy the following properties:

(I) $R_1(x) = \|x\|$ while for $k \geqslant 2$, $R_k(x) \leqslant \|x\|$ for all $x \in X$.
(II) $\lim_k R_k(x) = 0$ for all $x \in X$.
(III) If $x_n \to 0$ in τ and is norm-bounded, then for all $k \geqslant 1$

$$\limsup_n R_k(x_n) = \limsup_n \|x_n\|.$$

(IV) If $x_n \to 0$ in τ, is norm-bounded and $x \in X$, then

$$\limsup_n R_k(x_n + x) = \limsup_n R_k(x_n) + R_k(x)$$

for all $k \geqslant 1$.

Then we can state the following:

Theorem 1. *Let* $\{\gamma_k\}_k \subset (0,1)$ *be any nondecreasing sequence such that* $\lim_k \gamma_k = 1$ *and define*

$$|||x||| = \sup_{k\geqslant 1} \gamma_k R_k(x); \quad x \in X.$$

Then $|||\cdot|||$ *is an equivalent norm on* X *such that* $(X, |||\cdot|||)$ *satisfies the following property: for every nonempty closed convex bounded subset C which is* τ*-relatively sequentially compact and for every* $T: C \to C$ *nonexpansive, there exists a fixed point.*

That $|||\cdot|||$ is an equivalent norm on $(X, \|\cdot\|)$ is clear. In fact, $\gamma_1\|x\| \leqslant |||x||| \leqslant \|x\|$ for all $x \in X$.

We will prove Theorem 1 in the next section. Now we give several families of Banach spaces where our results can be applied.

Example 1. A first application of Theorem 1 is a generalization of P.K. Lin's example given in [20], where he proves that if $\gamma_k = 8^k/(1+8^k)$, then the renorming on ℓ_1 given by

$$|||x||| = \sup_k \frac{8^k}{1+8^k}\left\|\sum_{n=k}^{\infty} x_n e_n\right\|$$

has the FPP. Notice that Lin's result can be derived from Theorem 1 defining the seminorms $R_k(x) = \|\sum_{n=k}^{\infty} x_n e_n\|$ and τ the weak-star topology associated to the duality $\sigma(\ell_1, c_0)$. Since the unit ball is weak-star compact and c_0 is separable, every closed convex bounded subset is $\sigma(\ell_1, c_0)$-sequentially compact. Also, we obtain the renorming given in [10] where P.K. Lin's result is generalized by using any nondecreasing sequence $(\gamma_k) \in (0,1)$ with $\lim_k \gamma_k = 1$ and $\gamma_1 > 2/3$. Notice that in our approach the condition $\gamma_1 > 2/3$ can be dropped.

Example 2. If we consider again the sequence space ℓ_1 and change the family of seminorms, then we can obtain new renormings on ℓ_1 with the FPP. For instance, let $p > 1$ and for $k \geqslant 2$ define

$$R_k(x) = \sum_{n=2k}^{\infty} |x(n)| + \left(\sum_{n=k}^{2k-1} |x_n|^p\right)^{1/p}$$

and $R_1(x) = \|x\|_1$. It is easy to check that $\{R_k(\cdot)\}_k$ is a family of seminorms that verify properties (I), (II), (III) and (IV), so ℓ_1 with the norm generated by the seminorms $\{R_k(\cdot)\}_k$ satisfies the FPP.

Corollary 1. *Let* $\{X_n\}_n$ *be a sequence of finite dimensional Banach spaces and consider*

$$X = \oplus_1 \sum_n X_n = \left\{x = (x_n)_n\colon\ x_n \in X_n,\ \|x\| = \sum_n \|x_n\|_{X_n} < \infty\right\}.$$

Then X *can be renormed to have the FPP.*

Proof. Define the seminorms $R_k(x) = \sum_{n\geqslant k} \|x_n\|_{X_n}$ and let τ be the weak star topology where the predual of X is

$$E = \Bigl\{x = (x_n)\colon\ x_n \in X_n,\ \lim \|x_n\|_{X_n} = 0,\ \|x\| = \sup_n \|x_n\|_{X_n}\Bigr\}.$$

It is not difficult to check that the family $\{R_k(\cdot)\}_k$ satisfies properties (I), (II), (III) and (IV). Using the renorming given in Theorem 1, the Banach space $(X, |||\cdot|||)$ has the FPP. □

A first application of Corollary 1 is the following example:

Example 3. Consider $X_n = \ell_p^n$ for some $1 < p \leqslant +\infty$ and

$$X = \oplus_1 \sum_n \ell_p^n.$$

Then we obtain a nonreflexive Banach space that can be renormed to have the FPP and that is not isomorphic to any subspace of ℓ_1. Indeed, ℓ_p is finitely representable in X and the type and the cotype of X is equal to the type and the cotype of ℓ_p respectively. Notice that for every $1 < p \leqslant +\infty$, either the type or the cotype of ℓ_p is different from that of ℓ_1, since $\text{type}(\ell_p) = \min\{2, p\}$ and $\text{cotype}(\ell_p) = \max\{2, p\}$ (see [22], p. 73). Thus, X is not isomorphic to any subspace of ℓ_1 and we obtain new classes of nonreflexive Banach spaces with the FPP.

