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# On the spectral radius of graphs ☆

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**Abstract**

Let $G$ be a simple undirected graph. For $v \in V(G)$, the 2-degree of $v$ is the sum of the degrees of the vertices adjacent to $v$. Denote by $\rho(G)$ and $\mu(G)$ the spectral radius of the adjacency matrix and the Laplacian matrix of $G$, respectively. In this paper, we present two lower bounds of $\rho(G)$ and $\mu(G)$ in terms of the degrees and the 2-degrees of vertices.


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## 1. Introduction

Let $G = (V, E)$ be a simple undirected graph with $n$ vertices and $m$ edges. For $v_i \in V$, the degree of $v_i$, written by $d_i$, is the number of edges incident with $v_i$. Let $\delta(G) = \delta$ and $\Delta(G) = \Delta$ be the minimum degree and the maximum degree of vertices of $G$, respectively. A graph $G$ is called *regular* if every vertex of $G$ has equal

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degree. A bipartite graph is called *semiregular* if each vertex in the same part of a bipartition has the same degree.

The 2-*degree* of $v_i$ [2] is the sum of the degrees of the vertices adjacent to $v_i$ and denoted by $t_i$. We call $\frac{t_i}{d_i}$ the average-degree of $v_i$. A graph $G$ is called *pseudo-regular* if every vertex of $G$ has equal average-degree. A bipartite graph is called *pseudo-semiregular* if each vertex in the same part of a bipartition has the same average-degree. Obviously, any regular graph is a pseudo-regular graph and any semiregular bipartite graph is a pseudo-semiregular bipartite graph. Conversely, a pseudo-regular graph may be not a regular graph, such as $S(K_{1,3})$, and a pseudo-semiregular bipartite graph may be not a semiregular bipartite graph, such as $S(K_{1,n-1})$ $(n \geqslant 5)$, where $S(K_{1,t})$ is the graph obtained by subdividing each edge of $K_{1,t}$ one time.

Let $A(G)$ be the adjacency matrix of $G$ and $D(G) = \mathrm{diag}(d(v_1), d(v_2), \ldots, d(v_n))$ be the diagonal matrix of vertex degrees. Then the Laplacian matrix of $G$ is $L(G) = D(G) - A(G)$. Clearly, $A(G)$ and $L(G)$ are real symmetric matrices. Hence their eigenvalues are real numbers. We denote by $\rho(M)$ the largest eigenvalue of a symmetric matrix $M$. For a graph $G$, we denote by $\rho(G)$ the largest eigenvalue of $A(G)$ and call it the spectral radius of $G$; we denote by $\mu(G)$ the largest eigenvalue of $L(G)$ and call it the Laplacian spectral radius of $G$. When $G$ is connected, $A(G)$ is irreducible and so by Perron–Frobenius Theorem, $\rho(G)$ is simple.

Up to now, many bounds for $\rho(G)$ and $\mu(G)$ were given (see, for instance, [1–12]), but most of them are upper bounds. In the paper, we give two new lower bounds on $\rho(G)$ and $\mu(G)$ of $G$ in terms of the degrees and the 2-degrees of vertices of $G$, from which we can get some known results.

## 2. Lemmas and results

**Lemma 1** [5]. *Let $A$ be a nonnegative symmetric matrix and $x$ be a unit vector of $\mathscr{R}^n$. If $\rho(A) = x^{\mathrm{T}}Ax$, then $Ax = \rho(A)x$.*

**Lemma 2** [13]. *Let $d_1, d_2, \ldots, d_n$ be the degree sequence of a simple graph. Then*

$$\sum_{i=1}^{n} d_i^2 \leqslant \left(\sum_{i=1}^{n} \sqrt{d_i}\right)^2,$$

*with equality if and only if the graph is empty.*

**Lemma 3** [1]. *Let $G$ be a simple connected graph. Then*

$$\mu(G) \leqslant \max\{d(u) + d(v) : uv \in E(G)\},$$

*with equality if and only if $G$ is a regular or semiregular bipartite graph.*

The following theorem is one of our main results.