For the definitions of the Fourier algebra $A(G)$ and the Fourier–Stieltjes algebra $B(G)$ of a locally compact group see [9] or [19] and the references therein. When G is compact, $B(G) = A(G)$. Another application of Corollary 1 is the following.

Corollary 2. *Let G be a separable compact group and $B(G)$ its Fourier–Stieltjes algebra. Then $B(G)$ can be renormed to have the FPP.*

Proof. Using the arguments in the proof of Lemma 3.1 of [19], and having in mind that $B(G)$ is norm separable when G is a separable compact group [14, Corollary 6.9], the Banach space $B(G)$ can be written as

$$B(G) = \oplus_1 \sum_n \mathfrak{T}(H_n)$$

where H_n is a finite dimensional Hilbert space and $\mathfrak{T}(H_n)$ is the trass class operators on H_n. Applying Corollary 1 we obtain a renorming of $B(G)$ with the FPP. □

In the particular case that $G = \mathbb{T}$, the circle group, $B(G)$ is isometric to $\ell_1(\mathbb{Z})$ via Bochner's Theorem. Thus, Corollary 2 includes the sequence space ℓ_1 and the renorming given by P.K. Lin [20]. Also, recall that $B(G)$ with its usual norm has the FPP if and only if G is finite [18].

4. Proof of the main result

Before proving Theorem 1 we prove two technical lemmas:

Lemma 2. *Let X be a Banach space endowed with a linear topology τ and a family of seminorms $\{R_k(\cdot)_k\}$ satisfying properties* (I), (II), (III) *and* (IV) *stated above. Define the $|||\cdot|||$ norm as in Theorem* 1 *and let (x_n), (y_n) be two bounded sequences in X. Then the following statements are satisfied*:

(1) *If $x_n \to 0$ in τ, then*

$$\limsup_n |||x_n||| = \limsup_n \|x_n\|.$$

(2) *If $x_n \to x$ and $y_n \to y$ in τ then*

$$\limsup_m \limsup_n |||x_n - y_m||| \geqslant \limsup_n |||x_n - x||| + \limsup_m |||y_m - y|||.$$

Proof. (1) For every $k \geqslant 1$, using the definition of the $|||\cdot|||$ norm and property (III), we have

$$\limsup_n \|x_n\| \geqslant \limsup_n |||x_n||| \geqslant \gamma_k \limsup_n R_k(x_n) = \gamma_k \limsup_n \|x_n\|.$$

Taking limit as k goes to infinity we deduce (1).

(2) By property (IV) we have

$$\begin{aligned}\limsup_m \limsup_n R_k(x_n - y_m) &= \limsup_m \Big[\limsup_n R_k(x_n - x) + R_k(x - y_m)\Big] \\ &= \limsup_n R_k(x_n - x) + \limsup_m R_k(y_m - y) + R_k(x - y),\end{aligned}$$

for every $k \geqslant 1$.

Then, using again the definition of $|||\cdot|||$ and property (III),

$$\begin{aligned}\limsup_m \limsup_n |||x_n - y_m||| &\geqslant \gamma_k \Big[\limsup_n R_k(x_n - x) + \limsup_m R_k(y_m - y) + R_k(x - y)\Big] \\ &\geqslant \gamma_k \Big[\limsup_n |||x_n - x||| + \limsup_m |||y_m - y|||\Big].\end{aligned}$$

Taking limit as k goes to infinity we get the desired result. □

The following lemma is the key for the arguments in the proof of Theorem 1:

Lemma 3. *Consider the Banach space $(X, |||\cdot|||)$ and let C and T be as in Theorem* 1. *If T is fixed point free we find D as in Lemma* 1. *Let K be any closed convex T-invariant subset of D and denote*

$$\rho = \inf\Big\{\limsup_n |||x_n - x|||:\ (x_n) \subset K \text{ is an a.f.p.s. and } x_n \to x \text{ in } \tau\Big\} > 0.$$

Then for every a.f.p.s. $(x_n) \subset K$ *which is* τ*-convergent and for every* $z \in K$ *we have*

$$\limsup_n |||x_n - z||| \geqslant 2\rho.$$

Proof. Assume that there exist a τ-convergent approximate fixed point sequence (x_n) in K and $z \in K$ such that

$$r = \limsup_n |||x_n - z||| < 2\rho.$$

Then

$$K_1 = \Big\{ w \in K\colon \limsup_n |||x_n - w||| \leqslant r \Big\}$$

is a nonempty, convex, closed, bounded T-invariant subset of K. Choose an approximate fixed point sequence (y_n) in K_1 such that $y_n \overset{\tau}{\to} y$. Denote by x the τ-limit of the sequence (x_n). Then by (2) of Lemma 2, we have

$$\begin{aligned} r &\geqslant \limsup_m \limsup_n |||x_n - y_m||| \\ &\geqslant \limsup_n |||x_n - x||| + \limsup_n |||y_n - y||| \\ &\geqslant \rho + \rho = 2\rho, \end{aligned}$$

which is a contradiction. □

Now we prove Theorem 1.

Proof. Assume the contrary, that T has no fixed point. Let D be as in the conclusion of Lemma 1. Define

$$c = \inf\Big\{ \limsup_n |||x_n - x|||\colon\ (x_n) \subset D \text{ is an a.f.p.s. and } x_n \overset{\tau}{\to} x \Big\}$$

which is greater than zero by the remark made after Lemma 1.

Without loss generality we can assume that $c = 1$. Take $0 < \epsilon_1 < 1/2$ and an a.f.p.s. $(x_n) \subset D$ such that $x_n \overset{\tau}{\to} x$ and $\limsup_n |||x_n - x||| < 1 + \epsilon_1$. Again, by translation, we can assume that $x = 0$.

Let us consider now

$$K = \Big\{ z \in D\colon \limsup_n |||x_n - z||| \leqslant 2 + 2\epsilon_1 \Big\}.$$

The set K is closed, convex, T-invariant and nonempty. Indeed, we can find n_0 such that $x_n \in K$ for all $n \geqslant n_0$.

Define

$$\rho = \inf\Big\{\limsup_n |||y_n - y|||\colon\ (y_n) \subset K \text{ is an a.f.p.s. and } y_n \stackrel{\tau}{\to} y\Big\}.$$

It is clear that $1 \leqslant \rho \leqslant \limsup_n |||x_n||| < 1 + \epsilon_1$.

We are going to find an a.f.p.s. $(y_n) \subset K$ and $z \in K$ such that

$$\limsup_n |||y_n - z||| < 2\rho$$

and then we obtain a contradiction according to Lemma 3.

Notice the following: If $(y_n) \subset K$ is an a.f.p.s. and $y_n \stackrel{\tau}{\to} y$, then for all k,

$$\begin{aligned}
2 + 2\epsilon_1 &\geqslant \limsup_m \limsup_n |||x_n - y_m||| = \limsup_m \limsup_n \big|\big|\big|x_n - (y_m - y) - y\big|\big|\big| \\
&\geqslant \gamma_k \limsup_m \limsup_n R_k\big(x_n - (y_m - y) - y\big) \\
&= \gamma_k\Big[\limsup_n R_k(x_n) + \limsup_m R_k(y_m - y) + R_k(y)\Big] \\
&= \gamma_k\Big[\limsup_n |||x_n||| + \limsup_m |||y_m - y||| + R_k(y)\Big] \geqslant \gamma_k\big[2 + R_k(y)\big].
\end{aligned}$$

Consequently, if $(y_n) \subset K$ is an a.f.p.s. and $y_n \stackrel{\tau}{\to} y$, we have

$$R_k(y) \leqslant 2\left(\frac{1+\epsilon_1}{\gamma_k} - 1\right).$$

Let

$$p := 1 + \epsilon_1 + 2\left(\frac{1+\epsilon_1}{\gamma_1} - 1\right) > \rho, \qquad \delta \in (\epsilon_1, 1/2), \qquad 0 < \epsilon_2 < \rho - 2\delta.$$

Since, by Lemma 2(1), $\limsup_n \|x_n\| = \limsup_n |||x_n||| < 1 + \epsilon_1$, we can find $x \in K$ such that $\|x\| < 1 + \epsilon_1$. Also there exists $m \in \mathbb{N}$ such that if $k \geqslant m$

$$R_k(x) < \epsilon_2 \quad \big(\text{by property (II)}\big)$$

and

$$\frac{1+\epsilon_1}{1+\delta} < \gamma_k \quad \Big(\text{since } \lim_k \gamma_k = 1\Big).$$

We take $\lambda \in (0, 1)$ such that

$$\lambda < \frac{\rho(1-\gamma_m)}{\gamma_m(p-\rho)}.$$

Since

$$(2-\lambda)\rho + \lambda(\epsilon_2 + 2\delta) = 2\rho - \lambda\big(\rho - (2\delta + \epsilon_2)\big) < 2\rho$$

and

$$\gamma_m\big[(2-\lambda)\rho+\lambda p\big]<\rho(1+\gamma_m)<2\rho,$$

we can find $\epsilon_3>0$ such that

$$\text{(i)}\quad (2-\lambda)(\rho+\epsilon_3)+\lambda(\epsilon_2+2\delta)<2\rho$$

and

$$\text{(ii)}\quad \gamma_m\big[(2-\lambda)(\rho+\epsilon_3)+\lambda p\big]<2\rho.$$

Take $(y_n)\subset K$ to be an a.f.p.s. such that $y_n\xrightarrow{\tau} y$ and

$$\limsup_n\|y_n-y\|=\limsup_n|||y_n-y|||<\rho+\epsilon_3\quad\big(\text{using Lemma 2(1)}\big).$$

There exists $s\in\mathbb{N}$ such that $\|y_N-y\|<\rho+\epsilon_3$ for all $N\geqslant s$ and define

$$z=(1-\lambda)y_s+\lambda x$$

which belongs to K because K is convex.