**Theorem 4.** *Let G be a connected graph with degree sequence $d_1, d_2, \ldots, d_n$. Then*

$$\rho(G) \geqslant \sqrt{\frac{t_1^2 + t_2^2 + \cdots + t_n^2}{d_1^2 + d_2^2 + \cdots + d_n^2}},$$

*with equality if and only if G is a pseudo-regular graph or a pseudo-semiregular bipartite graph.*

**Proof.** Let $X = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$ be the unit positive eigenvector of $A$ corresponding to $\rho(A)$. Take

$$C = \sqrt{\frac{1}{\sum_{i=1}^n d_i^2}}(d_1, d_2, \ldots, d_n)^{\mathrm{T}}.$$

Noting that $C$ is a unit positive vector, we have

$$\rho(G) = \rho(A) = \sqrt{\rho(A^2)} = \sqrt{X^{\mathrm{T}}A^2X} \geqslant \sqrt{C^{\mathrm{T}}A^2C}.$$

Since

$$\begin{aligned} AC &= \sqrt{\frac{1}{\sum_{i=1}^n d_i^2}} \left( \sum_{j=1}^n a_{1j}d_j, \ldots, \sum_{j=1}^n a_{nj}d_j \right)^{\mathrm{T}} \\ &= \sqrt{\frac{1}{\sum_{i=1}^n d_i^2}}(t_1, \ldots, t_n)^{\mathrm{T}}, \end{aligned} \qquad (*)$$

we have

$$\rho(G) = \rho(A) \geqslant \sqrt{C^{\mathrm{T}}A^2C} = \sqrt{\frac{\sum_{i=1}^n t_i^2}{\sum_{i=1}^n d_i^2}}.$$

If the equality holds, then

$$\rho(A^2) = C^{\mathrm{T}}A^2C.$$

By Lemma 1, $A^2C = \rho(A^2)C$. If the multiplicity of $\rho(A^2)$ is one, then $X = C$, which implies $t_i = \rho(G)d_i$ $(1 \leqslant i \leqslant n)$. Hence $G$ is a pseudo-regular graph. Otherwise, the multiplicity of $\rho(A^2) = (\rho(A))^2$ is two, which implies that $-\rho(A)$ is also an eigenvalue of $G$. Then $G$ is a connected bipartite graph (see Theorem 3.4 in [3]). Without loss of generality, we assume

$$A = \begin{pmatrix} 0 & B \\ B^{\mathrm{T}} & 0 \end{pmatrix},$$

where $B = (b_{i,j})$ is an $n_1 \times n_2$ matrix with $n_1 + n_2 = n$. Let

$$X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

and

$$C = \sqrt{\frac{1}{\sum_{i=1}^{n} d_i^2}} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix},$$

where $X_1 = (x_1, x_2, \ldots, x_{n_1})^{\mathrm{T}}$, $X_2 = (x_{n_1+1}, x_{n_1+2}, \ldots, x_n)^{\mathrm{T}}$, $C_1 = (d_1, d_2, \ldots, d_{n_1})^{\mathrm{T}}$ and $C_2 = (d_{n_1+1}, d_{n_1+2}, \ldots, d_n)^{\mathrm{T}}$. Since

$$A^2 = \begin{pmatrix} BB^{\mathrm{T}} & 0 \\ 0 & B^{\mathrm{T}}B \end{pmatrix},$$

we have

$$BB^{\mathrm{T}}C_1 = \rho(A^2)C_1, \quad B^{\mathrm{T}}BC_2 = \rho(A^2)C_2$$

and

$$BB^{\mathrm{T}}X_1 = \rho(A^2)X_1, \quad B^{\mathrm{T}}BX_2 = \rho(A^2)X_2.$$

Noting that $BB^{\mathrm{T}}$ and $B^{\mathrm{T}}B$ have the same nonzero eigenvalues, $\rho(A^2)$ is the spectral radius of $BB^{\mathrm{T}}$ and its multiplicity is one. So $X_1 = p_1C_1$ ($p_1$ is a constant), which implies $\frac{t_i}{d_i} = \frac{t_j}{d_j}$ $(1 \leqslant i < j \leqslant n_1)$. Similarly, $X_2 = p_2C_2$ ($p_2$ is a constant), which implies $\frac{t_i}{d_i} = \frac{t_j}{d_j}$ $(n_1 + 1 \leqslant i < j \leqslant n)$. Hence $G$ is a pseudo-semiregular graph.

Conversely, if $G$ is pseudo-regular, then $\frac{t_i}{d_i} = p$ $(1 \leqslant i \leqslant n)$ is a constant, which implies $AC = pC$. It is known that for any positive eigenvector of a nonnegative matrix, the corresponding eigenvalue is the spectral radius of that matrix. Hence $\rho(G) = p = \sqrt{\frac{t_1^2+t_2^2+\cdots+t_n^2}{d_1^2+d_2^2+\cdots+d_n^2}}$.