Let us prove that $\limsup_n|||y_n-z|||<2\rho$. In order to do this, we will prove that there exists $M>0$ such that for all k and $N\geqslant s$ we have

$$\gamma_k R_k(y_N-z)<M<2\rho.$$

We split the proof into two cases:

Case 1: $k\geqslant m$:

$$\begin{aligned}
\gamma_k R_k(y_N-z)&=\gamma_k R_k\big(y_N-(1-\lambda)y_s-\lambda x\big)\\
&\leqslant R_k\big(y_N-y-(1-\lambda)(y_s-y)-\lambda(x-y)\big)\\
&\leqslant R_k(y_N-y)+(1-\lambda)R_k(y_s-y)+\lambda R_k(x-y)\\
&\leqslant \|y_N-y\|+(1-\lambda)\|y_s-y\|+\lambda R_k(x-y)\\
&\leqslant (\rho+\epsilon_3)+(1-\lambda)(\rho+\epsilon_3)+\lambda\big(R_k(x)+R_k(y)\big)\\
&\leqslant (2-\lambda)(\rho+\epsilon_3)+\lambda\big(\epsilon_2+R_k(y)\big)\\
&\leqslant (2-\lambda)(\rho+\epsilon_3)+\lambda\left(\epsilon_2+2\left(\frac{1+\epsilon_1}{\gamma_k}-1\right)\right)\\
&< (2-\lambda)(\rho+\epsilon_3)+\lambda(\epsilon_2+2\delta)<2\rho\quad\text{by (i).}
\end{aligned}$$

Case 2: $k \leqslant m$:

$$\begin{aligned}\gamma_k R_k(y_N - z) &\leqslant \gamma_m R_k\big(y_N - (1-\lambda)y_s - \lambda x\big)\\ &\leqslant \gamma_m\big[R_k\big(y_N - y - (1-\lambda)(y_s - y) - \lambda(x-y)\big)\big]\\ &\leqslant \gamma_m\big[R_k(y_N - y) + (1-\lambda)R_k(y_s - y) + \lambda R_k(x-y)\big]\\ &\leqslant \gamma_m\big[(\rho+\epsilon_3) + (1-\lambda)(\rho+\epsilon_3) + \lambda\big(R_k(x) + R_k(y)\big)\big]\\ &\leqslant \gamma_m\big[(2-\lambda)(\rho+\epsilon_3) + \lambda\big(1+\epsilon_1 + R_k(y)\big)\big]\\ &\leqslant \gamma_m\left[(2-\lambda)(\rho+\epsilon_3) + \lambda\left(1+\epsilon_1 + 2\left(\frac{1+\epsilon_1}{\gamma_1} - 1\right)\right)\right] < 2\rho \quad \text{by (ii).}\end{aligned}$$

Take

$$\begin{aligned}M = \max\bigg\{&(2-\lambda)(\rho+\epsilon_3) + \lambda(\epsilon_2 + 2\delta),\\ &\gamma_m\left[(2-\lambda)(\rho+\epsilon_3) + \lambda\left(1+\epsilon_1+2\left(\frac{1+\epsilon_1}{\gamma_1} - 1\right)\right)\right]\bigg\}.\end{aligned}$$

Then, for all $N \geqslant s$, $|||y_N - z||| < M < 2\rho$. Thus $\limsup_n |||y_n - z||| < 2\rho$ and this finishes the proof. □

5. Applications to the Lebesgue function space $L_1(\mu)$

In this section we are going to consider the Banach space $L_1(\mu)$, where (Σ, Ω, μ) is a σ-finite measure space and we will apply Theorem 1 to this space. As a consequence we will obtain new results about renorming and FPP for nonreflexive subspaces of $L_1(\mu)$.

In order to do that we will define a family of seminorms $\{R_k(\cdot)\}_{k\geqslant 1}$ which satisfies properties (I), (II), (III) and (IV) stated in Section 3.

We denote by $\|\cdot\|$ the usual norm on $L_1(\mu)$, that is

$$\|x\| = \int_\Omega |x|\,d\mu, \quad \text{for all } x \in L_1(\mu)$$

and $R_1(x) = \|x\|$ for all $x \in L_1(\mu)$.

Let $\Omega = \bigcup_n \Omega_n$ be such that $\mu(\Omega_n) < +\infty$ for all $n \in \mathbb{N}$ and denote $A_k = \bigcup_{n=1}^k \Omega_n$. For $k \geqslant 2$ define the seminorms

$$R_k(x) = \sup\left\{\int_{E\cap A_k} |x|\,d\mu\colon\ \mu(E) < \frac{1}{k}\right\} + \|x\chi_{A_k^c}\|. \tag{1}$$

Let τ be the topology of locally convergence in measure, which is given by the metric

$$d(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n}\frac{1}{\mu(\Omega_n)}\int_{\Omega_n} \frac{|x-y|}{1+|x-y|}\,d\mu; \quad x, y \in L_1(\mu).$$

This topology is related to the convergence almost everywhere in the following way: every sequence that converges almost everywhere also converges locally in measure to the same function. Moreover, if a sequence converges locally in measure, then it has a subsequence that converges almost everywhere [15], pp. 157–158.

Fix any nondecreasing sequence $(\gamma_k)_k \subset (0, 1)$ such that $\lim_k \gamma_k = 1$ and define the equivalent norm on $L_1(\mu)$ as

$$|||x||| = \sup_k \gamma_k R_k(x).$$

Now we have all the ingredients to state the following:

Theorem 2. *The seminorms $\{R_k(\cdot)\}_k$ defined above satisfy properties* (I), (II), (III) *and* (IV) *stated in Section* 3. *Thus the following holds*: *Let C be a convex bounded closed subset of $L_1(\mu)$ and $T : C \to C$ a $|||\cdot|||$-nonexpansive mapping. If every sequence in C has a subsequence which is almost everywhere convergent in $L_1(\mu)$, then T has a fixed point.*

Remark 1. Notice that the above fixed point result does not hold for the usual norm in $L_1(\mu)$. Indeed, consider (h_n) a disjointly supported normalized sequence in $L_1(\mu)$. Let

$$C = \left\{ \sum_n t_n h_n;\ t_n \geqslant 0,\ \sum_n t_n = 1 \right\} = \overline{co}(h_n).$$

The set C is closed convex bounded and every sequence in C has a convergent subsequence almost everywhere. Indeed, consider a sequence $(f_k) \subset C$. Then $f_k = \sum_n t_n(k) h_n$ where $t_n(k) \geqslant 0$ and $\sum_n t_n(k) = 1$ for every k. Define $t^k = (t_n(k))_n$. The sequence $(t^k)_k$ belongs to the unit ball of ℓ_1 which is $\sigma(\ell_1, c_0)$-compact. So there exists a subsequence, denoted again by t^k, which is $\sigma(\ell_1, c_0)$-convergent to some $t = (t_n)_n$ belonging to unit ball of ℓ_1. Now we can easily check that (f_k) is pointwise convergent to $f = \sum_n t_n h_n$ (notice that f is not, in general, in C).

Define $T : C \to C$ by

$$T\left(\sum_n t_n h_n\right) = \sum_n t_n h_{n+1}.$$

It is easy to check that T is $\|\cdot\|$-nonexpansive and has no fixed point in C.

Remark 2. If the measure space is not σ-finite and we consider the convergence almost everywhere in $L_1(\mu)$, we cannot renorm the space $L_1(\mu)$ so that Theorem 2 remains true. Indeed, in this case $\ell_1(\Gamma)$ is contained isometrically in $L_1(\mu)$ for some uncountable set Γ and every sequence in a bounded subset of $\ell_1(\Gamma)$ has a pointwise convergent subsequence. So if Theorem 2 holds then we would have a renorming in $\ell_1(\Gamma)$ with the FPP. This is impossible since every renorming of $\ell_1(\Gamma)$ contains an asymptotically isometric copy of ℓ_1 and then it fails the FPP [8].

Before proving Theorem 2 we give a simpler definition of the $|||\cdot|||$ norm in two special cases: when μ is finite and when μ is purely atomic.

Remark 3. Assume that the measure μ is finite. Consider $A_k = \Omega$ for all k. Then $A_k^c = \emptyset$ and $\|x\chi_{A_k^c}\| = 0$ for all $x \in L_1(\mu)$. Therefore

$$R_k(x) = \sup\left\{ \int_E |x|\,d\mu\colon\ \mu(E) < \frac{1}{k} \right\}$$

for all $k \in \mathbb{N}$.

In this case, the topology of the convergence locally in measure is given by the metric

$$d(x, y) = \int_\Omega \frac{|x-y|}{1+|x-y|}\,d\mu; \quad x, y \in L_1(\mu),$$

and the convergence with respect to this topology is the convergence in measure.

Remark 4. Assume now that $\Omega = \mathbb{N}$ and μ is the counting measure defined on the subsets of $\mathbb{N}$. Then the space $L_1(\mu)$ becomes the sequence space ℓ_1. Denote $\Omega_k = \{n\}$ and $A_k = \{1, \dots, n\}$. Then

$$R_k(x) = \|x\chi_{A_k^c}\| = \sum_{n=k+1}^{\infty} \left|x(n)\right|.$$

In this case we recover again the $|||\cdot|||$ norm defined by P.K. Lin in [20] for the particular case $\gamma_k = 8^k/(1+8^k)$. Notice that the convergence almost everywhere in ℓ_1 is the pointwise convergence and every bounded sequence in ℓ_1 has a pointwise convergent subsequence because the unit ball of ℓ_1 is $\sigma(\ell_1, c_0)$-compact.