If $G$ is a pseudo-semiregular bipartite graph, we assume

$$A = \begin{pmatrix} 0 & B \\ B^{\mathrm{T}} & 0 \end{pmatrix},$$

$\frac{t_i}{d_i} = p_1$ $(1 \leqslant i \leqslant n_1)$ and $\frac{t_i}{d_i} = p_2$ $(n_1 + 1 \leqslant i \leqslant n)$, where $B = (b_{i,j})$ is an $n_1 \times n_2$ matrix with $n_1 + n_2 = n$. Let $C_1 = (d_1, d_2, \ldots, d_{n_1})^{\mathrm{T}}$ and $C_2 = (d_{n_1+1}, d_{n_1+2}, \ldots, d_n)^{\mathrm{T}}$. So for each $i$ $(1 \leqslant i \leqslant n_1)$, the $i$th element of $BB^{\mathrm{T}}C_1$ is

$$\begin{aligned} r_i(BB^{\mathrm{T}}C_1) &= \sum_{j=1}^{n_1}\sum_{k=1}^{n_2} b_{ik}b_{jk}d_j = \sum_{k=1}^{n_2} b_{ik} \sum_{j=1}^{n_1} b_{jk}d_j \\ &= \sum_{k=1}^{n_2} b_{ik}p_2d_{n_1+k} = p_1p_2d_i. \end{aligned}$$

Similarly, $r_j(B^{\mathrm{T}}BC_2) = p_1p_2d_{n_1+j}$, for each $j$ $(1 \leqslant j \leqslant n_2)$. Hence $A^2C = p_1p_2C$, where $C = \sqrt{\frac{1}{d_1^2+d_2^2+\cdots+d_n^2}}(d_1, d_2, \ldots, d_n)^{\mathrm{T}}$. It is known that for any positive eigenvector of a nonnegative matrix, the corresponding eigenvalue is the spectral radius of that matrix. So

$$\rho(A^2) = p_1 p_2 = C^{\mathrm{T}} A^2 C.$$

From equality (∗), we have

$$\rho(A^2) = p_1 p_2 = \frac{t_1^2 + t_2^2 + \cdots + t_n^2}{d_1^2 + d_2^2 + \cdots + d_n^2}.$$

It follows that

$$\rho(G) = \sqrt{\frac{t_1^2 + t_2^2 + \cdots + t_n^2}{d_1^2 + d_2^2 + \cdots + d_n^2}}.$$

This completes the proof. □

**Corollary 5**

(1) *Let $G$ be a pseudo-regular graph with $t(v) = pd(v)$ for each $v \in V(G)$, then $\rho(G) = p$.*

(2) *Let $G$ be a pseudo-semiregular bipartite graph with the bipartition $(X, Y)$. If $t(v) = p_x d(v)$ for each $v \in X$ and $t(v) = p_y d(v)$ for each $v \in Y$, then $\rho(G) = \sqrt{p_x p_y}$.*

According to Corollary 5, it is very easy to compute the spectral radius of pseudo-regular graphs and pseudo-semiregular bipartite graphs.

**Example.** Let $S(K_{1,k})$ be the graph obtained by subdividing each edge of $K_{1,k}$ one time and $G_1$ and $G_2$ are the graphs shown in Fig. 1. Obviously, $G_1$ is a pseudo-regular graph and $G_2$ is a pseudo-semiregular bipartite graph. When $k = 3$, $S(K_{1,k})$ is a pseudo-regular graph; otherwise, $S(K_{1,k})$ is a pseudo-semiregular bipartite graph. Hence we have the following results:

(1) $\rho(G_1) = 4$.
(2) $\rho(S(K_{1,k})) = \sqrt{k + 1}$.
(3) $\rho(G_2) = 2\sqrt{2}$.

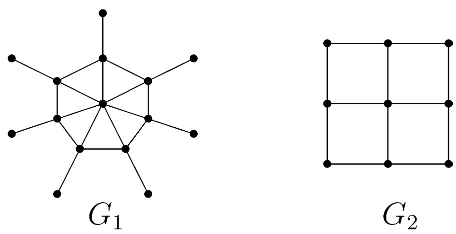


Fig. 1.