In general, if the measure space is σ-finite and purely atomic, $L_1(\mu)$ is isometric to ℓ_1. Thus by Theorem 2, it can be renormed to have the FPP.

Now, to prove Theorem 2 we only have to check that properties (I), (II), (III), (IV) are fulfilled for the family of seminorms $R_k(\cdot)$ defined on $L_1(\mu)$.

Proof. To simplify the notation we let

$$S_k(x) := \sup\left\{ \int_E |x|\,d\mu\colon\ \mu(E) < \frac{1}{k} \right\}$$

for $x \in L_1(\mu)$, therefore

$$R_k(x) = S_k(x\chi_{A_k}) + \|x\chi_{A_k^c}\|.$$

(I) Using the absolute continuity of the norm and that $\lim_k \|x\chi_{A_k^c}\| = 0$ we can check that $\lim_k R_k(x) = 0$ for all $x \in L_1(\mu)$.

(II) We have $S_k(x\chi_{A_k}) \leqslant \|x\chi_{A_k}\|$. Therefore $R_k(x) \leqslant \|x\chi_{A_k}\| + \|x\chi_{A_k^c}\| = \|x\|$ for every $x \in L_1(\mu)$.

(III) Fix $k \geqslant 1$ and let (x_n) be a sequence convergent to the null function locally in measure. Assume the contrary, that is, $\limsup_n R_k(x_n) < \limsup_n \|x_n\|$. We can take a sequence, again denoted by (x_n), such that $\lim_n \|x_n\|$, $\lim_n S_k(x_n \chi_{A_k})$, $\lim_n \|x_n \chi_{A_k}\|$ and $\lim_n \|x_n \chi_{A_k^c}\|$ exist, (x_n) converges to the null function almost everywhere and $\lim_n R_k(x_n) < \lim_n \|x_n\|$.

Let us prove that $\lim_n S_k(x_n \chi_{A_k}) = \lim_n \|x_n \chi_{A_k}\|$:

Using Egoroff's Theorem there exists a measurable set $E \subset A_k$ with $\mu(E) < 1/k$ and such that $x_n \to 0$ uniformly on $A_k \setminus E$. In particular $x_n \chi_{A_k \setminus E} \to 0$ in norm and $\lim_n \|x_n \chi_{A_k}\| = \lim_n \|x_n \chi_E\|$. Therefore

$$\lim_n \|x_n \chi_{A_k}\| \geqslant \lim_n S_k(x_n \chi_{A_k}) \geqslant \lim_n \|x_n \chi_E\| = \lim_n \|x_n \chi_{A_k}\|.$$

Now

$$\begin{aligned}
\lim_n R_k(x_n) &\geqslant \lim_n S_k(x_n \chi_{A_k}) + \lim_n \|x_n \chi_{A_k^c}\| \\
&= \lim_n \|x_n \chi_{A_k}\| + \lim_n \|x_n \chi_{A_k^c}\| \\
&= \lim_n \|x_n\|
\end{aligned}$$

which is a contradiction and property (III) holds.

(IV) If $k = 1$, since $R_1(\cdot) = \|\cdot\|$ and using [4] we obtain

$$\limsup_n \|x_n + x\| = \limsup_n \|x_n\| + \|x\|.$$

Assume that $k \geqslant 2$. Suppose by contradiction that property (IV) does not hold. We recall the following lemma for finite measure spaces [1]: Let (Ω, σ, μ) be a finite measure space and (h_n) be a bounded sequence in $L_1(\mu)$ converging to the null function in measure. Then there exists a subsequence (h_{n_l}) and a sequence of pairwise disjoint measurable sets (E_l) such that

$$\lim_l \|h_{n_l} - h_{n_l} \chi_{E_l}\| = 0.$$

In particular, for all $k \geqslant 1$, $\lim_l S_k(h_{n_l} - h_{n_l} \chi_{E_l}) = 0$.

Using the above result we can take a subsequence, again denoted by (x_n), such that there exists a sequence (h_n) of measurable functions defined in A_k which is disjointly supported, $\lim_n S_k((x_n - h_n)\chi_{A_k}) = 0$ and $\limsup_n R_k(x_n + x) < \limsup_n R_k(x_n) + R_k(x)$ for some $x \in L_1(\mu)$. Therefore

$$\begin{aligned}
\limsup_n R_k(x_n + x) &= \limsup_n S_k\big((x_n + x)\chi_{A_k}\big) + \limsup_n \big\|(x_n + x)\chi_{A_k^c}\big\| \\
&= \limsup_n S_k\big((h_n + x)\chi_{A_k}\big) + \limsup_n \|x_n \chi_{A_k^c}\| + \|x \chi_{A_k^c}\|.
\end{aligned}$$

Let us prove that $\limsup_n S_k((h_n + x)\chi_{A_k}) = \limsup_n S_k(h_n \chi_{A_k}) + S_k(x \chi_{A_k})$:

Denote by $E_n \subset A_k$ the support of the function h_n and let $\epsilon > 0$. By the definition of $S_k(\cdot)$, there exists a measurable set $A \subset A_k$ with $\mu(A) < \frac{1}{k}$ such that

$$\int_A |x|\,d\mu \geqslant S_k(x\chi_{A_k}) - \epsilon.$$

Since $\sum_n \mu(E_n) \leqslant \mu(A_k) < +\infty$ there exists n_0 such that $\mu(A) + \sum_{n \geqslant n_0} \mu(E_n) < \frac{1}{k}$. Therefore

$$\begin{aligned}
\limsup_n S_k(h_n + x) &\geqslant \limsup_n \int_{A \cup \bigcup_{n \geqslant n_0} E_n} |h_n + x|\,d\mu \\
&= \limsup_n \int_{A \cup \bigcup_{n \geqslant n_0} E_n} |h_n|\,d\mu + \int_{A \cup \bigcup_{n \geqslant n_0} E_n} |x|\,d\mu \\
&\geqslant \limsup_n \int_{E_n} |h_n|\,d\mu + \int_A |x|\,d\mu \\
&\geqslant \limsup_n \|h_n\chi_{A_k}\| + S_k(x\chi_{A_k}) - \epsilon \\
&= \limsup_n S_k(h_n\chi_{A_k}) + S_k(x\chi_{A_k}) - \epsilon.
\end{aligned}$$

Since ϵ is arbitrary we obtain the desired equality. Therefore:

$$\begin{aligned}
\limsup_n R_k(x_n + x) &= \limsup_n S_k(h_n\chi_{A_k}) + S_k(x\chi_{A_k}) + \limsup_n \|x_n\chi_{A_k^c}\| + \|x\chi_{A_k^c}\| \\
&= \limsup_n S_k(x_n\chi_{A_k}) + S_k(x\chi_{A_k}) + \limsup_n \|x_n\chi_{A_k^c}\| + \|x\chi_{A_k^c}\| \\
&= \limsup_n R_k(x_n) + R_k(x)
\end{aligned}$$

and we obtain (IV). □

Although it is well known that the space $L_1(\mu)$ does not have the FPP (it does not satisfies the weak fixed point property w-FPP [2]), in 1980, B. Maurey [23] proved that all reflexive subspaces of $L_1(\mu)$ do have the FPP. In 1997 P. Dowling and C. Lennard [7] proved that the converse holds, that is, a subspace X of $L_1(\mu)$ has the FPP if and only if X is reflexive. This leads us to the following question: Can a nonreflexive subspace of $L_1(\mu)$ be renormed to have the FPP? Theorem 2 lets us give a partial answer to this question:

Corollary 3. *Let X be a closed subspace of $L_1(\mu)$. If the unit ball of X is relatively sequentially compact for the topology of the convergence locally in measure, then $(X, |||\cdot|||)$ has the FPP.*

Corollary 4. *Let X be a closed subspace of $L_1(\mu)$. If X is a dual space such that the topology of the convergence locally in measure coincides with the weak star topology on the unit ball of X, then $(X, |||\cdot|||)$ has the FPP.*

Notice that this is the case of the sequence space ℓ_1. Here we present another example.

Example 4. Let $\mathbb{D}$ denote the open unit disc. The Bergman space $L_a(\mathbb{D})$ is defined as the subspace of $L_1(\mathbb{D})$ of all analytic functions on $\mathbb{D}$. This space is a dual space and for bounded sequences weak* convergence is equivalent to uniform convergence on compact sets [24]. This shows that the weak* topology is finer than the topology of convergence in measure on the unit ball of $L_a(\mathbb{D})$ and consequently these two topologies coincide for $B_{L_a(\mathbb{D})}$. Thus the Bergman space endowed with the $|||\cdot|||$ norm given in this section has the FPP. Notice that, from P.K. Lin's paper [20], it is deduced that the Bergman space can be renormed to have the FPP. Indeed, J. Lindenstrauss, A. Pelczynski [21] proved that the Bergman space and the sequence space ℓ_1 are isomorphic, although they did not give an explicit definition of the isomorphism. In fact, it turns out to be a difficult problem to find a system of functions which is a basis in $L_a(\mathbb{D})$ equivalent to the unit vector basis in ℓ_1 (see Notes and Remarks in Chapter III.A of [25] and the references therein). However, using Theorem 2 we can give explicitly the renorming on the Bergman space with the FPP.