**Corollary 6** [4, 5]. *Let $G$ be a connected graph with degree sequence $d_1, d_2, \ldots, d_n$. Then*

$$\rho(G) \geqslant \sqrt{\frac{1}{n}\sum_{i=1}^{n} d_i^2},$$

*with equality if and only if $G$ is either a regular connected graph or a semiregular connected bipartite graph.*

**Proof.** By Theorem 4 and the Cauchy–Schwarz inequality,

$$\rho(G) \geqslant \sqrt{\frac{t_1^2 + t_2^2 + \cdots + t_n^2}{d_1^2 + d_2^2 + \cdots + d_n^2}} \geqslant \sqrt{\frac{(t_1 + t_2 + \cdots + t_n)^2}{n(d_1^2 + d_2^2 + \cdots + d_n^2)}}.$$

Since

$$t_1 + t_2 + \cdots + t_n = d_1^2 + d_2^2 + \cdots + d_n^2,$$

we have

$$\rho(G) \geqslant \sqrt{\frac{1}{n}\sum_{i=1}^{n} d_i^2}.$$

If the equality holds, $G$ is a pseudo-regular graph or a pseudo-semiregular bipartite graph (by Theorem 4) with $t_i = t_j$ for all $1 \leqslant i < j \leqslant n$. Thus $G$ is a regular connected graph or a semiregular connected graph. Conversely, if $G$ is a regular connected graph, the equality holds immediately. If $G$ is a semiregular connected bipartite graph, we assume that $d(v_1) = \cdots = d(v_{n_1}) = \Delta$ and $d(v_{n_1+1}) = \cdots = d(v_n) = \delta$. Since $n_1\Delta = (n - n_1)\delta$, $\sqrt{\frac{1}{n}\sum_{i=1}^{n} d_i^2} = \sqrt{\Delta\delta}$. By Corollary 5, we have $\rho(G) = \sqrt{\Delta\delta}$. Thus the equality holds. □

**Corollary 7.** *Let $G$ be a simple connected graph. Then*

$$\rho(G) \geqslant \frac{2m}{n} \geqslant \delta.$$

**Proof.** By Corollary 6 and the Cauchy–Schwarz inequality,

$$\rho(G) \geqslant \sqrt{\frac{1}{n}\sum_{i=1}^{n} d_i^2} \geqslant \sqrt{\frac{\left(\sum_{i=1}^{n} d_i\right)^2}{n^2}} = \frac{2m}{n} \geqslant \delta. \qquad \square$$

By Lemma 2 and Theorem 4, we have the following:

**Corollary 8.** *Let $G$ be a connected graph with degree sequence $d_1, d_2, \ldots, d_n$. Then*

$$\rho(G) \geqslant \frac{\sqrt{t_1^2 + t_2^2 + \cdots + t_n^2}}{\sqrt{d_1} + \sqrt{d_2} + \cdots + \sqrt{d_n}}.$$

Now we show another main result of the paper.

**Theorem 9.** *Let G be a connected bipartite graph with degree sequence $d_1, d_2, \ldots, d_n$. Then*

$$\mu(G) \geqslant \sqrt{\frac{\sum_{i=1}^{n}(d_i^2 + t_i)^2}{\sum_{i=1}^{n} d_i^2}},$$

*where the equality holds if and only if G is a semiregular connected bipartite graph.*

**Proof.** Note that $D + A$ and $D - A$ have the same nonzero eigenvalues by $G$ being a bipartite graph and $D + A$ is a nonnegative irreducible positive semidefinite symmetric matrix.

Let $X = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$ be the unit positive eigenvector of $D + A$ corresponding to $\mu(G)$. Take

$$C = \sqrt{\frac{1}{\sum_{i=1}^{n} d_i^2}}(d_1, d_2, \ldots, d_n)^{\mathrm{T}}.$$

Then

$$\mu(G) = \sqrt{\rho((D + A)^2)} = \sqrt{X^{\mathrm{T}}(D + A)^2 X} \geqslant \sqrt{C^{\mathrm{T}}(D + A)^2 C}.$$

Since

$$(D + A)C = \sqrt{\frac{1}{\sum_{i=1}^{n} d_i^2}} \left(d_1^2 + \sum_{j=1}^{n} a_{1j} d_j, \ldots, d_n^2 + \sum_{j=1}^{n} a_{nj} d_j\right)^{\mathrm{T}}$$
$$= \sqrt{\frac{1}{\sum_{i=1}^{n} d_i^2}} \left(d_1^2 + t_1, \ldots, d_n^2 + t_n\right)^{\mathrm{T}},$$

we have

$$\mu(G) \geqslant \sqrt{C^{\mathrm{T}}(D + A)^2 C} = \sqrt{\frac{\sum_{i=1}^{n}(d_i^2 + t_i)^2}{\sum_{i=1}^{n} d_i^2}}.$$