The following example satisfies the hypothesis of Corollary 3 but does not fit in the scope of Corollary 4:

Example 5. In [11], Théorème 7, we can find an example of a subspace X of $L_1[0,1]$ such that its unit ball B_X is compact for the topology of convergence in measure but not locally convex for this topology. Then, by Corollary 3, $(X, |||\cdot|||)$ has the FPP.

To finish we show another example of a Banach space which can be renormed to satisfy the FPP:

Example 6. In [3] we can find an example of a Banach space E contained in L_1, over a probability space, and such that E fails to have the Radon–Nikodym property and every L_1-bounded sequence in E has a subsequence converging in measure. Applying again Theorem 2, we deduce that E can be renormed to have the FPP and by the failure of the Radon–Nikodym property, E is not isomorphic to any subspace of ℓ_1.

Acknowledgments

The authors would like to thank Professors T. Domínguez Benavides, B. Sims and A.T.-M. Lau for their valuable suggestions in the preparation of this paper.

References

[1] F. Albiac, N.J. Kalton, Topics in Banach Space Theory, Grad. Texts in Math., vol. 233, Springer-Verlag, New York, 2006.
[2] D.E. Alspach, A fixed point free nonexpansive map, Proc. Amer. Math. Soc. 82 (1981) 423–424.
[3] J. Bourgain, H.P. Rosenthal, Martingales valued in certain subspaces of L^1, Israel J. Math. 37 (1–2) (1980) 54–75.
[4] M. Brezis, E. Lieb, A relation between pointwise convergence of functions and convergence of functionals, Proc. Amer. Math. Soc. 88 (3) (1993) 486–490.
[5] F.E. Browder, Nonexpansive nonlinear operators in a Banach space, Proc. Natl. Acad. Sci. USA 54 (1965) 1041–1044.
[6] T. Domínguez Benavides, A renorming of some nonseparable Banach spaces with the Fixed Point Property, J. Math. Anal. Appl. 350 (2) (2009) 525–530.
[7] P.N. Dowling, C.J. Lennard, Every nonreflexive subspace of $L_1[0,1]$ fails the fixed point property, Proc. Amer. Math. Soc. 125 (2) (1997) 443–446.

[8] P.N. Dowling, C.J. Lennard, B. Turett, Reflexivity and the fixed point property for nonexpansive maps, J. Math. Anal. Appl. 200 (1996) 653–662.
[9] P. Eymard, L'algèbre de Fourier d'un groupe localement compact, Bull. Soc. Math. France 92 (1964) 181–236.
[10] H. Fetter, personal communication.
[11] G. Godefroy, N.J. Kalton, D. Li, Propriété d'approximation métrique inconditionnelle et sous-espaces de L^1 dont la boule est compacte en measure, C. R. Acad. Sci. Paris Sér. I 320 (1995) 1069–1073.
[12] K. Goebel, W.A. Kirk, Topics in Metric Fixed Point Theory, Cambridge University Press, 1990.
[13] D. Göhde, Zum Prinzip der kontraktiven Abbildung, Math. Nachr. 30 (1965) 251–258.
[14] C. Graham, A.T.-M. Lau, M. Leinert, Separable translation-invariant subspaces of $M(G)$ and other dual spaces on locally compact groups, Colloq. Math. 55 (1) (1988) 131–145.
[15] E. Hewitt, K. Stromberg, Real and Abstract Analysis, Springer-Verlag, 1965.
[16] W.A. Kirk, A fixed point theorem for mappings which do not increase distances, Amer. Math. Monthly 72 (1964) 1004–1006.
[17] W.A. Kirk, B. Sims, Handbook of Metric Fixed Point Theory, Kluwer Acad. Publ., Dordrecht, 2001.
[18] A.T.-M. Lau, M. Leinert, Fixed point property and the Fourier algebra of a locally compact group, Trans. Amer. Math. Soc. 360 (12) (2008) 6389–6402.
[19] A.T.-M. Lau, P.F. Mah, Fixed point property for Banach algebras associated to locally compact groups, J. Funct. Anal. 258 (2) (2010) 357–372.
[20] P.K. Lin, There is an equivalent norm on ℓ_1 that has the fixed point property, Nonlinear Anal. 68 (8) (2008) 2303–2308.
[21] J. Lindenstrauss, A. Pelczynski, Contributions of the theory of the classical Banach spaces, J. Funct. Anal. 8 (2) (1971) 225–249.
[22] J. Lindenstrauss, L. Tzafriri, Classical Banach spaces II, Springer-Verlag, 1979.
[23] B. Maurey, Points fixes des contractions sur un convexe ferme de L_1, in Seminaire d'Analyse Fonctionnelle, 1980–1981, Ecole Polytechnique.
[24] M. Nowak, Compact Hankel operators with conjugate analytic symbols, Rend. Circ. Mat. Palermo (2) XLVII (1998) 363–374.
[25] P. Wojtaszczyk, Banach Spaces of Analysts, Cambridge Stud. Adv. Math., vol. 25, 1991.