If the equality holds, then

$$\rho((D + A)^2) = C^{\mathrm{T}}(D + A)^2 C,$$

which implies that $(D + A)^2 C = \rho((D + A)^2)C$ (by Lemma 1). Since $D + A$ is a nonnegative irreducible positive semidefinite matrix, all eigenvalues of $D + A$ are nonnegative. By Perron–Frobenius Theorem, the multiplicity of $\rho(D + A)$ is one. Since $\rho((D + A)^2) = (\rho(D + A))^2$, we have the multiplicity of $\rho((D + A)^2)$ is one. Hence, if the equality holds then $X = C$. By $\rho(D + A)C = (D + A)C$, we have $\rho(D + A)d_i = d_i^2 + t_i$ for $i = 1, 2, \ldots, n$. Thus $d_i + t_i/d_i = d_j + t_j/d_j$ for all $i \neq j$. Assume, without loss of generality, that $d_1 = \Delta$, $d_2 = \delta$ and $\Delta \neq \delta$. Then we have

$$\Delta + t_1/\Delta = \delta + t_2/\delta.$$

Since $t_1 \geqslant \Delta\delta$ and $t_2 \leqslant \delta\Delta$,

$$\Delta + \delta \leqslant \Delta + t_1/\Delta = \delta + t_2/\delta \leqslant \Delta + \delta.$$

Thus we must have $t_1 = \Delta\delta = t_2$. This implies $d(v) = \Delta$ or $d(v) = \delta$ for all $v \in V(G)$ by $G$ being connected and $uv \notin E(G)$ if $d(u) = d(v)$. Let $Y_1 = \{v : d(v) = \Delta\}$ and $Y_2 = \{v : d(v) = \delta\}$. Then $G = (Y_1, Y_2; E)$ is a semiregular connected bipartite graph.

Conversely, assume that $G$ is a semiregular connected bipartite graph with $d(v_1) = \cdots = d(v_{n_1}) = \Delta$ and $d(v_{n_1+1}) = \cdots = d(v_n) = \delta$. Note that $n_1\Delta = (n - n_1)\delta$. Then

$$\sqrt{\frac{\sum_{i=1}^n (d_i^2 + t_i)^2}{\sum_{i=1}^n d_i^2}} = \sqrt{\frac{n_1(\Delta^2 + \Delta\delta)^2 + (n - n_1)(\delta^2 + \delta\Delta)^2}{n_1\Delta^2 + (n - n_1)\delta^2}} = \delta + \Delta.$$

By Lemma 3, $\mu(G) = \Delta + \delta$ and so the equality holds. □

**Corollary 10** [5]. *Let $G$ be a simple connected bipartite graph with degree sequence $d_1, d_2, \ldots, d_n$. Then*

$$\mu(G) \geqslant 2\sqrt{\frac{1}{n}\sum_{i=1}^n d_i^2},$$

*where the equality holds if and only if $G$ is a regular connected bipartite graph.*

**Proof.** By Theorem 9 and the Cauchy–Schwarz inequality, we have

$$\begin{aligned}\mu(G) &\geqslant \sqrt{\frac{\sum_{i=1}^n (d_i^2 + t_i)^2}{\sum_{i=1}^n d_i^2}} \\ &\geqslant \sqrt{\frac{(d_1^2 + t_1 + d_2^2 + t_2 + \cdots + d_n^2 + t_n)^2}{n(d_1^2 + d_2^2 + \cdots + d_n^2)}} \\ &= \sqrt{\frac{(2d_1^2 + 2d_2^2 + \cdots + 2d_n^2)^2}{n(d_1^2 + d_2^2 + \cdots + d_n^2)}} \\ &= 2\sqrt{\frac{1}{n}\sum_{i=1}^n d_i^2}.\end{aligned}$$

If the equality holds, $G$ is a semiregular connected bipartite graph (by Theorem 9) with $d_i^2 + t_i = d_j^2 + t_j$ for $1 \leqslant i < j \leqslant n$. Without loss of generality, assume that $d_1 = \Delta$ and $d_2 = \delta$. Then we have $\Delta^2 + \delta\Delta = \delta^2 + \delta\Delta$, which implies $\Delta = \delta$. Hence $G$ is a regular connected bipartite graph. Conversely, if $G$ is a regular connected bipartite graph, by Lemma 3, the equality holds immediately. □

**Corollary 11.** *Let G be a simple connected bipartite graph. Then*

$$\mu(G) \geqslant \frac{4m}{n} \geqslant 2\delta.$$

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